Sir,

I have taken the liberty of sending you a copy of the first number of the Mathematical Miscellany, and hope the design of the Work will meet with your approbation.

Should you feel inclined to contribute towards its support by becoming a subscriber, you are respectfully requested to inform me, by mail, of your intention, before the fifteenth of June.

If sufficient patronage be obtained to justify the continuance of the work, agents will be appointed in different parts of the Union to receive the amount of subscription, and the second number will be published on the first day of October next.

I am, Sir,

Yours very respectfully,

C. GILL.

Institute at Flushing, L. I.
March 24, 1836.

SOME of the Instructors of the INSTITUTE AT FLUSHING having understood that "on four fixed days in each year, the 21st of March, 21st of June, 21st of September, and 21st of December, (unless any of these days should fall on Sunday, in which case for the 21st, substitute the 22d,) horary observations of the Barometer, Thermometer, wet and dry Thermometer, clouds, winds, meteors, &c. were to be made by scientific men in different parts of the globe, at the commencement of each hour, (per clock,) mean time at the place, for 37 hours: beginning at 6 o'clock on the morning of the 21st, and ending at 6 o'clock on the evening of the 22d; and that it was deemed highly desirable that the points of observation should be multiplied by the co-operation of societies and individuals:" have made the following observations on the 21st and 22d of March, 1836.

The instruments used were made by the Messrs. Pike of New-York, but they have not been compared, by the observers, with any of acknowledged accuracy.

METEOROLOGICAL OBSERVATIONS,

MADE AT THE INSTITUTE, FLUSHING, L. I., FOR THIRTY-SEVEN SUCCESSIVE HOURS, COMMENCING AT SIX A. M. OF THE TWENTY-FIRST OF MARCH, EIGHTEEN HUNDRED AND THIRTY-SIX, AND ENDING AT SIX P. M. OF THE FOLLOWING DAY.

(*Lat.* 40° 44′ 58″ N.; *Long.* 73° 44′ 20″ W. *Height of barometer above low water mark of Flushing Bay, 54 feet.*)

Hour.	Barometer.	Attached Therm'ter.	Therm'ter.	Wet Bulb Therm'ter.	Winds —from—	Clouds —to—	Strength of wind.	REMARKS.
6	29,90	45	18	18	NE.		Hardly any.	Fine—thin misty vapours in the horizon.
7	29,91	45	21	20½	SW.		"	" "
8	29,92	46	25½	25	"		"	" mist cleared.
9	29,93	46	31½	30½	S.	NE.	Very light.	Light clouds rising in the west.
10	29,91	47	35	32	"	"	"	Streaked white clouds in the W. and NW.
11	29,90	46	38	33½	SE.	"	"	Large spread cloud rising in the W.
12	29,89	46	41½	37½	"	"	"	Grey clouds overspread.
1	29,88	48	39½	33	"	"	Gentle.	"
2	29,86	48	39½	34	"	"	"	Clouds a little darker.
3	29,83	47	37	33	S.	"	"	Black clouds coming up from the W.
4	29,82	47	36	32½	"	"	Fresh.	"
5	29,81	46	35	32	"	"	"	Clouds lighter—breaking in the W.
6	29,80	46	32½	30½	"	"	Gentle.	A few scattered clouds.
7	29,79	44	30½	29	"	"	"	"
8	29,78	43	31½	30½	"	"	"	Overcast—dark clouds.
9	29,78	43	32	31	"	"	Very light.	"
10	29,76	43	33	32	SE.	N.	"	"
11	29,73	42	33	32	"	"	Gentle.	Thin misty low clouds.
12	29,71	42	33½	32½	"	"	"	Fine—a few thin clouds in the NW.
1	29,70	42	32	31	"		"	Clear.
2	29,68	42	32	31	"	N.	Fresh.	Large cloud in the N.
3	29,65	47	32½	31½	"	"	"	Clouds spreading.
4	29,62	48	33	32	"	"	Brisk.	"
5	29,62	48	32	31½	"	"	"	Dark clouds all overspread.
6	29,59	48	33	32¼	"	"	"	" a break in the E.
7	29,58	48	34½	34	"	"	Gentle.	Clouds lowering and spread.
8	29,55	44	36½	36	"	NW.	"	" darker in the W.
9	29,55	44	34	34	S.		Hardly any.	Thick snow, Storm commenced at 8½ A. M. with gentle rain, at 12 there had fallen 5 in. of snow, and at 4 P. M. when the storm ended, there had fallen 7 inches.
10	29,55	43	33½	33½			Calm.	"
11	29,53	43	33½	33½	NW.		Very light.	"
12	29,51	44	32	32	"		Gentle.	"
1	29,50	45	32	32	"		"	Sleet.
2	29,48	45	31½	31½	"		Brisk.	"
3	29,48	45	31	31	"		Fresh.	"
4	29,48	45	31½	31	"	SE.	"	Clouds breaking in the W.
5	29,49	44	31½	31½	N.	"	Gentle.	Dark clouds mostly spread.
6	29,48	45	31	30½	"	"	"	Clouds broken in the W. dark in the NE.

THE

MATHEMATICAL

MISCELLANY.

NUMBER I.

CONDUCTED BY

C. GILL,

PROFESSOR OF MATHEMATICS IN THE INSTITUTE AT FLUSHING, LONG ISLAND.

PUBLISHED AT THE INSTITUTE.

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1836.

ADVERTISEMENT.

The first number of the Mathematical Miscellany is an experiment. It is presented to the public without the courtesy of a prospectus, in the belief that its character and claims to patronage will be better understood by giving a specimen, than a promise of what it will be.

The high estimation in which such works are held by the European mathematicians, and the fact that the great improvements in Analysis, and the vast variety of elegant problems scattered through the elementary works on science in present use, have generally first appeared in them, and are principally due to the discussions and investigations they are calculated to bring forward, are at least presumptive evidence that a miscellany of this kind might be made sufficiently interesting, were the talent of the country, of which there is certainly no want, concentrated in its aid.

The advantages of such a work, as a medium for valuable communications that might otherwise be lost to the public; as an index to mark the taste in science, and the progress in discovery, of the day and of the country; and as a field where the aspirant to mathematical distinction may try his strength with those of established reputation, will be perceived at once by all who would think of patronizing this undertaking.

The Editor has the assurance of assistance from individuals whose names would be a sufficient guarantee for the respectability of the work; and if he succeeds in establishing it, he has no doubt of enlisting in its aid much of the mathematical talent of the United States. He begs leave to commend his undertaking, in particular to gentlemen of the mathematical chairs in our colleges, with the suggestion, whether it might not be made a useful auxiliary in cherishing a spirit of science in their classes. Should this suggestion meet their view, there will be formed a distinct department adapted to this purpose; and pains will be taken to make this part of the work interesting, for it will be peculiarly gratifying to the Editor, if he can supply the means in any degree of fostering the emulation of American youth in a study which is peculiarly adapted to the enquiring mind, and which is daily becoming of more practical importance to the country.

The Mathematical Miscellany will appear semi-annually on the first days of March and October; thus making the summer interval of seven months, and the winter one of five; a distinction which will be at once appreciated by the *student*. The price of each number will be 50 cents, and as it is not designed to secure any profit from the publication, the size of the work will be increased to whatever extent its sale will allow.

Institute at Flushing, L. I.,
February, 1836.

THE

MATHEMATICAL MISCELLANY.

ARTICLE I.

INVESTIGATION OF A FORMULA FOR FINDING THE LONGITUDE AT SEA.

1. The two methods for this purpose in general use at sea, namely, those of Middle Latitude Sailing and Mercator's Sailing, are both of them liable to objection.

It is known that the middle latitude is in general less than the latitude of the parallel on which the departure should be estimated, and since

$$\text{Diff. long.} = \text{departure} \times \text{sec. middle latitude},$$

the difference of longitude found by this method is less than the true one, thus tending to make the ship's calculated place always behind her true place; certainly a less safe error than the opposite one; and when either the middle latitude or the difference of latitude is large, the error is altogether too great to render it safe to employ this method.

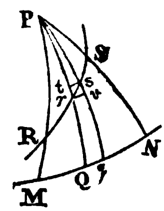
On the contrary, the principles of Mercator's Sailing are strictly true, and the errors in its use are caused only by the defects of the Tables of Meridional parts. These tables have only been computed to the nearest mile, and the consequence is, that the difference between two meridional parts, will sometimes have an error of nearly a whole mile. Now the formula in common use,

$$\text{Diff. long.} = \text{merid. diff. lat.} \times \text{tan. course},$$

shows that when the course is greater than 45° from the meridian, whatever error there may be in the meridional difference of latitude, it will be increased in finding the difference of longitude; and when the course approaches to 90°, the error is multiplied to an alarming extent. Thus, if the error in difference of latitude were ·7, and the course were 7 points from the meridian, or its tangent 5, the error in difference of longitude would be 3½ minutes.

2. My object in this article is to furnish a method combining accuracy in theory with simplicity and truth in practice; how far it may attain the object must be left for the determination of the reader.

The method is perfectly original, so far as my own reading may entitle me to make the assertion; although among the thousand plans and hints, for remedying the evils complained of, that have been scattered through the various periodicals of the last century, it would be singular indeed if none of them has anticipated me either in principle or manner. At any rate, I trust it will be new to the reader, and useful to the public.



3. Let PQ, Pq be any two meridians indefinitely near each other, intercepting the element rs of the rhumb line RS, the element Qq of the equator, and the elements ts, rv of the parallels of latitude through r and s.

Now for the infinitely small portion of the ship's path rs, we may estimate the departure as being made in the parallel of either extremity, and therefore if x represent the degrees, minutes, &c. in the variable latitude Qr, we shall have by the principles of parallel sailing,

$$\text{Diff. long. } (\mathrm{Q}q) = \text{departure} \times \text{sec. } x,$$

but, by plane sailing,

$$\text{departure} = \text{tan. course} \times \text{diff. lat.},$$

$$\therefore \mathrm{Q}q = \text{tan. course} \times rt \times \text{sec. } x;$$

but if R be the earth's radius $= 3437{,}74677078$ minutes of a great circle, or nautical miles, the small arc $rt = \mathrm{R}\,dx$,

$$\therefore \mathrm{Q}q = \text{tan. course} \times \mathrm{R}\,dx \text{ sec. } x \quad . \quad . \quad . \quad . \quad . \quad (1).$$

4. For any definite portion of the path RS, the difference of longitude MN = the sum of all the small arcs Qq, contained in it, or

$$\mathrm{MN} = \mathrm{R} \text{ tan. course} \times \int dx \text{ sec. } x \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Now the integral $\int dx$ sec. x = const. — hyp. log. tan. $\frac{1}{2}(90^\circ - x)$, and in order to convert these logarithmic tangents into the common artificial ones given in the tables, which are Briggian ones, we must multiply by the modulus $\mathrm{M} = 2{,}30258509299$ of the common system of logarithms, then

$$\int dx \text{ sec. } x = \text{const.} - \mathrm{M} \times \text{l. tan. } \tfrac{1}{2}(90^\circ - x),$$

and this taken between the limits $x = \mathrm{L}$ and $x = l$, the latitudes of the two extremities R and S of the ship's path, gives

$$\int dx \text{ sec. } x = \mathrm{M}\left\{\text{l. tan. } \tfrac{1}{2}(90^\circ - l) - \text{l. tan. } \tfrac{1}{2}(90^\circ - \mathrm{L})\right\}.$$

Hence equation (2) becomes

$$\mathrm{MN} = \mathrm{RM} \times \text{tan. course} \times \left\{\text{l. tan. } \tfrac{1}{2}(90^\circ - l) - \text{l. tan. } \tfrac{1}{2}(90^\circ - \mathrm{L})\right\} \,. \;(3).$$

5. The number $\mathrm{RM} = 7915{,}70446788$ and its logarithm $= 3{,}8984895$; and if we represent the difference between the tangents of the half complements of the latitude left and the latitude come to by D, (3) becomes

$$\text{Diff. long.} = \mathrm{RM} \times \text{tan. course} \times \mathrm{D} \quad . \quad . \quad . \quad . \quad (4),$$

or in logarithms,

$$\text{log. diff. long.} = 3{,}8984895 + \text{l. tan. course} + \text{log. D} \quad . \quad (5),$$

and hence the following rule :—

Find half the complements of the latitude left and the latitude come to, and take the difference between the logarithmic tangents of these arcs : to the logarithm of this difference add the logarithmic tangent of the course and the constant logarithm 3,8984895, and the sum will be the logarithm of the difference of longitude. If the logarithms are taken out in the usual way, that is, the tangents with the tabular radius, and when a number is less than unity removing the decimal point ten places to the right, it will be necessary to take 20 from the logarithmic sum thus found.

It must also be recollected that the two complements should be taken from the same pole ; so that if the two latitudes are of the same name, their complements will be found by either subtracting them both from 90°, or adding them both to 90° : but if they are of different names, that is, if one latitude be north and the other south, one must be taken from 90°, and the other added to it.

6. Example. A ship sails E.N.E. from the latitude 15° 37′ N. to the latitude 17° 53′ N., required her difference of longitude.

90°	0′	90°	0′
15	37	17	53
2)74	23	2)72	7
37	11½	36	3½

l. tan. 37° 11′½ = 9,8801342
l. tan. 36 3½ = 9,8621905

log. of the difference, ,0079437 = 7,9000228
l. tan. 67° 30′ = 10,3827757
constant logarithm = 3,8984895

log. of 151,8 = 2,1812880

Hence the diff. of long. = nearly 152′ or 2° 32′ E.

7. To find the course and distance from the ship to any place, the position of both being given by their latitudes and longitudes. From equation (4) we obtain

$$\text{cot. course} = \frac{\text{RM} \times \text{D}}{\text{diff. long.}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

or in logarithms,

$$\text{l. cot. course} = 3{,}8984895 + \text{log. D} - \text{log. diff. long.} \quad . \quad (7),$$

and hence the rule :—

Find the difference between the logarithmic tangents of the half complements of the latitude of the ship and the latitude of the place ; add the

logarithm of this difference to the constant logarithm 3,8984895, and from the sum take the logarithm of the difference of longitude expressed in nautical miles, the remainder will be the logarithmic cotangent of the course. The distance is found from the course and difference of latitude, by plane sailing.

8. Example. To find the course and distance from New-York to the Cape of Good Hope.

Latitude of N. York	40° 42′ N.		Longitude of N. York	73° 59′ W.	
" C. of G. Hope	34 24 S.		" C. of G. Hope	18 37 E.	
Diff. lat. = 4506′ =	75 6		Diff. long. = 5556′ =	92 36	

90° 0′	90° 0′
40 42	34 24
2)49 18	2)124 24
24 39	62 12

l. tan. 62° 12′ = 10,2779915
l. tan. 24 39 = 9,6617103

log. of the difference, ,6162812 = 9,7897789
constant logarithm = 3,8984895

13,6882684
log. 5556 = 3,7447622

l. cot. 48° 42′ 58″ = 9,9435062

l. sec. 48° 42′ 58″ = 10,1805940
log. 4506 = 3,6537912

log. 6829,54 = 3,8343852

Hence the course is S. 48° 42′ 58″, E. distance 6829½ nautical miles.

These are the principal cases that occur in practice; others are easily reducible to these.

9. If in equation (3) we make L=0°, which will be the case when the ship sails from the equator, it becomes

$$\text{Diff. long.} = \text{RM} \times \text{tan. course} \times \text{l. tan.}\ \tfrac{1}{2}(90^\circ - l) \quad . \quad . \quad (8).$$

It is evident from these equations, that the quantity

$$\text{RM} \times \text{l. tan.}\ \tfrac{1}{2}(90^\circ - l)$$

is the number that, in Mercator's sailing, is called the meridional parts of the latitude l, and that the quantity RM × D is nothing more than the meridional difference of latitude between the latitudes l and L. This fact is also evident from the formation of the numbers, since the sum $R \int dx \text{ sec. } x$

between $x=0$ and $x=l$, is the sum of the secants of all latitudes from the equator to the latitude l, which is also the meridional parts of that latitude.

A correct table of meridional parts could therefore very easily be calculated from the formula,

$$\text{log. merid. parts} = 3{,}898495 + \text{l. tan. } \tfrac{1}{2}(90^\circ - l);$$

but, it seems to me that these formulas and the rules deduced from them, will fully supply the place of such a table; since it is quite as easy to take out the logarithmic tangents of the half complements of latitude, as the meridional parts for the given latitudes, and the only additional labour is that of combining with them the constant logarithm of RM. There is also an advantage in dispensing with the use of the terms meridional parts and meridional difference of latitude, which are rarely understood by seamen.

10. I have employed the above method of investigating these formulas as being more elementary; but, by using the principles of Spherical Geometry established in a succeeding Article in the present number of the Miscellany, we shall arrive at them in a much more satisfactory manner. I shall here merely indicate so much of the operation as will be sufficient to confirm what has been already done, reserving for a future occasion a more complete investigation of the subject.

11. The rhumb line, or *Loxodromic* curve, is a line described on the surface of the sphere, cutting all meridians at equal angles.

If we take the pole of the earth for the origin of polar spherical co-ordinates, and the first meridian for the angular axis, the radius vector y, of any point will be the complement of latitude of that point, and the angle x will be its longitude. Now if ν be the angle the rhumb line makes with the radius vector, and which in the present problem will represent the ship's course, it will be seen in the article referred to, that

$$-\frac{dy}{dx}\,\text{cosec}\,y = \cot\nu \quad \ldots\ldots\ldots \quad (9),$$

$$\text{or } dx = -\tan\nu\,.\,\frac{dy}{\sin y} \quad \ldots\ldots \quad (10),$$

and integrating, putting c for a constant quantity,

$$x = \tan\nu\,.\,\text{h. log.}\,\frac{c}{\tan\frac{1}{2}y} \quad \ldots\ldots\ldots \quad (11).$$

If y_1 x_1 be a given point of the curve,

$$x_1 = \tan\nu\,.\,\text{h. log.}\,\frac{c}{\tan\frac{1}{2}y_1} \quad \ldots\ldots\ldots \quad (12);$$

and elimitating c,

$$x - x_1 = \tan\nu\,.\,\text{h. log.}\,\frac{\tan\frac{1}{2}y_1}{\tan\frac{1}{2}y} \quad \ldots\ldots \quad (13).$$

This equation might otherwise be written

$$\frac{\tan \frac{1}{2}y_{\imath}}{\tan \frac{1}{2}y} = \varepsilon^{(x - x_{\imath}) \cot \nu} = \delta^{x - x_{\imath}} \quad . \quad . \quad . \quad . \quad (14),$$

where ε represents the number whose hyperbolic logarithm is unity, and $\delta = \varepsilon^{\cot \nu}$.

Equations (13) and (14) are merely different forms for the equation of the loxodrome, referred to polar spherical co-ordinates. If the angular axis be made to pass through the point where the curve cuts the equator, so that $y = \frac{\pi}{2}$ when $x = 0$, these equations become

$$x = \tan \nu \,.\, \text{h. log.} \cot \tfrac{1}{2}y, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15),$$

$$\text{and, } \cot \tfrac{1}{2}y = \delta^{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

It will be seen at once that (3) and (13) are the same equation, differing only in the notation employed.

Δ.

ARTICLE II.

SOLUTION OF A GEOMETRICAL PROBLEM.

"In a given ellipse, it is required to inscribe the greatest possible equilateral triangle."

1. This question was proposed in Number XII. of the "Mathematical Diary," by Mr. James Macully, Richmond, Va.; and the elegant solution in the succeeding Diary, by Professor Pierce, would have left nothing further to desire on the subject, had it not been remarked in one of the other solutions that "It is evident that only two equilateral triangles can be inscribed in an ellipse;" an assertion which could not have been made, had the author, whose talents are sufficiently shown by his other solutions in the Diary, attentively considered the question. In the following investigation we shall endeavour to answer these questions:—

1°. Is every point in the periphery of the ellipse, the vertex of an inscribed equilateral triangle?

2°. Can any point in the periphery be the vertex of more than one inscribed equilateral triangle? and if so, of how many?

3°. In the variation of the triangles between the *maximum* and *minimum* positions already determined, are there other *maxima* or *minima* triangles?

2. The general equation of the second degree is,

$$A\,y^2 + B\,xy + C\,x^2 + D\,y + E\,x + F = 0 \quad . \quad . \quad . \quad . \quad (1).$$

If the axis of x is tangent to the curve, the two values of x, when $y=0$, must be equal, or the roots of the equation

$$C\,x^2 + E\,x + F = 0,$$

must be equal; therefore $\frac{E^2}{4C^2}-\frac{F}{C}=0$, or $E^2=4CF$; and if the origin of co-ordinates be in the curve, $x=0$ when $y=0$; $\therefore F=0$, and $E=0$. Hence the equation of a line of the second degree, referred to a tangent and a perpendicular to it through the point of contact as axes of co-ordinates, is

$$A\,y^2+B\,xy+C\,x^2+D\,y=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

3. Now, to apply this to the given ellipse whose semi-axes are a and b, it is shown by most writers on Analytical Geometry, that if y' x' be the co-ordinates of the ellipse's centre,

$$y'=\frac{-2CD}{4AC-B^2},\ x'=\frac{BD}{4AC-B^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

$$\frac{C+A+\sqrt{(C-A)^2+B^2}}{C+A-\sqrt{(C-A)^2+B^2}}=\frac{a^2}{b^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

$$\frac{2CD^2}{(4AC-B^2)^{\frac{3}{2}}}=ab \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

and that the equations of the axes are

$$y-y'+\frac{C-A\pm\sqrt{(C-A)^2+B^2}}{B}(x-x')=0 \quad . \quad . \quad . \quad (6).$$

Hence, if s be the angle the minor axis makes with the tangent axis of x, we get from (6),

$$\frac{C-A+\sqrt{(C-A)^2+B^2}}{D}=\cot. s,\ \frac{C-A-\sqrt{(C-A)^2+B^2}}{B}=-\tan. s \quad (7),$$

and by addition, $$\frac{C-A}{B}=\frac{\cot. s-\tan. s}{2}=\cot. 2s \quad . \quad . \quad . \quad . \quad . \quad (8).$$

By substituting this value of $\frac{C-A}{B}$ in (4), we derive

$$\frac{C+A}{B}=\frac{a^2+b^2}{a^2-b^2}\operatorname{cosec}. 2s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

Since one of the co-efficients in (2) is indeterminate, assume $B=2\sin. 2s$, and put $\frac{a^2+b^2}{a^2-b^2}=k$; then (8) and (9) give $A=k-\cos. 2s$, $C=k+\cos. 2s$; and these values of A, B, C being substituted in (5), we get $D=\frac{-c}{\sqrt{k+\cos. 2s}}$, where

$$c^2=4ab\,(k^2-1)^{\frac{3}{2}}=\frac{32a^4b^4}{(a^2-b^2)^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

and thus (2) becomes

$$(k-\cos. 2s)\,y^2+2\sin. 2s\,.\,xy+(k+\cos. 2s)\,x^2-\frac{cy}{\sqrt{k+\cos. 2s}}$$
$$=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

or, $$k(y^2+x^2)+\cos. 2\,s\,(x^2-y^2)+\sin. 2\,s\,.\,2\,xy-\frac{cy}{\sqrt{k+\cos. 2\,s}}$$
$$=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

Transforming this equation to polar co-ordinates, by making $y=r\sin. \varphi$, $x=r\cos. \varphi$, so that $y^2+x^2=r^2$, $x^2-y^2=r^2\cos. 2\varphi$, and $2xy=r^2\sin. 2\varphi$, it becomes

$$kr^2+r^2\cos. 2s\cos. 2\varphi+r^2\sin. 2s\sin. 2\varphi-\frac{cr\sin. \varphi}{\sqrt{k+\cos. 2s}}=0,$$

or, $$k\,r^2+r^2\cos. 2\,(\varphi-s)-\frac{cr\sin. \varphi}{\sqrt{k+\cos. 2\,s}}=0,$$

and therefore, $$r=\frac{c\sin. \varphi}{\{k+\cos. 2\,(\varphi-s)\}\sqrt{k+\cos. 2s}} \quad . \quad . \quad (13).$$

This is the polar equation of the ellipse, referred to a tangent as angular axis, the pole being at the point of contact. If s were the angle made by the major axis with the tangent, it would be,

$$r=\frac{c\sin. \varphi}{\{k-\cos. 2\,(\varphi-s)\}\sqrt{k-\cos. 2\,s}}.$$

It may be adapted to the hyperbola, by writing $-b^2$ for b^2, and therefore $k=\frac{a^2-b^2}{a^2+b^2}$, $c^2=\frac{32a^4b^4}{(a^2+b^2)^3}$; and for the parabola whose parameter is p, $k=1$, and $c^2=8p^2$; hence if s be the angle made by the tangent and diameter at the origin, the equation is,

$$r=\frac{2\,p\sqrt{2}\,.\sin. \varphi}{\{1-\cos. 2\,(\varphi-s)\}\sqrt{1-\cos. 2\,s}}=\frac{p\sin. \varphi}{\sin. s\sin.^2(\varphi-s)}.$$

These equations are remarkably symmetrical, and have not, to my knowledge, been published before.

4. Now, if two equal radius vectors include an angle of $\frac{\pi}{3}$, they will be the sides of an equilateral triangle inscribed in the ellipse; Let φ_1, φ_2 be the angles they make with the tangent, then from (13),

$$\frac{\sin. \varphi_1}{k+\cos. 2\,(\varphi_1-s)}=\frac{\sin. \varphi_2}{k+\cos. 2\,(\varphi_2-s)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

or $k(\sin. \varphi_2-\sin. \varphi_1)=\sin. \varphi_1\cos. 2\,(\varphi_2-s)-\sin. \varphi_2\cos. 2\,(\varphi_1-s)$; and by a common mode of transformation,

$$\begin{aligned}2k\cos. \tfrac{1}{2}(\varphi_2+\varphi_1)\sin. \tfrac{1}{2}(\varphi_2-\varphi_1)&\\ =\tfrac{1}{2}\sin. (2\varphi_2-2s+\varphi_1)&-\tfrac{1}{2}\sin. (2\varphi_2-2s-\varphi_1)\\ -\tfrac{1}{2}\sin. (2\varphi_1-2s+\varphi_2)&+\tfrac{1}{2}\sin. (2\varphi_1-2s-\varphi_2)\\ =\cos. \tfrac{1}{2}(3\varphi_2-4s+3\varphi_1)&\sin. \tfrac{1}{2}(\varphi_2-\varphi_1)\\ -\cos. \tfrac{1}{2}(\varphi_2-4s+\varphi_1)&\sin. \tfrac{1}{2}(3\varphi_2-3\varphi_1).\end{aligned}$$

But $\varphi_2-\varphi_1=\frac{\pi}{3}$, and if we put $\frac{1}{2}(\varphi_2+\varphi_1)=\theta$, the angle of the radius vector through the centre of the triangle, this equation becomes

$$k \cos.\theta = \tfrac{1}{2}\cos.(3\theta - 2\epsilon) - \cos.(\theta - 2\epsilon) \quad . \quad . \quad . \quad . \quad (15),$$

and developing this

$$(k + \cos.2\epsilon)\cos.\theta + \sin.2\epsilon\sin.\theta = \tfrac{1}{2}\cos.2\epsilon\cos.3\theta + \tfrac{1}{2}\sin.2\epsilon\sin.3\theta. \quad (16).$$

But $\cos.3\theta = 4\cos.^3\theta - 3\cos.\theta$,
$\sin.3\theta = 3\sin.\theta - 4\sin.^3\theta$;

By substituting these in (16), dividing by $\cos.^3\theta$, and properly ordering the terms, it reduces to

$$\tan.^3\theta + \frac{2k + 5\cos.2\epsilon}{3\sin.2\epsilon}\tan.^2\theta - \tfrac{1}{3}\tan.\theta + \frac{2k + \cos.2\epsilon}{3\sin.2\epsilon} = 0 \quad . \quad . \quad (17).$$

This equation, being of the third degree, must have, at least, *one* real root; the only exceptionable case is when $\sin.2\epsilon = 0$, and $\cos.2\epsilon = \pm 1$; that is, when the tangent axis is parallel or perpendicular to the minor axis of the ellipse, or the vertex of the triangle coincides with the vertex of either axis. But although in this case the equation (17) in $\tan.\theta$ becomes apparently of the second degree, and has imaginary roots, it is actually of the third degree, having $\tan.\theta = \infty$ for one of its roots; this may be made evident either by solving (17) in $\cot.\theta$ instead of $\tan.\theta$, or by writing $\sin.2\epsilon = 0$, and $\cos.2\epsilon = \pm 1$ in (16), it becomes

$$(k \pm 1)\cos.\theta \mp \tfrac{1}{2}\cos.3\theta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18),$$

of which $\cos.\theta = 0$, is one root, and therefore $\tan.\theta = \infty$ is a root of the corresponding equation (17). Whatever then be the value of ϵ, or whatever point of the circumference we take for the origin of co-ordinates, there is an inscribed equilateral triangle having its vertex at the same point.

5. We are naturally led to the enquiry, can more than one equilateral triangle have its vertex in that point? Now equation (17) has either one or three real roots, and there may, therefore, be one or three corresponding triangles. Let us examine the general equation

$$x^3 + ax^2 + bx + c = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19).$$

Assume $x = y - \frac{1}{3}a$, and it becomes

$$y^3 + (b - \tfrac{1}{3}a^2)y + c - \tfrac{1}{3}ab + \tfrac{2}{27}a^3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

Now it is well known that the three roots of the equation

$$y^3 + py + q = 0,$$

will be real when $-\frac{1}{27}p^3$ is either equal to, or greater than $\frac{1}{4}q^2$; hence the three roots of (20), and therefore of (19), will be real when

$$\tfrac{1}{27}(\tfrac{1}{3}a^2 - b)^3 = \text{or} > \tfrac{1}{4}(\tfrac{2}{27}a^3 - \tfrac{1}{3}ab + c)^2,$$

$$\text{or when } 27c^2 - 18abc + 4a^3c + 4b^3 - a^2b^2 = \text{or} < 0 \quad . \quad . \quad (21).$$

6. In equation (17), $a = \dfrac{2k + 5\cos.2\epsilon}{3\sin.2\epsilon}$, $b = -\frac{1}{3}$, $c = \dfrac{2k + \cos.2\epsilon}{3\sin.2\epsilon}$, and substituting these in (21), eliminating $\sin.2\epsilon$, and reducing the results, we shall find the three roots of equation (17) are real when

$$\cos.^2 2\epsilon + \frac{8k^3 + 25k}{4(k^2 + 2)}\cos.2\epsilon + \frac{16k^4 + 296k^2 - 3}{64(k^2 + 2)} = \text{or} < 0;$$

that is, when $\cos.2\epsilon$ is at or between the roots of the equation

$$x^2+\frac{8k^3+25k}{4(k^2+2)}\cdot x+\frac{16k^4+296k^2-3}{64(k^2+2)}=0;$$

$$\left.\begin{array}{r}\therefore \cos.2s \text{ at or between the limits } \dfrac{-k(8k^2+25)+(2k^2+1)^{\frac{3}{2}}\sqrt{6}}{8(k^2+2)} \\ \text{and } \dfrac{-k(8k^2+25)-(2k^2+1)^{\frac{3}{2}}\sqrt{6}}{8(k^2+2)}\end{array}\right\} \quad . \quad . \quad (22)$$

7. Now if either of the numbers in (22) be at or within the limits $+1$, and -1, there may be values of s that will fulfil these conditions. Since $k=\frac{a^2+b^2}{a^2-b^2}$, it is necessarily greater than unity, and therefore the number

$\frac{-k(8k^2+25)-(2k^2+1)^{\frac{3}{2}}\sqrt{6}}{9(k^2+2)}$ will always be $<\frac{-11-3\sqrt{2}}{8}$, or without the limits $+1$ and -1; but the number $\frac{-k(8k^2+25)+(2k^2+1)^{\frac{3}{2}}\sqrt{6}}{8(k^2+2)}$ may be equal to or greater than -1, and it will be so when

$$-k(8k^2+25)+(2k^2+1)^{\frac{3}{2}}\sqrt{6}= \text{ or } > -8(k^2+2),$$

or when $(2k^2+1)^{\frac{3}{2}}\sqrt{6}=$ or $>8k^3-8k^2+25k-16$;

that is, by squaring and reducing, when

$$16k^6-128k^5+392k^4-656k^3+845k^2-800k+250=0, \text{ or } \angle 0;$$

or, by dividing the first member into factors, when

$$(2k-1)(2k-5)^3(k^2+2)= \text{ or } \angle 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23).$$

Here k being always >1, this number is evidently $\angle 0$, until $k=2\frac{1}{2}$, when it is $=0$, and when $k>2\frac{1}{2}$, the number is always >0. Thus when the ellipse is so constituted that k is between the limits $2\frac{1}{2}$ and 1, or so long as $\frac{a}{b}$ is greater than $\sqrt{\frac{7}{3}}$, those points in the circumference of the ellipse that have $\cos.2s$ at or between the limits -1, and $\frac{-k(8k^2+25)+(2k^2+1)^{\frac{3}{2}}\sqrt{6}}{8(k^2+2)}$ are the vertices of three different inscribed equilateral triangles. It is evident that these points lie at and near the vertices of the minor axis, their limits on each side of it being defined by those two values of s that render $\cos.2s=\frac{-k(8k^2+25)+(2k^2+1)^{\frac{3}{2}}\sqrt{6}}{8(k^2+2)}$.

8. As an example, let us examine those triangles that have a vertex *at* the extremity of the minor axis. Here $\cos.2s=-1$, and equation (18) gives for this case

$$(k-1)\cos.\theta+\tfrac{1}{2}\cos.3\theta=0,$$

$$\text{or } 4 \cos.^3\theta-(5-2k)\cos.\theta=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

the roots of which are $\cos.\theta=0$, and $\cos.\theta=\pm\frac{1}{2}\sqrt{5-2k}$, which are all three real when $k=\text{or} \angle 2\frac{1}{2}$; the first of these designates the triangle whose centre is on the minor axis, and radii vectores through the centres of the other two triangles, make equal angles with this axis; these angles are less than 30° (θ being $>60°$) so long as k is greater than 2, and the two triangles will intersect each other; when $k=2$, or $\frac{a}{b}=\sqrt{3}$, these angles are each equal 30°, and the two triangles will have a common side which is the minor axis itself, the opposite vertices being in the two extremities of the major axis; and when $k\angle 2$, these angles are greater than 30°, and the two triangles will be wholly without each other, being inscribed in the opposite semi-ellipses.

9. Next, to find the *maxima* and *minima* triangles. The side will necessarily be a *maximum* or *minimum*; and its two values which include the angle $\frac{\pi}{3}$, are

$$\frac{c \sin.\varphi}{\{k+\cos.2(\varphi-s)\}\sqrt{k+\cos.2s}} \text{ and } \frac{c \sin.(\varphi+60°)}{\{k+\cos.2(\varphi+60°-s)\}\sqrt{k+\cos.2s}}.$$

By equating the differentials of these two expressions to zero, we have

$$\frac{d\varphi}{ds}=\frac{2\sin\varphi\sin 2(\varphi-2)(k+\cos 2s)-\sin\varphi\sin 2s\,(k+\cos 2(\varphi-s)),}{\{\cos\varphi\,(k+\cos 2(\varphi-s))+2\sin\varphi\sin 2(\varphi-s)\}(k+\cos 2s)} \quad . \quad (25),$$

$\frac{d\varphi}{ds}=$ the fraction formed by writing $\varphi+60°$ instead of φ in the second member of (25) . . . . (26),

but, since the expressions for the sides are equal,

$$k+\cos.2(\varphi-s+60°)=\frac{\sin.(\varphi+60°)\,(k+\cos.2(\varphi-s))}{\sin.\varphi},$$

and substituting this in (26) it becomes

$$\frac{2\sin\varphi\sin 2(\varphi-s+60°)(k+\cos 2s)-\sin(\varphi+60°)\sin 2s(k+\cos 2(\varphi-s))}{\{\cos(\varphi+60°)(k+\cos 2(\varphi-s))+2\sin\varphi\sin 2(\varphi-s+60°)\}(k+\cos 2s)}$$

$$=\frac{d\varphi}{ds} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27).$$

By equating the expressions for $\frac{d\varphi}{ds}$ in (25) and (27), clearing of fractions and dividing by the factor, $k+\cos.2(\varphi-s)$, which could never lead to a real root of the equation, since it could never equal zero, k being >1, the equation may be arranged thus,

$$2\sin.\varphi\,(k+\cos.2s)\{\cos.(\varphi+60°)\sin.2(\varphi-s)-\cos.\varphi\sin.2(\varphi-s+60°)\}$$

$$-2\sin.\varphi\sin.2\epsilon\{\sin.\varphi\sin.2(\varphi-\epsilon+60^\circ)-\sin.(\varphi+60^\circ)\sin.2(\varphi-\epsilon)\}$$
$$-\sin.2\epsilon(k+\cos.2(\varphi-\epsilon))\{\sin.\varphi\cos.(\varphi+60^\circ)-\cos.\varphi\sin.(\varphi+60^\circ)\}$$
$$=0 \quad \ldots \quad (28),$$

but $\sin.\varphi\cos.(\varphi+60^\circ)-\cos.\varphi\sin.(\varphi+60^\circ)=-\sin.60^\circ$, and by further ordering the terms the equation becomes

$$2k\sin.\varphi\{\cos.(\varphi+60^\circ)\sin.2(\varphi-\epsilon)-\cos.\varphi\sin.2(\varphi-\epsilon+60^\circ)\}$$
$$+2\sin.\varphi\,[\{\cos.2\epsilon\cos.(\varphi+60^\circ)+\sin.2\epsilon\sin.(\varphi+60^\circ)\}\sin.2(\varphi-\epsilon)$$
$$-\{\cos.2\epsilon\cos.\varphi+\sin.2\epsilon\sin.\varphi\}\sin.2(\varphi-\epsilon+60^\circ)]$$
$$+\sin.60^\circ\sin.2\epsilon\{k+\cos.2(\varphi-\epsilon)\}=0 \quad \ldots \quad (29).$$

Now $\cos.(\varphi+60^\circ)\sin.2(\varphi-\epsilon)-\cos.\varphi\sin.2(\varphi-\epsilon+60^\circ)$
$$=\tfrac{1}{2}\sin.(3\varphi-2\epsilon+60^\circ)+\tfrac{1}{2}\sin.(\varphi-2\epsilon-60^\circ)$$
$$-\tfrac{1}{2}\sin.(3\varphi-2\epsilon+120^\circ)-\tfrac{1}{2}\sin.(\varphi-2\epsilon+120^\circ)$$
$$=-\cos.(3\varphi-2\epsilon+90^\circ)\sin.30^\circ-\cos.(\varphi-2\epsilon+30^\circ)\sin.90^\circ$$
$$=\tfrac{1}{2}\sin.(3\varphi-2\epsilon)-\cos.(\varphi-2\epsilon+30^\circ),$$

and, $\{\cos.2\epsilon\cos.(\varphi+60^\circ)+\sin.2\epsilon\sin.(\varphi+60^\circ)\}\sin.2(\varphi-\epsilon)$
$$-\{\cos.2\epsilon\cos.\varphi+\sin.2\epsilon\sin.\varphi\}\sin.2(\varphi-\epsilon+60^\circ)$$
$$=\cos.(\varphi-2\epsilon+60^\circ)\sin.2(\varphi-\epsilon)-\cos.(\varphi-2\epsilon)\sin.2(\varphi-\epsilon+60^\circ)$$
$$=\tfrac{1}{2}\sin(3\varphi-4\epsilon+60^\circ)+\tfrac{1}{2}\sin(\varphi-60^\circ)$$
$$-\tfrac{1}{2}\sin(3\varphi-4\epsilon+120^\circ)-\tfrac{1}{2}\sin(\varphi+120^\circ)$$
$$=-\cos(3\varphi-4\epsilon+90^\circ)\sin 30^\circ-\cos(\varphi+30^\circ)\sin 90^\circ$$
$$=\tfrac{1}{2}\sin(3\varphi-4\epsilon)-\cos(\varphi+30^\circ),$$

and, by substituting in (29),

$$k\{\sin\varphi\sin(3\varphi-2\epsilon)-2\sin\varphi\cos(\varphi-2\epsilon+30^\circ)+\sin 60^\circ\sin 2\epsilon\}$$
$$+\sin\varphi\sin(3\varphi-4\epsilon)-2\sin\varphi\cos(\varphi+30^\circ)+\sin 2\epsilon\sin 60^\circ\cos 2(\varphi-\epsilon)$$
$$=0 \quad \ldots \quad (30).$$

By further reduction it takes the successive forms

$$k\sin 2(\varphi-\epsilon)(\sin 2\varphi-\sin 60^\circ)$$
$$+\sin 2(\varphi-\epsilon)\{\sin 2(\varphi-\epsilon)+\sin 2\epsilon-\sin 60^\circ\cos 2\epsilon\}=0 \quad \ldots \quad (31),$$

and,

$$\sin 2(\varphi-\epsilon)\{(k+\cos 2\epsilon)\sin(\varphi-30^\circ)\cos(\varphi+30^\circ)+\sin 2\epsilon\sin.^2\varphi\}$$
$$=0 \quad \ldots \quad (32).$$

10. Now equation (16), which arises from the equality of the two expressions for the side, becomes, by writing in it $\varphi+30^\circ$ for θ,

$$(k+\cos 2\epsilon)\cos(\varphi+30^\circ)+\sin 2\epsilon\sin(\varphi+30^\circ)-\tfrac{1}{2}\cos(3\varphi-2\epsilon+90^\circ)$$
$$=0 \quad \ldots \quad (33).$$

If this equation be multiplied by $\sin 2(\varphi-\epsilon)\sin(\varphi-30^\circ)$, and taken from (32), there will remain

$$\sin 2(\varphi-\epsilon)\{\tfrac{1}{4}\sin 2\epsilon+\tfrac{1}{2}\sin(\varphi-30^\circ)\cos(3\varphi-2\epsilon+90^\circ)\}=0 \quad . \quad (34),$$

because $\sin^2\varphi - \sin(\varphi+30^\circ)\sin(\varphi-30^\circ)=\frac{1}{4}$. Multiply this equation by 4, and after some slight and obvious reductions it will become, finally,

$$\sin 2(\varphi-\varepsilon)\cos(2\varphi-2\varepsilon+30^\circ)\{2\sin(2\varphi+30^\circ)-1\}=0 \quad . \quad (35).$$

Equations (33) and (35) will give the values of φ and ε for all the maxima and minima triangles, that can be inscribed in the ellipse. Let us first examine the factor

$$2\sin(2\varphi+30^\circ)-1=0, \text{ or } \sin(2\varphi+30^\circ)=\tfrac{1}{2};$$

then $2\varphi+30^\circ=30^\circ$ or 150°, and $\varphi=0^\circ$ or 60°. If we write $\varphi=0^\circ$ in (33) we shall find $k+\cos 2\varepsilon=0$, which is obviously impossible, since $k>1$; but $\varphi=60^\circ$ gives $\sin 2\varepsilon=0$; $\therefore \varepsilon=0$, or $\varepsilon=90^\circ$, which evidently indicate the two triangles that have their vertices in the extremities of the major and minor axes; by substitution in (13) we find their sides $\frac{c\sqrt{3}}{(2k-1)\sqrt{k+1}}=\frac{4ab^2\sqrt{3}}{a^2+3b^2}$, and $\frac{c\sqrt{3}}{(2k+1)\sqrt{k-1}}=\frac{4a^2b\sqrt{3}}{3a^2+b^2}$, the former being evidently a *minimum*, the latter a *maximum*.

11. Secondly, the factor of (35),

$$\cos(2\varphi-2\varepsilon+30^\circ)=0,$$

gives $2\varphi-2\varepsilon+30^\circ=90^\circ$ or 270°; $\therefore \varphi-\varepsilon=30^\circ$, or $\varphi-\varepsilon=120^\circ$, and without any further substitution we see at once that φ is the exterior angle of a triangle of which ε and 30° in the first case, and ε and 120° in the second case, are the opposite interior angles; now these are precisely the triangles which would be formed by a side, the axis passing through the vertex and a tangent at one of the other vertices, of the triangles already determined.

Hence this solution indicates the same two triangles as those determined in the last article, but each by one of its other vertices.

12. Thirdly, the remaining factor of (35),

$$\sin.\ 2(\varphi-\varepsilon)=0,$$

gives $\varphi-\varepsilon=0^\circ$ or 90°; and here again, since ε is the inclination of the minor axis with the tangent, when $\varphi=\varepsilon$, the side of the triangle is parallel to that axis, and when $\varphi=\varepsilon+90^\circ$, the side is perpendicular to it; hence their opposite vertices must be in the extremities of the major and minor axes, and the triangles must be those found in Art. 10. We thus arrive at the singular conclusion that, although there are six pairs of possible and apparently independent roots of the equations (33) and (35), they all belong to the same two triangles, indicating each of them by its three different vertices. We conclude also that the triangles vary from the minimum position to the maximum one without any change in the sign of variation, or that there are no *maxima* or *minima* triangles between the least and greatest.

Δ.

ARTICLE III.

ILLUSTRATIONS OF LAGRANGE.

The "*Mecanique Analytique*" of Lagrange has left us little to desire on the general problem of Mechanics. The student knows that every particular example which may come under his notice is included in the comprehensive formulas of Lagrange, and that, so far as the present power of analysis will carry him, the solution is within his reach. Still the very facility thus given to research, should inspire caution in applying these formulas; for since their use must almost entirely obviate the necessity of considering the usual relations of force, velocity, &c. it has a tendency to make us pass in a cursory manner over the consideration of the question, and thus mistake its nature and the mode of applying our analysis to it. There is, besides, great danger of misinterpreting the results, or of passing over important consequences, without detecting them in the analysis, and thus neglecting one of its greatest advantages.

We are of opinion that a number of familiar examples, where the equations of motion shall be deduced from Lagrange's formulas, and the consequences followed out, as far as possible, in the spirit of his analysis, might be made of more assistance to the student than any formal commentary could be; and it is our intention to devote a portion of the Miscellany to the purpose of effecting this object, so far as we are able.

We shall select our examples from any available source, and we shall not scruple to make use (with proper acknowledgments) of all or part of the solutions of others, when they seem properly adapted to our purpose.

We intend to continue the series from time to time, and we shall be obliged to any of our correspondents if they will transmit to us questions on Mechanical subjects, noticing the source whence they are derived, and accompanying them either with the original solutions, or new ones adapted to this department; a short history of the analogous class of questions would make them still more acceptable.

PROBLEM I.

Upon a horizontal plane, a rectilineal path is traced in which a body P is constrained to move uniformly. This body is connected by an inflexible and inextensible line, with another body M, which is posited on this plane, and which is supposed to have received some primitive impulse in the direction of this plane. It is required to find the nature of the curve described by the body M and the other circumstances of the motion, abstracting from friction.

We find this problem, with two or three solutions to it, copied into the Mathematical Repository (vol. V. No. 1.), from the "*Annales de Mathematiques*." M. Gergonne remarks that it was treated of by Clairault with

several analogous problems in the memoirs *de l'académie des sciences* of Paris, for 1736. Clairault seems to have been led into a discussion on the subject with Fontaine and others, who contended that the rod was always a tangent to the curve, and that the *Tractoire* was no other than the curve of equal tangents. By a note appended to Mr. Lowry's solution in the Repository, it appears that it was also published in the Ladies' Diary for 1778. After the equations of motion (5) are deduced, the following solution is mostly copied from that of M. Francais in the *Annales de Mathematiques*, which I find translated in the Repository.

SOLUTION.

Let the straight line in which the body P moves, be taken for the axis of x, and any straight line at right angles to this, for the axis of y.

Let, at the epoch t, x and y be the co-ordinates of the point M, and x' the abscissa of the point P; the rectilineal motion of this last point must be the effect of an accelerative force, directed according to the axis of x, and disturbed by the reaction of M upon P. Let this accelerative force be p.

The general formula of Dynamics (*Mecanique Analytique*, Part II. Sec. II. art. 5. ed. of 1811.), becomes

$$\mathrm{S}\left(\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y\right)m - \mathrm{P}p\delta x' = 0 \quad . \quad . \quad . \quad . \quad . \quad (1).$$

For the body M, the sum of the moments is

$$\left(\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y\right)\mathrm{M},$$

and, for the body P, since $y=0$, the sum of the moments would be

$$\mathrm{P}.\frac{d^2x'}{dt^2}\delta x';$$

but since the body P moves uniformly, $\frac{dx'}{dt}$ is constant, and $\frac{d^2x'}{dt^2} = 0$, and therefore the moments arising from the accelerating forces relative to this body are nothing; therefore

$$\mathrm{S}\left(\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y\right)m = \left(\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y\right)\mathrm{M},$$

and (1) becomes

$$\left(\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y\right)\mathrm{M} - \mathrm{P}p\delta x' = 0 \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Designating by a the length of the rod, the connection of the parts of the system is expressed by the single equation

$$(x-x')^2 + y^2 = a^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

which gives the relation among the variations

$$(x - x')\delta x - (x - x')\delta x' + y\delta y = 0;$$

and if this be multiplied by the indeterminate coefficient λ, (Part I. Sect. IV. § I.) and added to (2), all the circumstances of the rod's motion will be included in the equation,

$$\left\{M.\frac{d^2x}{dt^2}+\lambda(x-x')\right\}\delta x+\left\{M.\frac{d^2y}{dt^2}+\lambda y\right\}\delta y-\left\{Pp+\lambda(x-x')\right\}\delta x'=0 \quad (4),$$

and, since the variations δx, δy, $\delta x'$, may be now considered independent, on account of the introduction of the coefficient λ, we must have,

$$\left.\begin{array}{l} 1.\ M.\dfrac{d^2x}{dt^2}+\lambda(x-x')=0, \\ 2.\ M.\dfrac{d^2y}{dt^2}+\lambda y=0, \\ 3.\ Pp+\lambda(x-x')=0. \end{array}\right\} \quad \ldots\ldots \quad (5).$$

By eliminating the indeterminate λ, we have simply

$$M\frac{d^2x}{dt^2}=Pp,$$

$$M\frac{d^2y}{dt^2}=\frac{Ppy}{x-x'};$$

and eliminating dt, we find between x, y and x', the equation

$$yd^2x=(x-x')d^2y \quad \ldots\ldots\ldots \quad (6).$$

Since dx' is constant, we must try to find an equation between x' and y. For this purpose, by differentiating (3),

$$dx=dx'-\frac{ydy}{\sqrt{a^2-y^2}},$$

$$d^2x=\frac{-yd^2y}{\sqrt{a^2-y^2}}-\frac{a^2dy^2}{(a^2-y^2)^{\frac{3}{2}}};$$

but equation (6) gives

$$d^2x=\frac{(x-x')d^2y}{y}=\text{by (3)}\,\frac{d^2y\sqrt{a^2-y^2}}{y};$$

equating these two values, and reducing

$$\frac{d^2y}{\sqrt{a^2-y^2}}+\frac{ydy^2}{(a^2-y^2)^{\frac{3}{2}}}=0 \quad \ldots\ldots\ldots \quad (7),$$

an equation which has for its integral,

$$\frac{dy}{\sqrt{a^2-y^2}}+Cdx'=0 \quad \ldots\ldots\ldots \quad (8).$$

By integrating again,

$$\cos^{-1}\frac{y}{a}=Cx'+C';$$

or, since $x'=x-\sqrt{a^2-y^2}$,

$$\cos^{-1}\frac{y}{a}=C\left\{x-\sqrt{a^2-y^2}\right\}+C' \quad \ldots\ldots \quad (9).$$

For determining the constants C and C$'$, suppose first that the constant velocity of P is b, so that $\frac{dx'}{dt}=b$. Then equation (8) becomes

$$\frac{dy}{dt} + bC\sqrt{a^2 - y^2} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

Supposing, again, that at the origin of the time, the point P is the origin of co-ordinates, and that the rod a forms then an angle α with the axis of x. Suppose also that the initial velocity of M parallel to the axis of y is c, so that for $t = 0$, and $y = a \sin \alpha$, we have $\frac{dy}{dt} = c$: equation (10) becomes

$$c + abC \cos \alpha = 0,$$

$$\therefore C = \frac{-c}{ab \cos \alpha}.$$

The integral (9) relative to the same initial state, becomes

$$\cos^{-1} \sin \alpha = C', \text{ whence } C' = \frac{\pi}{2} - \alpha.$$

Thus we shall have, from (9),

$$x = \sqrt{a^2 - y^2} + \frac{ab}{c} \cos \alpha \left\{ \sin^{-1} \frac{y}{a} - \alpha \right\} \quad . \quad . \quad . \quad (11).$$

This is the equation of the curve described by the body M. We see that this curve is a cycloid relative to the straight line passed over by the centre of the generating circle: this circle has for its radius the length a of the rod; its centre is the extremity P of this rod, and the ratio of the velocities of translation of the centre, and of rotation of the parts of the circumference about this centre, is that of $b \cos \alpha$ to c; the cycloid is therefore *prolate*, *common*, or *curtate*, according as we have

$$b \cos \alpha > c, \; b \cos \alpha = c, \text{ or } b \cos \alpha < c.$$

Equation (11) contains, as one of the given quantities, the initial velocity of M in the direction of y; we may easily introduce its velocity in the direction of x. If, indeed, we put in the first integral (8) for $\frac{dy}{\sqrt{a^2 - y^2}}$, its value $\frac{dx' - dx}{y}$, we have

$$\frac{dx'}{dt} + Cy \frac{dx'}{dt} - \frac{dx}{dt} = 0.$$

Let now c' be the initial velocity of M in the direction of x, so that $\frac{dx}{dt} = c'$, this equation becomes, for that state,

$$b + abC \sin \alpha - c' = 0, \text{ and } C = \frac{c' - b}{ab \sin \alpha}.$$

Introducing this value into the equation of the curve, it becomes

$$x = \sqrt{a^2 - y^2} + \frac{ab \sin \alpha}{c' - b} \left\{ \sin^{-1} \frac{y}{a} - \alpha \right\} \quad . \quad . \quad . \quad (12);$$

the relation between the initial velocities c and c' being expressed by

$$c' \cos \alpha + c \sin \alpha = b \cos \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

We must notice here, that these initial velocities are not those which are imprinted on M by an impulsive force only; they are the resultants of the primitive impulse on M, and of the action of P upon M; so that, at the point of impulsion, they are due only to the action of P. The velocity of b is not wholly due to the action of the accelerative force p, but to this action modified by the effect of the impulse given to M. Equation (11) fails when we have $\alpha = \frac{1}{2}\pi$; but then we employ equation (12), which becomes

$$x = \sqrt{a^2 - y^2} - \frac{ab}{c' - b} \cdot \cos^{-1}\frac{y}{a}.$$

Similarly, if $\alpha = 0$, equation (12) fails, but equation (11) becomes

$$x = \sqrt{a^2 - y^2} + \frac{ab}{c} \cdot \sin^{-1}\frac{y}{a}.$$

For determining the velocity v of M, for any point of the curve, we have the equations

$$\frac{dx}{dt} = \frac{ab \cos \alpha - cy}{a \cos \alpha}, \frac{dy}{dt} = \frac{c\sqrt{a^2 - y^2}}{a \cos \alpha};$$

hence $v^2 = \dfrac{dx^2}{dt^2} + \dfrac{dy^2}{dt^2} = \dfrac{ab^2 \cos^2\alpha - 2bcy \cos \alpha + ac^2}{a \cos^2\alpha}.$

Also, according as we have $y = a$, or $y = -a$, we have

$$v = b - \frac{c}{\cos \alpha}, \text{ or } v = b + \frac{c}{\cos \alpha}.$$

It is very easy to see that these are the least and greatest velocities of the point M; the first has place at the highest and the second at the lowest point of each cycloid. Hence, in the common cycloid, for which we have $c = b \cos \alpha$, the velocity of the point M is nothing, each time it comes to the greatest elevation; and it is double that of the point P, each time it comes to the lowest.

The time is found from the formula $\dfrac{dy}{\sqrt{a^2 - y^2}} = \dfrac{cdt}{a \cos \alpha}$, which gives

$$\sin^{-1}\frac{y}{a} = \frac{ct}{a \cos \alpha} + C'';$$

and, as we have at the same time $y = a \sin \alpha$, and $t = 0$, it follows that $C'' = \alpha$, which gives

$$t = \frac{a \cos \alpha}{c}\left\{\sin^{-1}\frac{y}{a} - \alpha\right\} \quad . \quad . \quad . \quad . \quad . \quad (14).$$

Also, when $y = a$, $\sin^{-1}\dfrac{y}{a} = \sin^{-1} 1 = (n + \frac{1}{2})\pi$, n being any positive whole number, and therefore $t = \dfrac{a \cos \alpha}{c}\left\{(n + \frac{1}{2})\pi - \alpha\right\}$; whence it follows that the time employed in passing over the whole cycloid is $= \dfrac{\pi a \cos \alpha}{c}$.

The accelerative force $p = \frac{M}{P} \cdot \frac{d^2x}{dt^2}$; but

$$\frac{dx}{dt} = b - \frac{cy}{a \cos \alpha}, \text{ whence } \frac{d^2x}{dt^2} = -\frac{cdy}{adt \cos \alpha},$$

and since $\frac{dy}{dt} = \frac{c\sqrt{a^2 - y^2}}{a \cos \alpha}$, it follows that

$$p = -\frac{M}{P} \cdot \frac{c^2 \sqrt{a^2 - y^2}}{a^2 \cos^2 \alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15),$$

which gives, for the initial value of p, $p = -\frac{M}{P} \cdot \frac{c^2}{a \cos \alpha}$.

Some explanations of M. Francais' equation (13) *of conditions,*

By M. Dubuat.

The equation $c' \cos \alpha + c \sin \alpha = b \cos \alpha$, is no other than the general equation of condition $(x - x')(dx - dx') + ydy = 0$, in which we put for the variables dx', dx, dy, $x - x'$, y, the values bdt, $c'dt$, cdt, $a \cos \alpha$, $a \sin \alpha$, which they have at the origin of the motion.

Or, the general equation $(x - x')(dx - dx') + ydy = 0$, signifies that the variable velocities $\frac{dx}{dt}$, $\frac{dy}{dt}$ of the point M, in the direction of the axis of co-ordinates are such that, if from the velocity $\frac{dx}{dt}$ according to the axis of x, we subtract the velocity $\frac{dx'}{dt}$ of the point P, the remaining velocity $\frac{dx - dx'}{dt}$ forms with the velocity $\frac{dy}{dt}$ according to the axis of y, a resultant perpendicular to the radius vector P M: whence it follows that the velocity of the point M, considered either at the commencement or in the continuation of the motion, may always be decomposed into two velocities, the one parallel to the axis of x, constant and equal to b, the other perpendicular to the radius vector, the value of which may be anything.

Hence, if the velocity imprinted on the point M, at the origin of the motion, be not decomposable into two velocities, according to the same law, this velocity is not the initial velocity from which we must determine the constants in the integration.

Let, at the origin of motion, V be the velocity imprinted on the point M, in a direction making the angle β with the axis of x, the components are V $\cos \beta$ in the direction of x, and V $\sin \beta$ in the direction of y. The first component V $\cos \beta$ is equivalent to the two velocities b and V $\cos \beta - b$, of which the first b only subsists by virtue of the equation of condition; but the velocity V $\cos \beta - b$ is not totally destroyed by its decomposition into two velocities, the one according to the radius vector, and the other

perpendicular to this radius; this last, the expression for which is $(V \cos \beta - b) \sin \alpha$, subsists, while the other is destroyed.

The velocity $V \sin \beta$, imprinted in the way of y, being also decomposed into two velocities, the one according to the radius vector, and the other perpendicular to this radius; the second subsists only, and is expressed by $V \sin \beta \cos \alpha$.

The initial velocity resulting from the imprinted velocity V, is therefore composed of a velocity b, parallel to the axis of x, and a velocity $(V \cos \beta - b) \sin \alpha + V \sin \beta \cos \alpha$, perpendicular to the radius vector: this gives for the component c' of the initial velocity according to the axis of x,

$$c' = b \pm \left\{ V \sin (\alpha + \beta) - b \sin \alpha \right\} \sin \alpha,$$

and for the component c of the initial velocity according to the axis of y,

$$c = \pm \left\{ V \sin (\alpha + \beta) - b \sin \alpha \right\} \cos \alpha.$$

But there is another difficulty presented in equations (11) and (12). If we make in the first $c = 0$, or $c' - b = 0$ in the second, we have $x = \sqrt{a^2 - y^2} + \infty$, which has no signification. For removing this difficulty, I remark that by virtue of the equation of condition $(c' - b) \cos \alpha + c \sin \alpha = 0$, the hypothesis $c = 0$ gives $(c' - b) \cos \alpha = 0$, and consequently, $c' = b$, or $\cos \alpha = 0$.

Let us put $c = 0$, $c' = b$. These two equations signify that the initial velocity of the point M, parallel to the axis of y, is nothing, and that its initial velocity parallel to the axis of x is b, and of course equal to the velocity of the point P in the same direction: the two points M and P are therefore animated, at the origin of the motion with velocities equal and parallel. The equation of condition subsists for these two velocities, in the first instant and in the whole course of the motion. The point M therefore describes a straight line parallel to the axis of x, with a constant velocity and equal to b; this gives $y = constant$, and $x = bt + constant$.

Let, in the second place, $c = 0$, and $\cos \alpha = 0$. These two equations signify that the initial velocity of the point M parallel to y is nothing, and that the ordinate at the same point is also nothing at the origin of motion without determining the initial velocity parallel to x. The two points M and P at the origin of the motion are therefore on the axis of x, and the point P of which the velocity is b, by virtue of the equation of condition, can neither be augmented nor diminished. It is easy to conclude that the system of the two points moves in the first instant, and during the whole course of the motion on the axis of x, with a common velocity b, that is to say, we have $y = 0$, $x = bt + constant$.

We have only further to add, in order to show the use of the indeterminate coefficient λ, which is not introduced into the solution of M. Francais, that if we put, according to the notation of Lagrange in art. 5. of the section quoted,

$$\delta L = (x - x')(\delta x - \delta x') + y \delta y,$$

we shall have $\frac{\delta L}{\delta x'} = -(x - x')$. Hence the term $\lambda\delta L$ introduced into equation (4) is equivalent to a force $\lambda \cdot \frac{\delta L}{\delta x'} = -\lambda(x - x')$ acting on the body P, and exerted in a direction perpendicular to the surface whose equation is $\delta L = 0$, the co-ordinates of P alone varying; that is, the equation is $-(x - x')\,\delta x' = 0$, or $x' =$ *constant*, and this is evidently a straight line perpendicular to the axis of x. Moreover, since by the third of equations (5), $\lambda = \frac{-Pp}{x - x'}$, this force is $= -Pp$; it arises from the connection of the parts of the system expressed by equation (3), and simply shows the effect that the impulse impressed on the body M, produces on the body P, in the direction of x: we set out with supposing that this produced an accelerative force $= p$, and therefore a motive force $= Pp$, and we thus arrive at the same result again.

Similarly, with respect to the body M, whose co-ordinates are x and y; we have $\frac{\delta L}{\delta x} = x - x'$, and $\frac{\delta L}{\delta y} = y$; therefore the connection of the system is equivalent to a force on the body $M = \lambda\sqrt{\left(\frac{\delta L}{\delta x}\right)^2 + \left(\frac{\delta L}{\delta y}\right)^2} = a\lambda =$ $\frac{-aPp}{x - x'} = \frac{-aPp}{\sqrt{a^2 - y^2}} = \frac{Mc^2}{a\cos^2\alpha}$, exerted in the direction of a perpendicular to the surface whose equation is $\delta L = 0$, or our equation (3), the co-ordinates of M (x and y) alone varying, which is evidently a circle whose centre is P and radius a; that is, it is directed along the line or rod connecting the two points, and may therefore represent the tension of the line in every position of the system. This tension is therefore constant and independent of the magnitude of P, as it evidently ought to be.

PROBLEM II.

A uniform straight rod AB is placed in an assigned position, upon a smooth horizontal plane, and one end of it, B, is drawn uniformly along a straight line of that plane, with a given velocity: it is proposed to find the position of the rod at any time, and its angular velocity.

This question was proposed in the Ladies' Diary for 1826, by Mr. Mason, of Scoulton, a gentleman whose labours have enriched the English periodicals for several years. There are two solutions by the proposer inserted in the succeeding Diary, neither of which have any resemblance to the following one.

SOLUTION.

Let the straight line on which the point B moves be taken for the axis of x, and a perpendicular to it through the first and given position of B for

the axis of y. Let, at the time t from the origin of motion, x and y be the co-ordinates of any point in the rod, which is supposed to be very slender, inflexible, and inextensible, let x' be the abscissa of the point B, and p be the accelerative force required to draw the end B along the axis of x, m being the mass of the rod.

The dynamical formula will then be

$$S\left(\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\right)Dm + pm\delta x' = 0 \quad . \quad . \quad . \quad . \quad (1);$$

the three characteristics d, δ, D, are employed here in conformity with the observations made in Art. 7, Sect. II. Part II. Mec. Analytique, the integral characteristic S corresponding to D.

Let φ be the angle the rod makes with the axis of x, and z the length of that part of the rod comprised between the point whose co-ordinates are x, y, and the end B, so that we shall have

$$\left.\begin{aligned} x &= x' + z\cos\varphi \\ y &= z\sin\varphi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

The quantity z does not vary with the position of the rod, but only with respect to the characteristics S and D, therefore $dz = 0$, and $\delta z = 0$; moreover, the velocity of the point B is uniform, therefore $\frac{dx'}{dt} =$ constant, and $\frac{d^2x'}{dt} = 0$. Hence we shall have, from (2),

$$\left.\begin{aligned} \delta x &= \delta x' - z\delta\varphi\sin\varphi \\ \delta y &= z\delta\varphi\cos\varphi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

$$\left.\begin{aligned} dx &= dx' - zd\varphi\sin\varphi \\ dy &= zd\varphi\cos\varphi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

$$\left.\begin{aligned} d^2x &= -zd^2\varphi\sin\varphi - zd\varphi^2\cos\varphi \\ d^2y &= zd^2\varphi\cos\varphi - zd\varphi^2\sin\varphi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (5).$$

Substituting these values in equation (1), and passing without the characteristic S, the quantities x', φ, and their differentials, since they do not vary in the sense of the differentials D, it becomes

$$\left\{pm - \left(\frac{d^2\varphi}{dt^2}\sin\varphi + \frac{d\varphi^2}{dt^2}\cos\varphi\right)Szdm\right\}dx' + \frac{d^2\varphi}{dt^2}\delta\varphi \times Sz^2Dm = 0 \quad . \quad (6).$$

If k be the mass of an unit's length of the rod, and l its length; then $Dm = kDz$, $SzDm = kSzDz =$ (from $z = 0$ to $z = l$) $\frac{1}{2}kl^2$, and $Sz^2Dm = kSz^2Dz = \frac{1}{3}kl^3$; but $m = kl$, therefore $SzDm = \frac{1}{2}lm$, and $Sz^2Dm = \frac{1}{3}l^2m$; hence (6) becomes, after dividing it by m,

$$\left\{p - \tfrac{1}{2}l\left(\frac{d^2\varphi}{dt^2}\sin\varphi + \frac{d\varphi^2}{dt^2}\cos\varphi\right)\right\}\delta x' + \tfrac{1}{3}l^2 \cdot \frac{d^2\varphi}{dt^2}\delta\varphi = 0 \quad . \quad (7);$$

and, since the variations $\delta x'$, $\delta\varphi$, are independent of each other,

$$p - \tfrac{1}{2}l\left(\frac{d^2\varphi}{dt^2}\sin\varphi + \frac{d\varphi^2}{dt^2}\cos\varphi\right) = 0 \quad . \quad . \quad . \quad . \quad (8),$$

$$\frac{d^2\varphi}{dt^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

By integrating equation (9) twice, we have

$$\frac{d\varphi}{dt} = \mathrm{A}, \text{ and } \varphi = \mathrm{A}t + \mathrm{A}' \quad . \quad . \quad . \quad . \quad . \quad (10).$$

Where A and A′ are constant quantities, to be determined.

Let φ' be the value of the angle φ when $t = 0$, in the assigned position of the rod on the plane, so that $\mathrm{A}' = \varphi'$, and u the given uniform velocity communicated to the point B, so that $\frac{dx'}{dt} = u$. Now at the commencement of motion, the force that draws B will be of the nature of an impulsive force, and may be regarded as equivalent to the motion impressed. Suppose that, at this first instant, the impulse at B communicates to the point of the rod whose co-ordinates are x and y, the finite velocities $\dot{x}$ and $\dot{y}$ in the direction of these co-ordinates, we shall have, as in Art. 11. of the section cited,

$$\mathrm{S}\,(\dot{x}\delta x + \dot{y}\delta y)\mathrm{D}m + pm\delta x' = 0 \quad . \quad . \quad . \quad . \quad (11).$$

But, from equations (4), φ' and A being the initial values of φ and $\frac{d\varphi}{dt}$,

$$\dot{x} = \frac{dx}{dt} = u - z\mathrm{A}\sin\varphi',$$

$$\dot{y} = \frac{dy}{dt} = z\mathrm{A}\cos\varphi'.$$

Subsituting these, with equations (3) and the integrals before found, in (11), it becomes, relative to the motion of rotation,

$$(\tfrac{1}{2}\,lu\sin\varphi' - \tfrac{1}{3}\,l^2\mathrm{A})\delta\varphi = 0 \quad . \quad . \quad . \quad . \quad . \quad (12);$$

$$\therefore \tfrac{1}{2}\,u\sin\varphi' - \tfrac{1}{3}\,l\mathrm{A} = 0,$$

$$\text{and } \mathrm{A} = \frac{3u}{2l}\,.\,\sin\varphi' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

This is the angular velocity impressed on the rod in the first instant, and which, by (10), it continues to have during the motion. Hence, at the epoch t, we shall have

$$\left.\begin{aligned} x' &= ut \\ \varphi &= \frac{3ut}{2l}\,.\,\sin\varphi' + \varphi' \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14),$$

which completely determines the position of the rod.

The force p, employed to draw B along the axis of x, is found from equation (8); for, since $\frac{d^2\varphi}{dt^2} = 0$, $\frac{d\varphi}{dt} = \mathrm{A} = \frac{3u}{2l}\,.\,\sin\varphi'$, and $\varphi = \frac{3ut}{2l}\,.\,\sin\varphi' + \varphi'$, we find by substitution,

$$p = \frac{9u^2}{8l}\,.\,\sin^2\varphi'\cos\left\{\frac{3ut}{2l}\,.\,\sin\varphi' + \varphi'\right\} \quad . \quad . \quad . \quad (15);$$

and therefore the initial value of $p = \frac{9u^2}{8l}\,.\,\sin^2\varphi'\cos\varphi'$. This force always vanishes when

$$\frac{3ut}{2l} . \sin \varphi' + \varphi' = (n + \tfrac{1}{2})\pi, \text{ or when } t = \frac{2l\left\{(n+\frac{1}{2})\pi - \varphi'\right\}}{3u \sin \varphi'},$$

n being any integer. The rod is then evidently perpendicular to the axis of x. If the motion of the rod were caused by a force P in the direction of the rod, the components of this force would be p and p', one in the direction of x, the other in that of y; then P $\cos \varphi = p$, and P $\sin \varphi = p'$; whence $p' = p \tan \varphi$, and by substitution,

$$p' = \frac{9u^2}{8l} . \sin {}^2\varphi' \sin \left\{\frac{3ut}{2l} . \sin \varphi' + \varphi'\right\} \quad . \quad . \quad . \quad (16).$$

This force, multiplied by m, would be the pressure of the rod upon the line in which its end B moves, in a direction perpendicular to the line, and the point B can only be preserved in the line by an equal reaction in an opposite direction; this force also vanishes when the rod coincides with the axis of x, or when

$$\frac{3ut}{2l} \sin \varphi' + \varphi' = n\pi, \text{ and } t = \frac{2l\,(n\pi - \varphi')}{3u \sin \varphi'},$$

n being any whole number.

To find the velocities of translation of any point of the rod at any time, z being its distance from the point B, equations (4), and the values already determined, give us

$$\left.\begin{aligned} \frac{dx}{dt} &= \frac{dx'}{dt} - z . \frac{d\varphi}{dt} . \sin \varphi \\ &= u - \frac{3zu}{2l} \sin \varphi' \sin \left\{\frac{3ut}{2l} . \sin \varphi' + \varphi'\right\} \\ \frac{dy}{dt} &= z . \frac{d\varphi}{dt} . \cos \varphi = \frac{3zu}{2l} . \sin \varphi' \cos \left\{\frac{3ut}{2l} \sin \varphi' + \varphi'\right\} \end{aligned}\right\} . \quad (17).$$

We see then that at every moment there may be one point of the rod which has no motion of translation in the direction of x; that is, where $\frac{dx}{dt} = 0$, or where

$$z = \tfrac{2}{3}\, l \operatorname{cosec} \varphi' \operatorname{cosec} \left\{\frac{3ut}{2l} \sin \varphi' + \varphi'\right\},$$

and this point will be the centre of rotation of the rod at the instant t; it will be nearest to the extremity B when the rod is perpendicular to x, and its nearest distance is expressed by $\frac{2}{3} l \operatorname{cosec} \varphi'$; it will be at the other extremity (A) of the rod when $\sin \varphi' \sin\left\{\frac{3ut}{2l} \sin \varphi' + \varphi'\right\} = \frac{2}{3}$, or when $t = \frac{2l}{3u \sin \varphi'}\left\{\sin^{-1} \frac{2}{3 \sin \varphi'} - \varphi'\right\}$, and beyond this limit the centre of rotation is without the rod, being at an infinite distance from it when the rod coincides in position with the axis of x; in fact, the ordinate of any point being, by (2), $y = z \sin \varphi$, for the centre of rotation $y = \frac{2l}{3 \sin \varphi'}$, a

constant quantity; and therefore the centre of rotation of the rod moves in a straight line parallel to x, and at the distance $\frac{2l}{3 \sin \varphi'}$ from it.

These facts are already sufficient to show that every point in the rod describes a species of cycloid. In order to exhibit their equations, eliminate t from the equations (14), and solve for x', then

$$x' = \frac{2l(\varphi - \varphi')}{3 \sin \varphi'},$$

and from the second of equations (2), we find,

$$\sin \varphi = \frac{y}{z}, \; \cos \varphi = \frac{\sqrt{z^2 - y^2}}{z}, \text{ and } \varphi = \sin^{-1}\frac{y}{z};$$

$$\therefore x' = \frac{2l}{3 \sin \varphi'}\left\{\sin^{-1}\frac{y}{z} - \varphi'\right\}.$$

These being substituted in the first of equations (2), gives

$$x = \sqrt{z^2 - y^2} + \frac{2l}{3 \sin \varphi'}\left\{\sin^{-1}\frac{y}{z} - \varphi'\right\} \quad . \quad . \quad . \quad (18).$$

This is the equation of the curve described by that point of the rod whose distance from B is z. It will evidently be

prolate when $z < \frac{2l}{3 \sin \varphi'}$,

common when $z = \frac{2l}{3 \sin \varphi'}$,

curtate when $z > \frac{2l}{3 \sin \varphi'}$.

ARTICLE IV.

ON SPHERICAL GEOMETRY.

The singular analogy which exists between many of the results in Plane and Spherical Geometry, has been noticed by several writers on these subjects; and it has no doubt occurred to many that the system of Analysis, which has been gradually perfecting by the successors of Leibnitz, might possibly be adapted to the investigation of lines on the sphere. A short paper on this subject in the Ladies' Diary for 1835, by T. S. Davis, Esq. is introduced, with the following remarks:—

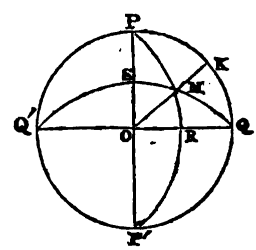
"The fertile mind of Euler seems to have been the first that conceived the idea of *spherical co-ordinates;* and it occurred to him as the means of eluding a difficulty of a kind that was then peculiar, (in his first paper on

the Halleyan Lines, *Berlin Mémoires* for 1756,) and, with the occasion that gave rise to it, he laid it aside, and seems never more to have resumed the employment of the method, or the investigation of its principles. About the close of the last century, several mathematicians of great eminence in this country also entered upon the inquiry; but, owing to the awkwardness of the trigonometrical notation that then prevailed, they did not find the results of such a kind as to encourage them to proceed to any great extent. * * * *

"About 1827, I was led to consider the nature of the hour lines upon the antique sun-dials. The successful application of the method to this previously intricate question, led me to investigate the principles of spherical geometry more carefully. In two papers in vol. XII. of the Transactions of the Royal Society of Edinburgh, I have given the results of these researches, so far as *polar spherical co-ordinates* are concerned, with several applications of the method,—as well as in one or two other places; but especially in the number now printing of the Monthly Repository, by my excellent friend professor Leybourn."

Some idea may be formed of the light in which Mr. Davies has viewed the subject, from the following definitions, in the same article.

"Conceive that O is the pole (or "spherical centre") of the great circle PQP′Q′; and that M is a point on the spherical surface which we desire to refer to spherical co-ordinates.



1. If we take OR as a primitive meridian through O the origin of co-ordinates, then an equation between the angle QOK (or arc KQ) and the polar distance OM, (spherical radius vector), will express the nature of the curve in which M may be supposed to be situated. This system is that of *spherical polar co-ordinates*.

Sometimes it is convenient to consider the relation between QK and KM, in preference to the other system.

2. We may consider the point M defined by the relation between OS and OR. This system is called the *longitudinal system of spherical co-ordinates*, the arcs OS and OR being measured in the same manner as *longitudes* on the earth are measured.

3. The point M may also be defined by an equation between MS and MR. This system, from the arcs being measured as terrestrial latitudes, I have called the *latitudinal system of spherical co-ordinates*."

He then gives formulas for transforming the co-ordinates from one system to either of the others, and promises to conclude the subject in a succeeding number of the Diary. I have endeavoured, but in vain, to procure the volume of the Edinburgh Transactions to which Mr. D. alludes, and I believe the number of the Repository (I presume he means the "Mathematical Repository") that he refers to as being then printed, has never yet

been published. I have, therefore, been induced to take up the subject myself; and, presuming that other American readers will find the same difficulty in procuring access to Mr. Davies' researches as I have done, I have here endeavoured to arrange the results at which I have arrived, as well as many that have laid by me for many years, into something like a regular system, and shall offer them to the readers of the Miscellany in this and the succeeding numbers.

I have used entirely the "*spherical polar co-ordinates*," not only because the subject first presented itself to my own view in this light, but because there still appears a greater facility of investigation by this system than by either of the others. I shall, however, take occasion to show the mode of transforming an equation from one system to another, and point out some remarkable analogies exhibited in the results.

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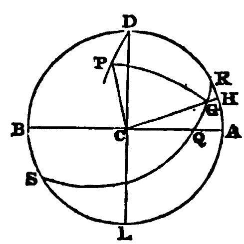
§. I.

Great circles and their intersections.

1. Let BCA be the angular axis, or prime meridian, C the origin of co-ordinates, the pole of the circle ADBL; let G be a point in the circle RQS, and P its pole, so that $PG = \frac{\pi}{2}$; put the radius vector $CG = y$, the angle $ACG = x$, $CP = \omega$, and $ACP = \varphi$.



Then, in the quadrantal triangle CPG, we have $CP = \omega$, $CG = y$, $PG = \frac{\pi}{2}$, and angle $PCG = \varphi - x$; therefore, by spherical trigonometry,

$$\cos(\varphi - x) = -\cot y \cot \omega;$$

$$\therefore \cot y + \cot \omega \cos(\varphi - x) = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is the equation of the circle.

When $y = \frac{\pi}{2}$, $\cos(\varphi - x) = 0$, or $\varphi - x = \pm \frac{\pi}{2}$, therefore the co-ordinates of the point R are $\frac{\pi}{2}$ and $\varphi - \frac{\pi}{2}$, and those of S, $\frac{\pi}{2}$ and $\varphi + \frac{\pi}{2}$. When $x = 0$, $\cot y + \cot \omega \cos \varphi = 0$, hence the circle whose equation is (1), intersects the axis at the distances $\cot^{-1}(-\cot \omega \cos \varphi)$ and $\pi + \cot^{-1}(-\cot \omega \cos \varphi)$ from C.

2. The equation of a great circle, then, is always of the form

$$\cot y + a \cos x + b \sin x = 0, \quad . \quad . \quad . \quad . \quad . \quad (2),$$

as will be seen by developing $\cos(\varphi - x)$ in (1), where the values of the constants a and b, in functions of the co-ordinates of its pole, are

$$a = \tan\omega \cos\varphi,\ b = \tan\omega \sin\varphi \quad . \quad . \quad . \quad . \quad (3).$$

Conversely, if the equation of a great circle be given, it may, in general, be put under the form (2), and the co-ordinates of its poles will be, from (3),

$$\tan\omega = \pm\sqrt{a^2 + b^2},\ \tan\varphi = \frac{b}{a} \quad . \quad . \quad . \quad . \quad (4).$$

We have taken the obvious course of defining the equation of the circle by the co-ordinates of its poles, but any other two constants, that determine the circle's position, may be used for the same purpose; thus, the distance CQ $= \psi$, and the angle RQA, or arc PD $= \theta$, which measures it, might be used, and we should have the relations

$$\left.\begin{aligned} \cot\psi &= -\tan\omega\cos\varphi = -a, \\ \cos\theta &= \sin\omega\sin\varphi = b\cos\omega = \frac{b}{\sqrt{1 + a^2 + b^2}} \end{aligned}\right\} \quad . \quad . \quad (5).$$

3. Cor. 1. If, in (1), $\omega = \frac{\pi}{2}$, P will be in the circumference ADB, and the circle will pass through C, and be perpendicular to ADB; hence, the equation of such circles is

$$\cos(\varphi - x) = 0, \text{ or } x = \varphi - \frac{\pi}{2} \quad . \quad . \quad . \quad . \quad (6).$$

Cor. 2. If $\varphi = 0$, or the pole is in the axis AB, the circle will be a secondary to the axis, and will pass through D and L; the equation of all these secondaries is

$$\cot y + \tan\omega\cos x = 0, \quad . \quad . \quad . \quad . \quad . \quad (7),$$

and they intersect AB at the distances $\omega \pm \frac{\pi}{2}$ from C.

Cor. 3. If $\varphi = \frac{\pi}{2}$, or the pole is in CD, the circle is secondary to DL, and will pass through A and B; the equation of such circles is

$$\cot y + \tan\omega\sin x = 0 \quad . \quad . \quad . \quad . \quad . \quad (8),$$

and they intersect DL at the distances $\omega \pm \frac{\pi}{2}$ from C.

4. If the circle whose equation is

$$\cot y + \tan\omega\cos(\varphi - x) = 0,$$

pass through the given point $(y_1 x_1)$, then also

$$\cot y_1 + \tan\omega\cos(\varphi - x_1) = 0,$$

and eliminating ω, the equation will be

$$\cot y - \cot y_1 \cdot \frac{\cos(\varphi - x)}{\cos(\varphi - x_1)} = 0 \quad . \quad . \quad . \quad . \quad (9).$$

COR. If $(y_1\, x_1)$ be the pole $(\omega_1\, \varphi_1)$ of the circle

$$\cot y + \tan \omega_1 \cos (\varphi_1 - x) = 0,$$

the equation of all secondaries to that circle will be

$$\cot y - \cot \omega_1 \,.\, \frac{\cos (\varphi - x)}{\cos (\varphi - \varphi_1)} = 0 \quad . \quad . \quad . \quad . \quad . \quad (10).$$

5. If the circle pass through a second given point $(y_2\, x_2)$, then also

$$\cot y - \cot y_2 \,.\, \frac{\cos (\varphi - x)}{\cos (\varphi - x_2)} = 0 \quad . \quad . \quad . \quad . \quad . \quad (11);$$

and eliminating φ between (9) and (11), we shall get, after a little reduction, the very symmetrical equation,

$$\cot y \sin (x_1 - x_2) + \cot y_1 \sin (x_2 - x) + \cot y_2 \sin (x - x_1) = 0 \quad . \quad (12);$$

the analogy between this and the equation of a straight line passing through two given points, is very obvious.

For some purposes, a better form for (12), and one easily deducible from it, is

$$\cot y - \frac{\cot y_1 + \cot y_2}{2 \cos \frac{1}{2}(x_1 - x_2)} \,.\, \cos \left\{ x - \tfrac{1}{2}(x_1 + x_2) \right\}$$
$$- \frac{\cot y_1 - \cot y_2}{2 \sin \frac{1}{2}(x_1 - x_2)} \,.\, \sin \left\{ x - \tfrac{1}{2}(x_1 + x_2) \right\} = 0 \quad . \quad . \quad . \quad (13).$$

By putting equation (12) into the form (2), we shall find

$$\left. \begin{aligned} a &= \frac{\cot y_1 \sin x_2 - \cot y_2 \sin x_1}{\sin (x_1 - x_2)} \\ b &= \frac{\cot y_2 \cos x_1 - \cot y \;\; \cos x_2}{\sin (x_1 - x_2)} \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad (14);$$

and hence, by (4), the poles of the great circle passing through the two given points $(y_1\, x_1)$, $(y_2\, x_2)$, are determined by

$$\left. \begin{aligned} \tan \omega &= \pm \frac{\sqrt{\cot^2 y_1 + \cot^2 y_2 - 2 \cot y_1 \cot y_2 \cos (x_1 - x_2)}}{\sin (x_1 - x_2)} \\ \tan \varphi &= \frac{\cot y_2 \cos x_1 - \cot y \;\; \cos x_2}{\cot y_1 \sin x_2 - \cot y_2 \sin x_1} \end{aligned} \right\} . \; (15).$$

6. COR. 1. If the prime meridian AB be the equinoctial colure, $(y\, x)$, $(y_1\, x_1)$, $(y_2\, x_2)$, will represent the polar distances and right ascensions of three stars, if C be the pole of the heavens, or their co-latitudes and longitudes, if C be the pole of the ecliptic; and equation (12), or (13), will express the relation between these elements when the three stars are situated in the same great circle.

7. COR. 2. If the three great circles

$$\cot y + \tan \omega_1 \cos (\varphi_1 - x) = 0,$$
$$\cot y + \tan \omega_2 \cos (\varphi_2 - x) = 0,$$
$$\cot y + \tan \omega_3 \cos (\varphi_3 - x) = 0,$$

intersect in the same point, their poles $(\omega_1\, \varphi_1)$, $(\omega_2\, \varphi_2)$, $(\omega_3\, \varphi_3)$, must neces-

sarily be found in the same great circle; hence the co-ordinates must fulfil equation (9), that is, they must have the relation

$$\cot\omega_1 \sin(\varphi_2-\varphi_3)+\cot\omega_2 \sin(\varphi_3-\varphi_1)+\cot\omega_3 \sin(\varphi_1-\varphi_2)=0 \quad (16).$$

If the three equations had been given in the form (2), a_1, b_1; a_2, b_2; a_3, b_3; being the several constants, then by substituting the values from (4),

$$\cot\omega_1=\frac{1}{\sqrt{a_1^2+b_1^2}},\ \sin\varphi_1=\frac{b_1}{\sqrt{a_1^2+b_1^2}},\ \cos\varphi_1=\frac{a_1}{\sqrt{a_1^2+b_1^2}},$$

and so for the others, the relation among the constants would be

$$b_1(a_2-a_3)+b_2(a_3-a_1)+b_3(a_1-a_2)=0 \quad . \quad . \quad . \quad . \quad (17),$$

which is precisely the relation among the co-ordinates of three points on a plane, when they are situated in the same straight line.

8. Cor. 3. The equations of two great circles being

$$\left.\begin{aligned}\cot y+\tan\omega_1\cos(\varphi_1-x)=0,\\ \cot y+\tan\omega_2\cos(\varphi_2-x)=0;\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (18),$$

the great circle perpendicular to them both, will pass through their poles $(\omega_1\,\varphi_1)$, and $(\omega_2\,\varphi_2)$, and its equation will therefore be

$$\cot y\sin(\varphi_1-\varphi_2)+\cot\omega_1\sin(\varphi_2-x)+\cot\omega_2\sin(x-\varphi_1)=0 \quad . \quad (19),$$

and, if the equations of the two circles be

$$\left.\begin{aligned}\cot y+a_1\cos x+b_1\sin x=0,\\ \cot y+a_2\cos x+b_2\sin x=0;\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (20),$$

the equation of the circle perpendicular to them both will be

$$\cot y+\frac{b_2-b_1}{b_1a_2-b_2a_1}\cdot\cos x+\frac{a_1-a_2}{b_1a_2-b_2a_1}\cdot\sin x=0 \quad . \quad (21).$$

9. Cor. 4. The points of intersection of the two great circles (18), are the poles of the great circle passing through their two poles, of which the equation is (19); and therefore, by (15), the co-ordinates of these points will be

$$\left.\begin{aligned}\tan y&=\frac{\pm\sqrt{\cot^2\omega_1+\cot^2\omega_2-2\cot\omega_1\cot\omega_2\cos(\varphi_1-\varphi_2)}}{\sin(\varphi_1-\varphi_2)}\\ \tan x&=\frac{\cot\omega_2\cos\varphi_1-\cot\omega_1\cos\varphi_2}{\cot\omega_1\sin\varphi_2-\cot\omega_2\sin\varphi_1}\end{aligned}\right\} \quad . \quad (22)$$

and if the equations be those of (20), their points of intersection will be

$$\left.\begin{aligned}\tan y&=\frac{\pm\sqrt{(a_1-a_2)^2+(b_1-b_2)^2}}{b_1a_2-b_2a_1}\\ \tan x&=\frac{a_1-a_2}{b_2-b_1}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (23).$$

10. Cor. 5. If, in (12), the point $y_2\,x_2$ be the pole $(\omega\,\varphi)$ of the great circle (1), then,

$$\cot y\sin(x_1-\varphi)+\cot y_1\sin(\varphi-x)+\cot\omega\sin(x-x_1)=0 \quad . \quad (24),$$

is the equation of a great circle through the point $(y_1 x_1)$, and perpendicular to the circle

$$\cot y + \tan \omega \cos (\varphi - x) = 0.$$

11. Let there be two given points $(y_1 x_1)$, $(y_2 x_2)$; to find their distance (Δ) from each other, counted on the arc of a great circle.

It is evident that Δ will be the third side of a spherical triangle, of which y_1, y_2 are two sides, and $x_1 - x_2$ their included angle; hence, by trigonometry,

$$\cos \Delta = \cos y_1 \cos y_2 + \sin y_1 \sin y_2 \cos (x_1 - x_2) \quad . \quad . \quad (25).$$

12. Cor. 1. The length of a perpendicular (Π), drawn as in Art. 10., from a given point $(y_1 x_1)$, on a given circle, is the complement of the arc between the given point and the pole $(\omega \varphi)$ of the given circle; therefore

$$\sin \Pi = \cos y_1 \cos \omega + \sin y_1 \sin \omega \cos (x_1 - \varphi) \quad . \quad . \quad (26),$$

and if the point $y_1 x_1$ be the origin, so that $y_1 = 0$;

$$\sin \Pi = \cos \omega, \text{ or } \Pi = \frac{\pi}{2} - \omega, \text{ or } \omega - \frac{\pi}{2} \quad . \quad . \quad . \quad . \quad (27).$$

Cor. 2. If two points are situated a quadrant's distance from each other, $\Delta = \frac{\pi}{2}$, and their co-ordinates must have the relation,

$$0 = \cos y_1 \cos y_2 + \sin y_1 \sin y_2 \cos (x_1 - x_2)$$

$$\text{or, } \cot y_1 \cot y_2 + \cos (x_1 - x_2) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28).$$

13. Cor. 3. If two points are situated diametrically opposite to each other, or $\Delta = \pi$, their co-ordinates must have the relation,

$$-1 = \cos y_1 \cos y_2 + \sin y_1 \sin y_2 \cos (x_1 - x_2) \quad . \quad . \quad (29).$$

But, since $\cos y_1 \cos y_2 = \frac{1}{2} \cos (y_1 + y_2) + \frac{1}{2} \cos (y_1 - y_2)$

$$= -1 + \cos^2 \tfrac{1}{2}(y_1 + y_2) + \cos^2 \tfrac{1}{2}(y_1 - y_2);$$

and, $\sin y_1 \sin y_2 = \frac{1}{2} \cos (y_1 - y_2) - \frac{1}{2} \cos (y_1 + y_2)$

$$= \cos^2 \tfrac{1}{2}(y_1 - y_2) - \cos^2 \tfrac{1}{2}(y_1 + y_2);$$

substituting these in (29),

$$\cos^2 \tfrac{1}{2}(y_1+y_2)(1-\cos \overline{x_1-x_2})+\cos^2 \tfrac{1}{2}(y_1-y_2)(1+\cos \overline{x_1-x_2})=0;$$

$$\text{or } \cos^2 \tfrac{1}{2}(y_1+y_2) \sin^2 \tfrac{1}{2}(x_1-x_2)+\cos^2 \tfrac{1}{2}(y_1-y_2) \cos^2 \tfrac{1}{2}(x_1-x_2)=0 \ . \ (30),$$

an equation which can only exist when each of the terms in its first member is equal to zero, that is,

$$\cos \tfrac{1}{2}(y_1 + y_2) \sin \tfrac{1}{2}(x_1 - x_2) = 0,$$

$$\cos \tfrac{1}{2}(y_1 - y_2) \cos \tfrac{1}{2}(x_1 - x_2) = 0;$$

$$\left.\begin{array}{ll} \text{therefore,} & y_1 + y_2 = \pi, \text{ and } x_1 - x_2 = \pi; \\ \text{or,} & x_1 - x_2 = 0, \text{ and } y_1 - y_2 = \pi. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (31),$$

either of which show that the two points are situated in a circle passing through the origin at the distance π from each other.

Since the position of the origin of co-ordinates is arbitrary with respect to the two given points, it follows that an infinite number of circles may be drawn through the two given points, and that a circle which passes

through one point, must necessarily pass through another one diametrically opposite to it. These well known properties, which might also have been shown from equation (13), have been deduced here for the purpose of showing the facility with which the first principles of the doctrine of the Sphere might be exhibited by analysis.

14. The angle contained between two great circles,

$$\cot y + \tan \omega_1 \cos (\varphi_1 - x) = 0,$$
$$\cot y + \tan \omega_2 \cos (\varphi_2 - x) = 0,$$

being measured by the arc subtended between their poles $(\omega_1 \varphi_1)$, and $(\omega_2 \varphi_2)$; if i be the angle of intersection, we have, from (25),

$$\cos i = \cos \omega_1 \cos \omega_2 + \sin \omega_1 \sin \omega_2 \cos (\varphi_1 - \varphi_2) \quad . \quad . \quad (32).$$

If the equations be given in the form (20), we shall find, by substituting the relations (4) in (32),

$$\cos i = \frac{1 + a_1 a_2 + b_1 b_2}{\sqrt{(1 + a_1^2 + b_1^2)(1 + a_2^2 + b_2^2)}} \quad . \quad . \quad . \quad (33).$$

This is the same as the expression for the angle of inclination between two planes, whose equations are

$$z = a_1 x + b_1 y + c_1,$$
$$z = a_2 x + b_2 y + c_2.$$

Cor. If one of the given circles be the axis, or $\omega_2 = \frac{\pi}{2}$, $\varphi_2 = \frac{\pi}{2}$, then

$$\cos i = \sin \omega_1 \sin \varphi_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34).$$

15. Let the circle whose equation is

$$\cot y + \tan \omega_1 \cos (\varphi_1 - x) = 0, \quad . \quad . \quad . \quad . \quad (35),$$

pass through the given point $y_1 x_1$, and make a given angle i with the given circle,

$$\cot y + \tan \omega \cos (\varphi - x) = 0.$$

Then, to determine ω_1 and φ_1, we have the two equations,

$$\left.\begin{array}{r} \cot y_1 + \tan \omega_1 \cos (\varphi_1 - x_1) = 0 \\ \cos i = \cos \omega \cos \omega_1 + \sin \omega \sin \omega_1 \cos (\varphi - \varphi_1) \end{array}\right\} \quad . \quad . \quad (36).$$

In order to facilitate the solution of these equations, let the distance from the given point to the pole of the given circle be δ, so that

$$\cos \delta = \cos \omega \cos y_1 + \sin \omega \sin y_1 \cos (\varphi - x_1) \quad . \quad . \quad (37),$$

and assume the arcs s and ζ, so that

$$\left.\begin{array}{l} \sin \delta \sin s = \sin \omega \sin (\varphi - x_1) \\ \sin \delta \cos \zeta = \cos i \end{array}\right\} \quad . \quad . \quad . \quad . \quad (38),$$

we shall then find, with very little trouble,

$$\left.\begin{array}{l} \cos \omega_1 = \sin y_1 \cos (s - \zeta) \\ \cot (\varphi_1 - x_1) = - \cos y_1 \cot (s - \zeta) \end{array}\right\} \quad . \quad . \quad . \quad . \quad (39).$$

Then, since $\cos (\varphi_1 - x) = \cos \{(\varphi_1 - x_1) + (x_1 - x)\}$

$$= \cos (\varphi_1 - x_1) \cos (x_1 - x) - \sin (\varphi_1 - x_1) \sin (x_1 - x)$$
$$= \sin (\varphi_1 - x_1) \{\cot (\varphi_1 - x_1) \cos (x_1 - x) - \sin (x_1 - x)\}$$
$$= - \sin (\varphi_1 - x_1) \{\cos y_1 \cot (s - \zeta) \cos (x_1 - x) + \sin (x_1 - x)\}$$

$$\text{and } \tan\omega_1 \sin(\varphi_1 - x_1) = \frac{\sqrt{1-\sin^2 y_1 \cos^2(s-\zeta)}}{\sin y_1 \cos(s-\zeta)} \times \frac{1}{\sqrt{1+\cos^2 y_1 \cot^2(s-\zeta)}}$$
$$= \frac{\sqrt{1-\sin^2 y_1 \cos^2(s-\zeta)}}{\sin y_1 \cos(s-\zeta)} \times \frac{\sin(s-\zeta)}{\sqrt{1-\sin^2 y_1 \cos^2(s-\zeta)}}$$
$$= \frac{\tan(s-\zeta)}{\sin y_1}.$$

Thus the equation (35) of the required circle becomes,

$$\cot y - \cot y_1 \cos(x_1 - x) - \operatorname{cosec} y_1 \tan(s-\zeta)\sin(x_1 - x) = 0 \quad . \quad (40).$$

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§. II.

Less-Circles and their Tangents.

16. If the distance PG of the circle from its pole be greater or less than $\frac{\pi}{2}$, the circle will be a lesser circle; calling this distance r, the triangle PGC will give us,

$$\cos r = \cos\omega\cos y + \sin\omega\sin y\cos(\varphi - x) \quad . \quad . \quad (41).$$

This equation may be changed into

$$\cot y - \cos r \sec\omega \operatorname{cosec} y + \tan\omega\cos(\varphi - x) = 0 \quad . \quad (42),$$

in which it differs from the equation of a great circle (1), by its second term alone. It may always be put, as in Art. 2. under the form,

$$\cot y + c\operatorname{cosec} y + a\cos x + b\sin x = 0 \quad . \quad . \quad . \quad (43),$$

where, $\qquad a = \tan\omega\cos\varphi,\ b = \tan\omega\sin\varphi,\ c = -\frac{\cos r}{\cos\omega} \quad . \quad . \quad (44);$

and, therefore,

$$\tan\omega = \pm\sqrt{a^2+b^2},\ \tan\varphi = \frac{b}{a},\ \cos r = \frac{-c}{\sqrt{1+a^2+b^2}} \quad . \quad (45).$$

17. Cor. 1. If $\omega = \varphi = \frac{\pi}{2}$, the circle will be parallel to the axis, and its equation will be

$$\sin y\sin x = \cos r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46).$$

Cor. 2. When $\varphi = 0$, the pole will be in the axis, and the circle will be perpendicular to it; its equation is

$$\cot y - \cos r\sec\omega\operatorname{cosec} y + \tan\omega\cos x = 0 \quad . \quad . \quad (47).$$

Cor. 3. And when also $\omega = 0$, or the pole is C, the equation is

$$y = r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48).$$

18. If the circle pass through the given point $x_1\ y_1$, then

$$\cos r = \cos\omega\cos y_1 + \sin\omega\sin y_1\cos(\varphi - x_1) \quad . \quad . \quad (49);$$

and, eliminating r between this equation and (41),

$$\cos\omega(\cos y-\cos y_1)+\sin\omega\left\{\sin y\cos(\varphi-x)-\sin y_1\cos(\varphi-x_1)\right\}=0 \quad (50),$$

is the equation of the circle; it may also be written,

$$\cot y-(\cos y_1+\tan\omega\sin y_1\cos\overline{\varphi-x_1})\operatorname{cosec} y+\tan\omega\cos(\varphi-x)=0 \quad . \ (51).$$

Cor. If the circle pass through the origin, or $y_1=0$; then $r=\omega$, and the equation is,

$$\cot y-\operatorname{cosec} y+\tan r\cos(\varphi-x)=0;$$

$$\text{or, } \tan\tfrac{1}{2}y=\tan r\cos(\varphi-x) \quad . \ . \ . \ . \ . \ . \ (52).$$

If also, $\varphi=\frac{\pi}{2}$, it will touch the axis in the origin, and the equation is,

$$\tan\tfrac{1}{2}y=\tan r\sin x \quad . \ . \ . \ . \ . \ . \ . \ . \ (53).$$

19. If the circle pass through the two given points $(x_1\, y_1)$, $(x_2\, y_2)$, then, besides equations (49) and (50), we shall have,

$$\cos r=\cos\omega\cos y_2+\sin\omega\sin y_2\cos(\varphi-x_2) \quad . \ . \ (54),$$

$$\cos\omega(\cos y_1-\cos y_2)+\sin\omega\left\{\sin y_1\cos(\varphi-x_1)-\sin y_2\cos(\varphi-x_2)\right\}=0 \quad (55),$$

$$\cos\omega(\cos y-\cos y_2)+\sin\omega\left\{\sin y\cos(\varphi-x)-\sin y_2\cos(\varphi-x_2)\right\}=0 \quad (56),$$

and eliminating ω between (50) and (56), we get

$$\frac{\cos y-\cos y_1}{\cos y-\cos y_2}=\frac{\sin y\cos(\varphi-x)-\sin y_1\cos(\varphi-x_1)}{\sin y\cos(\varphi-x)-\sin y_2\cos(\varphi-x_2)} \quad . \ (57).$$

This equation, by a little reduction becomes,

$$(\cos y-\cos y_1)\sin y_2\cos(\varphi-x_2)-(\cos y-\cos y_2)\sin y_1\cos(\varphi-x_1)$$
$$+(\cos y_1-\cos y_2)\sin y\cos(\varphi-x)=0 \quad . \ . \ (58).$$

And again,

$$\cot y\left(\frac{\cos(\varphi-x_2)}{\sin y_1}-\frac{\cos(\varphi-x_1)}{\sin y_2}\right)$$
$$-\left\{\cot y_1\cos(\varphi-x_2)-\cot y_2\cos(\varphi-x_1)\right\}\operatorname{cosec} y$$
$$+\frac{\cos y_1-\cos y_2}{\sin y_1\sin y_2}.\cos(\varphi-x)=0 \quad . \ . \ . \ (59).$$

20. Equation (55) shows that the poles $(\omega\,\varphi)$ of all circles passing through two given points $(y_1\, x_1)$, $(y_2\, x_2)$ lie in a great circle, whose equation is

$$\cot\omega+\frac{\sin y_1\cos x_1-\sin y_2\cos x_2}{\cos y_1-\cos y_2}\cos\varphi$$
$$+\frac{\sin y_1\sin x_1-\sin y_2\sin x_2}{\cos y_1-\cos y_2}.\sin\varphi=0 \quad . \ . \ . \ (60);$$

therefore, by (4), the co-ordinates of its poles are determined by

$$\left.\begin{aligned}\tan\omega_1 &= \pm\frac{\sqrt{\sin^2 y_1+\sin^2 y_2-2\sin y_1\sin y_2\cos(x_1-x_2)}}{\cos y_1-\cos y_2}\\ \tan\varphi_1 &= \frac{\sin y_1\sin x_1-\sin y_2\sin x_2}{\sin y_1\cos x_1-\sin y_2\cos x_2}\end{aligned}\right\}\ .\ (61).$$

These co-ordinates will be found to satisfy the equation (12) of the circle passing through the two given points $(y_1\ x_1)$, $(y_2\ x_2)$; the circle (60) is therefore perpendicular to that circle, and it bisects the arc of it included between the given points; because, if we put δ for the distance between the two points, we shall have, by (25),

$$\cos\delta=\cos y_1\cos y_2+\sin y_1\sin y_2\cos(x_1-x_2),$$

and if N be the numerator of the value of $\tan\omega_1$ in (61),

$$\begin{aligned}N^2&=\sin^2 y_1+\sin^2 y_2-2\cos\delta+2\cos y_1\cos y_2\\ &=4\sin^2\tfrac{1}{2}\delta-(\cos y_1-\cos y_2)^2;\end{aligned}$$

$$\therefore\ \cos\omega_1=\frac{\cos y_1-\cos y_2}{2\sin\frac{1}{2}\delta},\ \sin\omega_1=\frac{N}{2\sin\frac{1}{2}\delta}$$

$$\cos\varphi_1=\frac{\sin y_1\cos x_1-\sin y_2\cos x_2}{N},\ \sin\varphi_1=\frac{\sin y_1\sin x_1-\sin y_2\sin x_2}{N};$$

$$\therefore\ \cos(\varphi_1-x_1)$$
$$=\frac{\sin y_1\cos^2 x_1-\sin y_2\cos x_1\cos x_2+\sin y_1\sin^2 x_1-\sin y_2\sin x_1\sin x_2}{N}$$
$$=\frac{\sin y_1-\sin y_2\cos(x_1-x_2)}{N}.$$

Then, if we put $\varDelta$ for the distance between the pole $\omega_1\ \varphi_1$ and the point $y_1\ x_1$, we have

$$\begin{aligned}\cos\varDelta&=\cos\omega_1\cos y_1+\sin\omega_1\sin y_1\cos(\varphi_1-x_1)\\ &=\frac{\cos^2 y_1-\cos y_1\cos y_2+\sin^2 y_1-\sin y_1\sin y_2\cos(x_1-x_2)}{2\sin\frac{1}{2}\delta}\\ &=\frac{1-\cos\delta}{2\sin\frac{1}{2}\delta}\\ &=\sin\tfrac{1}{2}\delta,\end{aligned}$$

or $\varDelta=\frac{1}{2}\pi-\frac{1}{2}\delta$. The same thing might be shown by placing the two given points on the axis or prime meridian; then $x_1=0$, $x_2=0$, and the locus (60) will intersect the axis when $\varphi=0$;

$$\therefore\ \cot\omega=\frac{\sin y_2-\sin y_1}{\cos y_1-\cos y_2}=\cot\tfrac{1}{2}(y_1+y_2);$$

$\therefore\ \omega=\frac{1}{2}(y_1+y_2)$, and the locus bisects the arc joining the two points.

21. If the circle pass through three given points $(y_1\ x_1)$, $(y_2\ x_2)$, $(y_3\ x_3)$; then also we shall have,

$$\frac{\cos y-\cos y_1}{\cos y-\cos y_3}=\frac{\sin y\cos(\varphi-x)-\sin y_1\cos(\varphi-x_1)}{\sin y\cos(\varphi-x)-\sin y_3\cos(\varphi-x_3)}\ .\ (62);$$

for, the circle passing through $y_1\ x_1$ and $y_3\ x_3$; and eliminating φ between

the equations (57) and (62), the process of which need not be repeated here, we obtain the remarkably symmetrical equation

$$\begin{aligned}&(\cos y-\cos y_1)\sin y_2\sin y_3\sin(x_3-x_2)\\ &+(\cos y-\cos y_2)\sin y_1\sin y_3\sin(x_1-x_3)\\ &+(\cos y-\cos y_3)\sin y_1\sin y_2\sin(x_2-x_1)\\ &+(\cos y_1-\cos y_2)\sin y\sin y_3\sin(x_3-x)\\ &+(\cos y_1-\cos y_3)\sin y\sin y_2\sin(x-x_2)\\ &+(\cos y_2-\cos y_3)\sin y\sin y_1\sin(x_1-x)\\ &=0\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\qquad(63).\end{aligned}$$

It may also take the form

$$\begin{aligned}&\left\{\frac{\sin(x_3-x_1)}{\sin y_1}+\frac{\sin(x_1-x_3)}{\sin y_2}+\frac{\sin(x_2-x_1)}{\sin y_3}\right\}\cot y\\ &-\left\{\cot y_1\sin(x_3-x_2)+\cot y_2\sin(x_1-x_3)+\cot y_3\sin(x_2-x_1)\right\}\operatorname{cosec} y\\ &+\frac{\cos y_1-\cos y_2}{\sin y_1\sin y_2}\,.\,\sin(x_3-x)+\frac{\cos y_1-\cos y_3}{\sin y_1\sin y_3}\,.\,\sin(x-x_2)\\ &+\frac{\cos y_2-\cos y_3}{\sin y_1\sin y_3}\,.\,\sin(x_1-x)=0\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\qquad(64),\end{aligned}$$

in which it can easily be assimilated with (43), and we shall see at once that when the three given points are in the same great circle, $c=0$, or the second term will disappear from the equation; because then, by (12),

$$\cot y_1\sin(x_3-x_2)+\cot y_2\sin(x_1-x_3)+\cot y_3\sin(x_2-x_1)=0.$$

22. We have seen (art. 20.), that the poles of all circles passing through the points $(y_1\,x_1)$, $(y_2\,x_2)$, lie in the great circle

$$\begin{aligned}\cot y&+\frac{\sin y_1\cos x_1-\sin y_2\cos x_2}{\cos y_1-\cos y_2}\cos x\\ &+\frac{\sin y_1\sin x_1-\sin y_2\sin x_2}{\cos y_1-\cos y_2}\sin x=0\quad.\;.\;.\quad(65).\end{aligned}$$

Similarly, the poles of all circles passing through $(y_1\,x_1)$, $(y_3\,x_3)$, lie in the circle

$$\begin{aligned}\cot y&+\frac{\sin y_1\cos x_1-\sin y_3\cos x_3}{\cos y_1-\cos y_3}\cos x\\ &+\frac{\sin y_1\sin x_1-\sin y_3\sin x_3}{\cos y_1-\cos y_3}\sin x=0\quad.\;.\;.\quad(66),\end{aligned}$$

and the poles of all circles passing through $(y_2\,x_2)$, $(y_3\,x_3)$, lie in the circle

$$\begin{aligned}\cot y&+\frac{\sin y_2\cos x_2-\sin y_3\cos x_3}{\cos y_2-\cos y_3}\cos x\\ &+\frac{\sin y_2\sin x_2-\sin y_3\sin x_3}{\cos y_2-\cos y_3}\sin x=0\quad.\;.\;.\quad(67).\end{aligned}$$

Now, the circle (63) which passes through all the three points, must have its poles in all these three circles; hence, the three great circles drawn perpendicular to three given great circles, and bisecting severally

the arcs intercepted between their intersections, will meet in the same point; hence also, to find the pole of the circle whose equation is (63) or (64), we have only to find the point of intersection of (65) and (66). For this purpose, let

$$a_1 = \frac{\sin y_1 \cos x_1 - \sin y_2 \cos x_2}{\cos y_1 - \cos y_2},$$

$$b_1 = \frac{\sin y_1 \sin x_1 - \sin y_2 \sin x_2}{\cos y_1 - \cos y_2},$$

$$a_2 = \frac{\sin y_1 \cos x_1 - \sin y_3 \cos x_3}{\cos y_1 - \cos y_3},$$

$$b_2 = \frac{\sin y_1 \sin x_1 - \sin y_3 \sin x_3}{\cos y_1 - \cos y_3};$$

then, $(\omega\,\varphi)$ being the pole of the circle (64), we have, by (23),

$$\left.\begin{aligned} \tan\varphi &= \frac{a_1 - a_2}{b_1 - b_2}, \\ \tan\omega &= \frac{\sqrt{(a_1-a_2)^2+(b_1-b_2)^2}}{b_1\,a_2 - b_2\,a_1} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (68).$$

23. If now we put, for greater conveniency,

$$\sin y_1 \sin y_2 \sin(x_1-x_2) + \sin y_1 \sin y_3 \sin(x_3-x_1) + \sin y_2 \sin y_3 \sin(x_2-x_3) = \mathrm{P},$$

$$\cos y_3 \sin y_1 \sin y_2 \sin(x_1-x_2) + \cos y_2 \sin y_1 \sin y_3 \sin(x_3-x_1) + \cos y_1 \sin y_2 \sin y_3 \sin(x_2-x_3) = \mathrm{N};$$

we shall find, by writing (64) in the form (43), $c = -\frac{\mathrm{N}}{\mathrm{P}}$; and by (44),

$$\cos r = -c\cos\omega = \frac{\mathrm{N}\cos\omega}{\mathrm{P}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69).$$

$$\text{Now, } \cos\omega = \frac{b_1\,a_2 - b_2\,a_1}{\sqrt{(b_1\,a_2-b_2\,a_1)^2+(a_1-a_2)^2+(b_1-b_2)^2}}$$

$$= \frac{b_1\,a_2 - b_2\,a_1}{\sqrt{\left\{(a_1^2+b_1^2+1)\,(a_2^2+b_2^2+1)-(a_1\,a_2+b_1\,b_2+1)^2\right\}}} \quad . \quad (70).$$

Let δ_3 = distance between the points $(y_1\,x_1)$ and $(y_2\,x_2)$, δ_2 = distance from $(y_1\,x_1)$ to $(y_3\,x_3)$, and δ_1 = distance from $(y_2\,x_2)$ to $(y_3\,x_3)$, so that

$$\cos\delta_3 = \cos y_1 \cos y_2 + \sin y_1 \sin y_2 \cos(x_1 - x_2),$$

$$\cos\delta_2 = \cos y_1 \cos y_3 + \sin y_1 \sin y_3 \cos(x_3 - x_1),$$

$$\cos\delta_1 = \cos y_2 \cos y_3 + \sin y_2 \sin y_3 \cos(x_2 - x_3).$$

Then we shall find by substitution,

$$\left.\begin{aligned}
a_1^2+b_1^2+1&=\frac{2(1-\cos\delta_3)}{(\cos y_1-\cos y_2)^2}=\frac{4\sin^2\frac{1}{2}\delta_3}{(\cos y_1-\cos y_2)^2},\\
a_2^2+b_2^2+1&=\frac{2(1-\cos\delta_2)}{(\cos y_1-\cos y_3)^2}=\frac{4\sin^2\frac{1}{2}\delta_2}{(\cos y_1-\cos y_3)^2},\\
a_1\,a_2+b_1\,b_2+1&=\frac{1-\cos\delta_1-\cos\delta_2-\cos\delta_3}{(\cos y_1-\cos y_2)(\cos y_1-\cos y_3)},\\
b_1\,a_2-b_2\,a_1&=\frac{\mathrm{P}}{(\cos y_1-\cos y_2)(\cos y_1-\cos y_3)},\\
\sqrt{\left\{(a_1^2+b_1^2+1)(a_2^2+b_2^2+1)-(a_1\,a_2+b_1\,b_2+1)^2\right\}}&\\
=\frac{\mathrm{M}}{(\cos y_1-\cos y_2)(\cos y_1-\cos y_3)},&
\end{aligned}\right\}\quad .\ (71);$$

where

$$\mathrm{M}=\sqrt{\left\{(1-\cos\delta_1-\cos\delta_2-\cos\delta_3)^2-2\cos^2\delta_1-2\cos^2\delta_2-2\cos^2\delta_3+2\right\}}\ (72),$$

a symmetrical function of the three distances.

Without at present going into any further detail of the many curious relations, such as those in (71), that exist among these quantities, it will be enough to say, that by a mode of transformation on the quantity we have called N almost precisely similar, we get

$$\mathrm{N}=\sqrt{\left\{1-\cos^2\delta_1-\cos^2\delta_2-\cos^2\delta_3+2\cos\delta_1\cos\delta_2\cos\delta_3\right\}}\ .\ (73),$$

another symmetrical function of the three distances.

Hence, by substitution in (70) and (69), we find

$$\cos\omega=\frac{\mathrm{P}}{\mathrm{M}},\ \therefore\ \cos r=\frac{\mathrm{N}}{\mathrm{M}},\ \text{and}\ \tan r=\frac{\sqrt{\mathrm{M}^2-\mathrm{N}^2}}{\mathrm{N}};$$

But $\mathrm{M}^2-\mathrm{N}^2=2(1-\cos\delta_1)(1-\cos\delta_2)(1-\cos\delta_3)=16\sin^2\frac{1}{2}\delta_1\sin^2\frac{1}{2}\delta_2\sin^2\frac{1}{2}\delta_3$;

$$\therefore\ \tan r=\frac{4\sin\frac{1}{2}\delta_1\sin\frac{1}{2}\delta_2\sin\frac{1}{2}\delta_3}{\mathrm{N}}\ .\ .\ .\ .\ .\ .\ (74).$$

This expression for the radiating arc of the circle, passing through the angular points of the triangle whose sides are δ_1, δ_2, δ_3, has been long known; it was, I believe, first given by Lagrange.

24. If the circles (65) and (66) which are drawn through the pole of (63), perpendicular to the arcs δ_3 and δ_2, include the angle i_1; then, by (33),

$$\cos i_1=\frac{1+a_1\,a_2+b_1\,b_2}{\sqrt{(1+a_1^2+b_1^2)(1+a_2^2+b_2^2)}}=\frac{1-\cos\delta_3-\cos\delta_2+\cos\delta_1}{4\sin\frac{1}{2}\delta_2\sin\frac{1}{2}\delta_3}\quad(75),$$

and similarly, if the arcs through the pole, perpendicular to δ_3 and δ_1 include the angle i_2, and those perpendicular to δ_2 and δ_1 include the angle i_3, we find in like manner,

$$\cos i_2=\frac{1-\cos\delta_3-\cos\delta_1+\cos\delta_2}{4\sin\frac{1}{2}\delta_1\sin\frac{1}{2}\delta_3}$$

$$\cos i_3=\frac{1-\cos\delta_1-\cos\delta_2+\cos\delta_3}{4\sin\frac{1}{2}\delta_1\sin\frac{1}{2}\delta_2}.$$

I am not aware that these results have been found before. All the values

which come out here as functions of the distances alone, do not depend on the position of the origin of co-ordinates, and we could have found them much more easily by placing two of the given points on the prime meridian, one of them being at the origin; but the beautiful relations which are elicited in the general investigation between the co-ordinates and distances of the given points, fully repay us for the additional trouble.

25. To find the points of intersection of the two circles

$$\cot y - \cos r_1 \sec \omega_1 \operatorname{cosec} y + \tan \omega_1 \cos (\varphi_1 - x) = 0 \quad . \quad . \quad (76),$$

$$\cot y - \cos r_2 \sec \omega_2 \operatorname{cosec} y + \tan \omega_2 \cos (\varphi_2 - x) = 0 \quad . \quad . \quad (77).$$

We find, by eliminating $\cot y$,

$$\operatorname{cosec} y = \frac{\sin \omega_1 \cos \omega_2 \cos (\varphi_1 - x) - \sin \omega_2 \cos \omega_1 \cos (\varphi_2 - x)}{\cos r_1 \cos \omega_2 - \cos r_2 \cos \omega_1} \quad . \ (78),$$

and substituting this in (76),

$$\cot y = \frac{\sin \omega_1 \cos r_2 \cos (\varphi_1 - x) - \sin \omega_2 \cos r_1 \cos (\varphi_2 - x)}{\cos r_1 \cos \omega_2 - \cos r_2 \cos \omega_1} \quad . \ (79).$$

By writing these in the identical equation

$$\operatorname{cosec}^2 y - \cot^2 y = \cos^2 x + \sin^2 x,$$

after expanding the cosines of the double angles, dividing by $\cos^2 x$, and putting for brevity,

$$\left.\begin{aligned} a &= \sin \omega_1 \cos \omega_2 \cos \varphi_1 - \cos \omega_1 \sin \omega_2 \cos \varphi_2, \\ b &= \sin \omega_1 \cos \omega_2 \sin \varphi_1 - \cos \omega_1 \sin \omega_2 \sin \varphi_2, \\ c &= \sin \omega_1 \cos r_2 \cos \varphi_1 - \sin \omega_2 \cos r_1 \cos \varphi_2, \\ d &= \sin \omega_1 \cos r_2 \sin \varphi_1 - \sin \omega_2 \cos r_1 \sin \varphi_2, \\ k &= \cos r_1 \cos \omega_2 - \cos r_2 \cos \omega_1, \end{aligned}\right\} \quad . \quad . \quad (80);$$

we shall obtain the equation

$$\tan^2 x - \frac{2(ab - cd)}{k^2 + d^2 - b^2} \cdot \tan x + \frac{k^2 + c^2 - a^2}{k^2 + d^2 - b^2} = 0 \quad . \quad . \quad (81);$$

$$\therefore \tan x = \frac{ab - cd \pm \sqrt{\{(ad - bc)^2 + k^2(a^2 + b^2) - k^2(c^2 + d^2) - k^4\}}}{k^2 + d^2 - b^2} \quad (82).$$

Now, if we put δ for the distance between the poles $(\omega_1 \varphi_1)$ and $(\omega_2 \varphi_2)$ of the given circles, so that

$$\cos \delta = \cos \omega_1 \cos \omega_2 + \sin \omega_1 \sin \omega_2 \cos (\varphi_1 - \varphi_2) \quad . \quad . \quad (83),$$

we shall find, by substitution,

$$\left.\begin{aligned} a^2 + b^2 &= \cos^2 \omega_1 + \cos^2 \omega_2 - 2 \cos \omega_1 \cos \omega_2 \cos \delta \\ c^2 + d^2 &= \cos^2 r_1 + \cos^2 r_2 - 2 \cos r_1 \cos r_2 \cos \delta - k^2 \\ ad - bc &= k \sin \omega_1 \sin \omega_2 \sin (\varphi_1 - \varphi_2) \end{aligned}\right\} \quad . \ (84);$$

and therefore, substituting again, and making some slight reductions,

$$\begin{aligned} &\sqrt{\{(ad - bc)^2 + k^2(a^2 + b^2) - k^2(c^2 + d^2 + k^2)\}} \\ &\quad = k\sqrt{(1 - \cos^2 r_1 - \cos^2 r_2 - \cos^2 \delta + 2 \cos r_1 \cos r_2 \cos \delta)} \\ &\quad = 2k\sqrt{\sin s \sin (s - r_1) \sin (s - r_2) \sin (s - \delta)} \quad . \quad . \quad . \quad (85), \end{aligned}$$

where $s = \frac{1}{2}(r_1 + r_2 + \delta)$, hence

$$\tan x = \frac{ab - cd \pm 2k\sqrt{\sin s \sin (s-r_1) \sin (s-r_2) \sin (s-\delta)}}{k^2 + d^2 - b^2} \quad . \quad (86).$$

Wherefore, by combining (78) and (79),

$$\cos y = \frac{\sin \omega_1 \cos r_2 \cos (\varphi_1 - x) - \sin \omega_2 \cos r_1 \cos (\varphi_2 - x)}{\sin \omega_1 \cos \omega_2 \cos (\varphi_1 - x) - \sin \omega_2 \cos \omega_1 \cos (\varphi_2 - x)}$$

$$= \frac{c \cos x + d \sin x}{a \cos x + b \sin x} = \frac{c + d \tan x}{a + b \tan x}$$

$$= \frac{ck + b \sin \omega_1 \sin \omega_2 \sin (\varphi_1 - \varphi_2) \pm 2dN}{ak + d \sin \omega_1 \sin \omega_2 \sin (\varphi_1 - \varphi_2) \pm 2bN} \quad . \quad . \quad . \quad (87);$$

Where $N = \sqrt{\sin s \sin (s - r_1) \sin (s - r_2) \sin (s - \delta)}$ . . . (88).

26. If we observe that the two radii r_1, r_2 to the point of intersection, and the distance δ form the sides of a spherical triangle, we shall be at no loss in interpreting the results of the last article. In fact, the two circles will intersect when $N > 0$, that is, when $s > r_1 > r_2$ and $> \delta$ at the same time, or when $r_1 < r_2 + \delta$, $r_2 < r_1 + \delta$, and $\delta < r_1 + r_2$, or when the triangle in question can be constructed; that there is but one point of contact, or the circles are tangent to each other, when $N = 0$, or when $s = r_1 = r_2$ or $= \delta$; that is,

$$\text{when } \delta = r_1 + r_2, \text{ or } \delta = r_1 \sim r_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (89),$$

the contact being external in the first case, and internal in the second: lastly, that the circles do not intersect when $N < 0$, that is, when $s < r_1 < r_2$ or $< \delta$, or when the triangle in question cannot be constructed. To state these results in other words :—

The circles will be without each other when $\delta > r_1 + r_2$;
" touch externally when $\delta = r_1 + r_2$;
" intersect when $\delta < r_1 + r_2 > r_1 \sim r_2$;
" touch internally when $\delta = r_1 \sim r_2$;
" one within the other when $\delta < r_1 \sim r_2$.

27. When $k = 0$, that is, when (80)

$$\frac{\cos r_1}{\cos \omega_1} = \frac{\cos r_2}{\cos \omega_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90),$$

there is but one value of x in (86), but as y has still two values in (87), the circles have contact in two places; it is easily seen that this merely indicates the case where the two points of contact and the origin are in the same great circle. In all cases (Art. 20.) the circle through the two poles $(\omega_1\ \varphi_1)$, $(\omega_2\ \varphi_2)$ is perpendicular to the great circle through the two points of contact, and bisects the arc of it between the two points. Hence also when the circles are tangent to each other, the two poles and the point of contact are in the same great circle.

28. Let the circle

$$\cot y - \cos r \sec \omega \operatorname{cosec} y + \tan \omega \cos (\varphi - x) = 0 \quad . \quad . \quad (91),$$

touch the great circle

$$\cot y+\tan \omega_1 \cos (\varphi_1-x)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (92),$$

then, by (89), $\delta=r+\frac{\pi}{2}$, or $\delta=r\sim\frac{\pi}{2}$, and by (83),

$$\mp \sin r=\cos \omega \cos \omega_1+\sin \omega \sin \omega_1 \cos (\varphi-\varphi_1) \quad . \quad . \quad (93);$$

the ambiguous sign merely denotes the two cases where the circle (91) is on the opposite or on the same side of the great circle as the pole to which we refer it; keeping this in mind, (93) may always be written

$$\sin r=\cos \omega \cos \omega_1+\sin \omega \sin \omega_1 \cos (\varphi-\varphi_1) \quad . \quad . \quad . \quad (94).$$

29. If the circle (91) touch a second great circle

$$\cot y+\tan \omega_2 \cos (\varphi_2-x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95),$$

then also we shall have

$$\sin r=\cos \omega \cos \omega_2+\sin \omega \sin \omega_2 \cos (\varphi-\varphi_2) \quad . \quad . \quad . \quad (96),$$

and eliminating r between (94) and (96), we have the equation

$$\cot \omega + \frac{\sin \omega_1 \cos \varphi_1-\sin \omega_2 \cos \varphi_2}{\cos \omega_1-\cos \omega_2} \cos \varphi$$
$$+ \frac{\sin \omega_1 \sin \varphi_1-\sin \omega_2 \sin \varphi_2}{\cos \omega_1-\cos \omega_2} \sin \varphi=0 \quad . \quad . \quad . \quad (97);$$

hence, the poles of all circles that touch the two great circles (92) and (95), lie in the great circle (97). We could show in the same manner as in Art. 20, that this locus passes through the pole of the circle in which $(\omega_1\ \varphi_1)$, $(\omega_2\ \varphi_2)$ are situated; that is, through the intersection of the two circles (92) and (95); and that it is perpendicular to, and bisects the arc contained between $(\omega_1\ \varphi_1)$ and $(\omega_2\ \varphi_2)$, or, since by what was said in the last article, these two poles are in the opposite hemispheres made by the circles to which they belong, it must bisect the angle made by the two circles (92) and (95).

30. If the circle (91) touch a third great circle,

$$\cot y+\tan \omega_3 \cos (\varphi_3-x)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98),$$

then also,

$$\sin r=\cos \omega \cos \omega_3+\sin \omega \sin \omega_3 \cos (\varphi-\varphi_3) \quad . \quad . \quad (99);$$

and eliminating r between (94) and (99), the poles of all circles touching the great circles (92) and (98), lie in the great circle

$$\cot \omega + \frac{\sin \omega_1 \cos \varphi_1-\sin \omega_3 \cos \varphi_3}{\cos \omega_1-\cos \omega_3} \cos \varphi$$
$$+ \frac{\sin \omega_1 \sin \varphi_1-\sin \omega_3 \sin \varphi_3}{\cos \omega_1-\cos \omega_3} \sin \varphi=0 \quad . \quad . \quad (100),$$

which passes through the intersection of (92) and (98), and bisects the angles formed at their intersection. Similarly eliminating r between (96) and (99), the poles of all circles touching the great circles (95) and (98) lie in the great circle

$$\cot\omega+\frac{\sin\omega_2\cos\varphi_2-\sin\omega_3\cos\varphi_3}{\cos\omega_2-\cos\omega_3}\cos\varphi$$
$$+\frac{\sin\omega_2\sin\varphi_2-\sin\omega_3\sin\varphi_3}{\cos\omega_2-\cos\omega_3}\sin\varphi=0 \quad . \quad . \quad (101),$$

which bisects the angle made by those two circles.

31. Now, the circle that touches all the three circles (92), (95) and (98), must have its pole in all the three circles (97), (100) and (101); hence the three great circles bisecting the angles formed by the intersection of three great circles meet in the same point; and to find this point, which is the pole of the circle touching the three given great circles, we have only to find the intersection of the circles (97) and (100). Now, if we compare these two equations with the two equations (65) and (66), we shall find that they involve the quantities $\omega_1\ \varphi_1$, $\omega_2\ \varphi_2$, $\omega_3\ \varphi_3$, in precisely the same manner as (65) and (66) involve $y_1\ x_1$, $y_2\ x_2$, $y_3\ x_3$; therefore the coordinates of the intersection ($\omega\ \varphi$) of these circles will be the same function of the quantities $\omega_1\ \varphi_1$, $\omega_2\ \varphi_2$, $\omega_3\ \varphi_3$, that the intersection of (65) and (66) are of $y_1\ x_1$, $y_2\ x_2$, $y_3\ x_3$, and if we write the former for the latter in the values of a_1, b_1, a_2, b_2, of Art. 22, the equations (68) will be the values of $\omega\ \varphi$. Moreover, since $\sin r$ in (94) is the same function of $\omega\ \varphi$, $\omega_1\ \varphi_1$ that $\cos r$ in (49) is of $\omega\ \varphi$, $y_1\ x_1$; it follows that $\frac{\pi}{2}-r$ is the same function of the sides of the triangle whose angular points are $\omega_1\ \varphi_1$, $\omega_2\ \varphi_2$, $\omega_3\ \varphi_3$, that r is of the sides of the triangle whose angular points are $y_1\ x_1$, $y_2\ x_2$, $y_3\ x_3$; that is, if δ_3 be the arc joining the points $\omega_1\ \varphi_1$, $\omega_2\ \varphi_2$; δ_2 the arc joining $\omega_1\ \varphi_1$, $\omega_3\ \varphi_3$; and δ_1 the arc joining $\omega_2\ \varphi_2$, $\omega_3\ \varphi_3$; then by (74),

$$\cot r=\frac{4\sin\frac{1}{2}\delta_1\sin\frac{1}{2}\delta_2\sin\frac{1}{2}\delta_3}{\mathrm{N}} \quad . \quad . \quad . \quad . \quad . \quad (102),$$

the value of N being as in (73). But the angular points of this triangle being the poles of the circles (92), (95), (98), it is supplemental to the triangle formed by the intersection of these circles; hence, if the circles whose poles are $\omega_1\ \varphi_1$, $\omega_2\ \varphi_2$, make an angle i_3; the circles whose poles are $\omega_1\ \varphi_1$, $\omega_3\ \varphi_3$, make an angle i_2; and the circle whose poles are $\omega_2\ \varphi_2$, $\omega_3\ \varphi_3$, make an angle i_1; we shall have $\delta_3=\pi-i_3$, $\delta_2=\pi-i_2$, $\delta_1=\pi-i_1$, and (102) becomes

$$\cot r=\frac{4\cos\frac{1}{2}i_1\cos\frac{1}{2}i_2\cos\frac{1}{2}i_3}{\mathrm{N}} \quad . \quad . \quad . \quad . \quad . \quad (103),$$

where $\mathrm{N}=\sqrt{\left\{1-\cos^2 i_1-\cos^2 i_2-\cos^2 i_3-2\cos i_1\cos i_2\cos i_3\right\}}$. (104).

32. Cor. The pole of the circle that touches the three sides of a triangle is also the pole of the circle passing through the angular points of its supplemental triangle. I do not think that this has been remarked before. The manner in which the other circles that touch the three given great circles are deduced by the help of the ambiguous signs in equation (93), and its analogous ones, will be sufficiently obvious. I have already so far encroached on the limits of the Miscellany that it will be necessary, for the present, to omit much that is interesting on this part of the subject, in order to make room for other discussions, that will better enable the reader

to judge of the power and extent of the analysis, and the inexhaustible mine of research it opens to his investigation.

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§. III.

On the area and length of lines traced on the surface of the sphere; on the tangent circles of these lines, &c.

33. It is evident that the nature of any curve on the surface of the sphere could be expressed by an equation of the form $y = f(x)$ between the polar spherical co-ordinates of any point in it, in precisely the same manner it is done on a plane, or as we have done for circles on the sphere. Let that part of the surface of the sphere which is included between two spherical radius vectors and the curve be represented by Σ. Then $d\Sigma$ will be that part of the elementary lune whose angle at the origin is dx, which is bounded by the curve. Now, the lune may be conceived as made up of the elements of parallel circles having the origin for their pole, and if y be the variable distance of the parallel circle, and therefore $\sin y$ its radius, the element of it intercepted by the sides of the lune will be $\sin y\, dx$, and therefore the element of the area of the lune will be $\sin y\, dx\, dy$, or

$$d^2\Sigma = \sin y\, dx\, dy \quad \ldots \ldots \quad (105).$$

Integrating between $y = 0$, and $y = y$, we have

$$d\Sigma = (1 - \cos y)\, dx \quad \ldots \ldots \quad (106).$$

It will of course be understood here that the unit is the surface of the tri-rectangular triangle, or one-eighth of the surface of the sphere.

34. To find the arc of a great circle equal in length to the arc of any curve on the sphere.

Let yx and $y_1\, x_1$ be any two points on the curve, and s the length of the great circle arc between them; by (25),

$$\cos s = \cos y \cos y_1 + \sin y \sin y_1 \cos (x - x_1)$$
$$= \cos (y - y_1) - \sin y \sin y_1 (1 - \cos (x - x_1)),$$
$$\therefore 1 - \cos s = 1 - \cos (y - y_1) + \sin y \sin y_1 (1 - \cos (x - x_1)),$$
$$\text{and } \sin^2 \tfrac{1}{2} s = \sin^2 \tfrac{1}{2}(y - y_1) + \sin y \sin y_1 \sin^2 \tfrac{1}{2} (x - x_1);$$
$$\text{hence } \frac{\sin^2 \frac{1}{2} s}{\sin^2 \frac{1}{2}(x - x_1)} = \frac{\sin^2 \frac{1}{2}(y - y_1)}{\sin^2 \frac{1}{2}(x - x_1)} + \sin y \sin y_1 \quad \ldots \ldots \quad (107).$$

Now, when the two points are indefinitely near each other, the arc of the curve between them coincides with the arc of a great circle, and if we write the indefinitely small arcs for their sines, we shall have for that position,

$$\frac{\sin \frac{1}{2} s}{\sin \frac{1}{2}(x - x_1)} = \frac{ds}{dx},$$
$$\frac{\sin \frac{1}{2} (y - y_1)}{\sin \frac{1}{2} (x - x_1)} = \frac{dy}{dx},$$

and (107) becomes

$$\frac{ds^2}{dx^2} = \frac{dy^2}{dx^2} + \sin^2 y,$$

$$\text{or, } ds^2 = dy^2 + \sin^2 y \, dx^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (108).$$

35. To find a great circle tangent to a line on the sphere, at a given point $(y_1 \, x_1)$ of that line.

The principles of contact established by Lagrange in his "Théorie des Fonctions Analytiques," part II. Chap. I., are altogether independent of the nature of the co-ordinates y and x, and his reasoning would apply as well to polar co-ordinates as to rectangular ones, and as well to two lines drawn on a curve surface, provided they are sufficient to determine the position of the point, as to the analogous ones on a plane. These principles, which are summed up in Art. 10, may be thus translated into the notation of the *Differential Calculus*.

36. Let $y \, x$ be the co-ordinates of any point in one curve, and $y_1 \, x_1$ the co-ordinates of any point in another curve. The two curves will have a common point if $y = y_1$ and $x = x_1$ at the same time; if, besides this, they have at the same place $\frac{dy}{dx} = \frac{dy_1}{dx_1}$, they will have *contact of the first order;* if, moreover, they have at the same place, $\frac{d^2y}{dx^2} = \frac{d^2y_1}{dx_1^{\,2}}$, they will have *contact of the second order*, and so on. In order to have contact of the first order, the equations of the two curves must have at least two constants involved in them; to have contact of the second order there must be at least three constants in each equation, &c., and contact of the mth order requires $m + 1$ constants.

37. Now, to apply these principles to the contact of lines on the sphere. If the great circle whose equation is

$$\cot y + \tan \omega \cos (\varphi - x) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (109),$$

touch a given curve at the point $y_1 \, x_1$, it is necessary that

$$\cot y_1 + \tan \omega \cos (\varphi - x_1) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (110),$$

and also that $\frac{dy}{dx} = \frac{dy_1}{dx_1}$, when y_1 and x_1 are written for y and x; therefore differentiating (110) for y_1 and x_1,

$$\frac{dy_1}{dx_1} = \sin^2 y_1 \tan \omega \sin (\varphi - x_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad (111).$$

Eliminating ω between (110) and (111), we find

$$\tan (\varphi - x_1) = -\frac{dy_1}{dx_1} \cdot \frac{1}{\sin y_1 \cos y_1} \quad . \quad . \quad . \quad . \quad . \quad (112),$$

and, from (110),

$$\tan \omega = -\cot y_1 \sec (\varphi - x_1),$$

$$\text{or } \cos \omega = \frac{1}{\sqrt{(1 + \cot^2 y_1 \sec^2 (\varphi - x_1))}}$$

$$= \frac{\sin y_1 \cdot}{\sqrt{(1 + \cos^2 y_1 \tan^2(\varphi - x_1))}}$$

$$= \frac{\sin^2 y_1}{\sqrt{(\sin^2 y_1 + \frac{dy_1^2}{dx_1^2})}}$$

$$= \frac{dx_1}{ds} . \sin^2 y_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (113),$$

using the value of ds in equation (108).

By eliminating ω between (109) and (110), we get

$$\cot y - \cot y_1 \frac{\cos(\varphi - x)}{\cos(\varphi - x_1)} = 0;$$

But $\cos(\varphi - x) = \cos(\overline{\varphi - x_1} + \overline{x_1 - x}) = \cos(\varphi - x_1)\cos(x_1 - x) - \sin(\varphi - x_1)\sin(x_1 - x)$, and substituting this it becomes

$$\cot y - \cot y_1 \cos(x_1 - x) + \cot y_1 \tan(\varphi - x_1)\sin(x_1 - x) = 0,$$

or, by (112), the equation of the tangent circle will be,

$$\cot y - \cot y_1 \cos(x_1 - x) - \frac{dy_1}{dx_1} . \frac{\sin(x_1 - x)}{\sin^2 y_1} = 0 \quad . \quad . \quad (114).$$

The position of the poles of this circle is determined from equations (112) and (113).

38. The tangent circle will intersect the axis when $x = 0$, and therefore, for this point,

$$\cot y = \cot y_1 \cos x_1 + \frac{dy_1}{dx_1} . \frac{\sin x_1}{\sin^2 y_1} \quad . \quad . \quad . \quad . \quad (115),$$

and the arc corresponding to the polar subtangent, will be the value of y when $x = \frac{\pi}{2} + x'$, or when y is perpendicular to y_1, and if σ represent this arc,

$$\cot \sigma = -\frac{dy_1}{dx_1} . \frac{1}{\sin^2 y_1} = \frac{d \cot y_1}{dx_1} \quad . \quad . \quad . \quad . \quad (116).$$

Moreover, if i represent the angle between the radius vector y_1 and the tangent circle (114), we shall have

$$\cot i = \sin y_1 \cot \sigma = -\frac{dy_1}{dx_1} \operatorname{cosec} y_1 \quad . \quad . \quad . \quad . \quad (117).$$

39. A normal circle to the curve through the point $y_1\, x_1$ is necessarily perpendicular to the tangent circle (114), and therefore its equation, by Art. 10, will be

$$\cot y - \cot y_1 \cdot \frac{\sin(\varphi - x)}{\sin(\varphi - x_1)} + \cot \omega . \frac{\sin(x_1 - x)}{\sin(\varphi - x_1)} = 0 \quad . \quad . \quad (118).$$

But $\sin(\varphi - x) = \sin\{(\varphi - x_1) + (x_1 - x)\}$

$$= \sin(\varphi - x_1)\cos(x_1 - x) + \cos(\varphi - x_1)\sin(x_1 - x),$$

and from (110),

$$\cot \omega = -\tan y_1 \cos (\varphi - x_1);$$

these substituted in (118), it becomes

$$\cot y - \cot y_1 \cos (x_1 - x) - \cot (\varphi - x_1) \cdot \frac{\sin (x_1 - x)}{\sin y_1 \cos y_1} = 0,$$

and substituting again the value of $\cot (\varphi - x_1)$ from (112), we get the equation of the normal circle

$$\cot y - \cot y_1 \cos (x_1 - x) + \frac{dx_1}{dy_1} \sin (x_1 - x) = 0 \quad . \quad . \quad (119).$$

This circle will intersect the axis when $x = 0$, and when

$$\cot y = \cot y_1 \cos x_1 - \frac{dx_1}{dy_1} \sin x_1 \quad . \quad . \quad . \quad . \quad . \quad (120).$$

And if ν represent the polar subnormal, or the value of y when $x = \frac{\pi}{2} + x'$, we shall have

$$\cot \nu = \frac{dx_1}{dy_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (121).$$

40. It may not be amiss, in order to confirm what has been done on this subject, to find the equation of the normal circle in a different manner.

By Art. 20, it appears that the poles of all circles passing through the two given points $y_1\, x_1$ and $y_2\, x_2$, lie in the great circle whose equation is

$$\cot y + \frac{\sin y_1 \cos x_1 - \sin y_2 \cos x_2}{\cos y_1 - \cos y_2} \cos x$$
$$+ \frac{\sin y_1 \sin x_1 - \sin y_2 \sin x_2}{\cos y_1 - \cos y_2} \sin x = 0,$$

or, as it may be otherwise written,

$$\cot y + \frac{\sin y_1 \cos (x_1 - x) - \sin y_2 \cos (x_2 - x)}{\cos y_1 - \cos y_2} = 0 \quad . \quad . \quad (122).$$

Now, $\sin y_1 = \sin \left\{ \tfrac{1}{2}(y_1 + y_2) + \tfrac{1}{2}(y_1 - y_2) \right\}$
$= \sin \tfrac{1}{2}(y_1 + y_2) \cos \tfrac{1}{2}(y_1 - y_2) + \cos \tfrac{1}{2}(y_1 + y_2) \sin \tfrac{1}{2}(y_1 - y_2)$,
$\sin y_2 = \sin \left\{ \tfrac{1}{2}(y_1 + y_2) - \tfrac{1}{2}(y_1 - y_2) \right\}$
$= \sin \tfrac{1}{2}(y_1 + y_2) \cos \tfrac{1}{2}(y_1 - y_2) - \cos \tfrac{1}{2}(y_1 + y_2) \sin \tfrac{1}{2}(y_1 - y_2)$,
and $\cos y_1 - \cos y_2 = -2 \sin \tfrac{1}{2}(y_1 + y_2) \sin \tfrac{1}{2}(y_1 - y_2)$;

hence equation (122) becomes

$$\cot y - \tfrac{1}{2} \cot \tfrac{1}{2}(y_1 + y_2) \left\{ \cos (x_1 - x) + \cos (x_2 - x) \right\}$$
$$- \tfrac{1}{2} \cot \tfrac{1}{2}(y_1 - y_2) \left\{ \cos (x_1 - x) - \cos (x_2 - x) \right\} = 0 \quad . \quad (123).$$

But $\cos (x_1 - x) + \cos (x_2 - x) = 2 \cos \tfrac{1}{2}(x_1 - x_2) \cos \left\{ \tfrac{1}{2}(x_1 + x_2) - x \right\}$,

and $\cos (x_1 - x) - \cos (x_2 - x) = -2 \sin \tfrac{1}{2}(x_1 - x_2) \sin \left\{ \tfrac{1}{2}(x_1 + x_2) - x \right\}$;

and equation (123) will therefore be,

$$\cot y - \cot \tfrac{1}{2}(y_1+y_2) \cos \tfrac{1}{2}(x_1-x_2) \cos \left\{ \tfrac{1}{2}(x_1+x_2) - x \right\}$$
$$+ \cot \tfrac{1}{2}(y_1-y_2) \sin \tfrac{1}{2}(x_1-x_2) \sin \left\{ \tfrac{1}{2}(x_1+x_2) - x \right\} = 0 \quad . \quad (124).$$

41. From what was done in Art. 20, it appears that this circle bisects the arc joining the two points, and is perpendicular to that arc. Hence, when these points become consecutive points in a curve, it will be a normal to that curve, and for this ultimate state we shall have $\frac{1}{2}(y_1+y_2)=y_1$, $\frac{1}{2}(x_1+x_2)=x_1$, $y_1-y_2=dy_1$, and $x_1-x_2=dx_1$; substituting these in (124), writing the small arcs for their sines and unity for their cosines, it becomes

$$\cot y - \cot y_1 \cos (x_1-x) + \frac{dx_1}{dy_1} \sin (x_1-x) = 0,$$

which is the same as before.

The pole ($\omega\ \varphi$) of this circle is necessarily in the tangent circle (114), therefore,

$$\cot \omega - \cot y_1 \cos (x_1-\varphi) - \frac{dy_1}{dx_1} \cdot \frac{\sin (x_1-\varphi)}{\sin^2 y_1} = 0,$$

and its distance from the point $y_1\ x_1$ is $\frac{1}{2}\pi$; therefore,

$$\cot \omega \cot y_1 + \cos (x_1-\varphi) = 0.$$

From these two equations we shall find

$$\left.\begin{aligned} \tan (x_1-\varphi) &= -\frac{dx_1}{dy} \tan y_1 \\ \cos \omega &= \frac{dy_1}{ds} \cdot \sin y_1 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (125),$$

ds being the element of the curve.

42. To find a less circle which has the same curvature as a given curve at the point $y_1\ x_1$.

Let the equation of the osculating circle be

$$\cos r = \cos \omega \cos y + \sin \omega \sin y \cos (\varphi - x) \quad . \quad . \quad (126);$$

then, first, at the point $y_1\ x_1$,

$$\cos r = \cos \omega \cos y_1 + \sin \omega \sin y_1 \cos (\varphi - x_1) \quad . \quad . \quad (127);$$

second, $\frac{dy}{dx} = \frac{dy_1}{dx_1}$, at the same point, therefore differentiating (127),

$$\left\{ -\cos \omega \sin y_1 + \sin \omega \cos y_1 \cos (\varphi - x_1) \right\} \frac{dy_1}{dx_1}$$
$$+ \sin \omega \sin y_1 \sin (\varphi - x_1) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (128),$$

this equation may be written

$$\cot \omega - \cot y_1 \cos (\varphi - x_1) - \frac{dx_1}{dy_1} \sin (\varphi - x_1) = 0 \quad . \quad . \quad (129),$$

and therefore the pole of the osculating circle is in the normal circle (119); third, $\frac{d^2y}{dx^2} = \frac{d^2y_1}{dx_1^2}$, at the same point; therefore, differentiating (129), we

shall find

$$\cot(\varphi - x_1)\left(\frac{dy_1}{dx_1}\operatorname{cosec}^2 y_1 + \frac{dx_1}{dy_1}\right) = \cot y_1 - \frac{d^2y_1}{dx_1^2}\cdot\frac{dx_1^2}{dy_1^2} \quad . \quad (130).$$

Equations (129) and (130), together with the given equation of the curve, enable us to determine the locus of the pole of the osculating circle, or the *evolute* of the curve on the sphere.

By eliminating ω and φ from the equations (127), (129) and (130), we shall find, after some reduction, for the polar distance r of the osculating circle,

$$\cot r = \left\{\cos y_1 \sin^2 y_1 + 2\,.\,\frac{dy_1^2}{dx_1^2}\cos y_1 - \frac{d^2y_1}{dx_1^2}\sin y_1\right\}.\,\frac{dx_1^3}{ds^3} \quad . \quad (131),$$

where ds is the element of the curve.

43. If we introduce into this expression, the radius R of the sphere, it may be written

$$\tan r = \frac{ds^3}{dx_1^3}\,.\,\frac{\sec y_1}{R} \div \left\{\sin^2 y_1 + 2\frac{dy_1^2}{dx_1^2} - \frac{d^2y_1}{dx_1^2}\tan y_1\right\};$$

if, now, the radius of the sphere become infinite, its surface will be a plane, the arcs will become straight lines indefinitely small with respect to the radius, so that $\tan r = r$, $\sec y_1 = R$, $\sin y_1 = \tan y_1 = y_1$, and the equation will become

$$r = \frac{ds^3}{dx_1^3} \div \left\{y_1^2 + 2\frac{dy_1^2}{dx_1^2} - y_1\frac{d^2y_1}{dx_1^2}\right\},$$

which is a well-known expression for the radius of the osculating circle of a plane curve.

It will be sufficiently apparent that a change of inflexion will in general take place in the curve, when r, the radius of the osculating circle, passes through the magnitude $\frac{1}{2}\pi$; that is, when $\cos y_1 \sin^2 y_1 + 2\frac{dy_1^2}{dx_1^2}\cos y_1 - \frac{d^2y_1}{dx_1^2}\sin y_1 = 0$.

(*To be continued.*)

*** The Editor begs to thank Professor Davies, of West Point, for copies of his new Editions of Legendre's Geometry, and Bourdon's Algebra; he trusts they will become as popular as their merits deserve. To Professor Pierce, of Harvard University, the Editor is obliged for a copy of his Elements of Plane Trigonometry;—as a Text-Book for such a course of instruction as is usually taught in our Colleges, it seems to be superior to any before published on that subject; and if the projected course of elementary treatises be carried out in the same spirit and style, there is no doubt they will be highly useful to both teachers and pupils.

ARTICLE V.

NEW QUESTIONS TO BE ANSWERED IN NUMBER II.

QUESTION I. BY P.

How many diagonals can there be drawn in a polygon of n sides?

QUESTION II. BY A.

Let x_1, x_2, x_3 be any three angles, prove that

$$1^\circ.\quad \sin^2(x_1-x_2)+\sin^2(x_1-x_3)+\sin^2(x_2-x_3)$$
$$+2\cos(x_1-x_2)\cos(x_1-x_3)\cos(x_2-x_3)=2.$$

$$2^\circ.\quad 1-\cos^2x_1\cos^2x_2-\cos^2x_1\cos^2x_3-\cos^2x_2\cos^2x_3$$
$$+2\cos^2x_1\cos^2x_2\cos^2x_3$$
$$-\sin^2x_1\sin^2x_2-\sin^2x_1\sin^2x_3-\sin^2x_2\sin^2x_3$$
$$+2\sin^2x_1\sin^2x_2\sin^2x_3=0.$$

QUESTION III. BY A. B.

Given the three equations

$$y+z=2\sqrt{(xy+xz-x^2)}$$
$$2y-z=2\sqrt{(2xy-xz-x^2)}$$
$$3y-x=5z-\sqrt{z},$$

to find x, y and z.

QUESTION IV. BY THE EDITOR.

Prove that, in the two inequalities

$$x^2+ax+b<0,$$
$$x^2+ax+b>0;$$

the first has place when x is within the limits of the roots of the equation

$$x^2+ax+b=0,$$

and the second when x is without the limits of those roots.

QUESTION V. BY MR. JAMES F. MACULLY, RICHMOND, VA.

To find such a positive value of x as will make

$$1-8x=\text{a square number},$$
$$x-4x^2+4=\text{a square number};$$

or prove that it is impossible.

QUESTION VI. BY THE EDITOR.

If a be a prime integer, find how many numbers less than a^n, are divisible by a, how many by a^2, how many by a^3, &c.

QUESTION VII. BY MR. GERARDUS B. DOCHARTY, INSTITUTE, FLUSHING.

Two circles touch each other internally; to find the sum of the areas of all the circles that shall touch each other, and also the two given circles.

QUESTION VIII. BY Q. Q.

Suppose the axis of the planet Venus to be inclined to the plane of her orbit at an angle of 15°, the time of rotation round that axis 23h. 21m., and her periodic time round the sun 224,7008 days; admitting also the planet to be spherical, her orbit a perfect circle, and her axis remaining parallel to itself: it is required to show the change of the seasons, the lengths of the day and night at the different parts of her surface in any time of her year, the diurnal change of the sun's declination, &c., and to compare these results with those exhibited in Wallace's "New Treatise on the Use of the Globes," pp. 276, 277.

QUESTION IX. BY INVESTIGATOR.

Let there be three straight lines on a plane, whose equations are

$$y = a_1 x + b_1,$$
$$y = a_2 x + b_2,$$
$$y = a_3 x + b_3;$$

prove that the three lines will have a common point of intersection if the three points whose co-ordinates are

$$b_1 \text{ and } ca_1,\ b_2 \text{ and } ca_2,\ b_3 \text{ and } ca_3,$$

be situated in the same straight line, c being a given line.

QUESTION X. BY Δ.

A point being given on the sphere, how must a second point be situated, at a given distance from it, so that the surface, included between the arcs of a great circle and a loxodromic curve, both passing through the two points, may be the greatest possible?

QUESTION XI. BY MR. DAVID LANGDON, SCHENECTADY, N. Y.

A given sphere has a given rectangular perforation through its centre, what is the solidity of the remaining part?

QUESTION XII. BY P.

Find a curve and its involute, such that the intercept of their common axis, between the ordinates to any two corresponding points of the curves, may be a constant line. Find, also, the areas and lengths of the two curves.

QUESTION XIII. BY P.

If from a given point in the axis of a parabola, perpendiculars be let fall upon the tangents of the curve, these right angles will be in a curve whose equation is $y^2(x-a) = x^2(b-x)$. Now, if from the same point, perpendiculars be let fall upon the tangents of this second curve, it is required to find the locus of the right angles so formed;—determine its inflexions and length, and, in the case where $b = 0$, determine its area.

QUESTION XIV. BY S. S.

Find the sum of n terms of the series

$$\cos\varphi\cos 2\varphi + 2\cos 2\varphi\cos 4\varphi + 3\cos 3\varphi\cos 6\varphi + \&c.$$

QUESTION XV. BY MR. JAMES F. MACULLY.

Three circles tangent to each other, are given on a plane; to find the greatest or least ellipse that touches all the three circles.

QUESTION XVI. BY Δ.

If the quadrantal arc of a great circle revolve so that its extremities are always in two given great circles, to find the equation of the curve traced on the surface of the sphere by the centre of rotation of the revolving arc.

QUESTION XVII. BY INVESTIGATOR.

A plane intersects the axes of co-ordinates at the distances x', y', z', from the origin, so that $x'y' + x'z' + y'z' =$ a given rectangle. To find the surface to which this plane is always a tangent.

QUESTION XVIII. BY RICHARD TINTO, ESQ., GREENVILLE, OHIO.

A given sphere is viewed in perspective from a given point; it is required to find the nature and position of a surface such, that in whatever position the picture be placed upon it, the image of the sphere may have the same given magnitude.

QUESTION XIX. BY PETRARCH, NEW-YORK.

A given semi-prolate-spheroid is placed with its base on the horizontal

plane, and its axis vertical; with what velocity must a body be projected vertically along its interior surface, so that it may pass through the focus?

QUESTION XX. BY INVESTIGATOR.

A perfectly smooth plane is made to oscillate according to a given law, round one of its own lines placed in a given position as an axis; to find the circumstances of the motion of a body on this plane when acted upon by gravity.

⁂ All communications for the second number of the Miscellany must be post paid; and, in order to ensure the publication of the number at the specified time, they must arrive before the first of August, 1836.

An accident in the printing office has delayed the publication of this number much beyond the intended time; it is hoped that such a detention will not again occur.

TO SUBSCRIBERS.

On the fifteenth of June the sum arising from the subscriptions for the Mathematical Miscellany did not amount to one-fourth of its probable expenditure. The design of continuing it would therefore have been abandoned, had not several gentlemen from different parts of the United States, and independently of each other, proposed that the subscription price of the work should be raised to $1 per number, or $2 per annum; and that any deficiency which might still exist should be sustained by those interested in the continuance of the publication, in the proportion of certain annual sums to be named by themselves individually.

The following is a list of the gentlemen who pledge themselves to support the work to the extent of the sums placed opposite to their names:—

Rev. W. A. Muhlenberg, D. D. Principal of the Institute, Flushing,		-	$50	per annum.
C. Gill, Professor of Mathematics,	-	do.	20	"
L. Van Bokkelen, Mathematical Instructer,		do.	10	"
G. B. Docharty, - do.	-	do.	10	"
Professor C. Avery, Hamilton College, Clinton, N. Y.,			15	"
Professor M. Catlin, do.	do.		10	"
O. Root, Mathematical Tutor, do.	do.		10	"
G. R. Perkins, Liberal Institute,	do.		5	"
William Lenhart, Esq., York, Penn.	-	-	10	"
J. F. Macully, Esq., New-York,	-	-	5	"
Evans Hollis, Esq., Rye, Westchester Co., N. Y.		-	5	"

The advanced price of subscription is certainly not more than in justice it ought to be; since the cost of printing such a work is more than double that of ordinary matter. If this advance is not objected to by the subscribers, the Miscellany may be considered as firmly established.

I have found it impossible to procure agents for the work in the different states of the Union, the circulation being necessarily so small as to hold out no inducement to men of business. It will be therefore necessary for subscriptions to be sent by mail, and where two or more subscribers reside in a place, by enclosing the joint amount, they will not find this method inconvenient. In the case of a single subscriber, he is requested to transmit $5 on the receipt of the 2d, 7th, 12th, &c., numbers.

In the peculiar circumstances under which this work is published, the necessity of having its accounts settled at stated periods will be immediately seen. In all cases, therefore, where subscriptions are not remitted by the first of January of every year, the subscription will be considered as having ceased, and the work will be no longer forwarded.

C. GILL.

Flushing, L. I. October, 1836.

METEOROLOGICAL OBSERVATIONS,

MADE AT THE INSTITUTE, FLUSHING, L. I., FOR THIRTY-SEVEN SUCCESSIVE HOURS, COMMENCING AT SIX A. M. OF THE TWENTY-FIRST OF JUNE, EIGHTEEN HUNDRED AND THIRTY-SIX, AND ENDING AT SIX P. M. OF THE FOLLOWING DAY.

(*Lat.* 40° 44′ 58″ N. *Long.* 73° 44′ 20″ W. *Height of Barometer above low water mark of Flushing Bay*, 54 *feet.*)

Hour.	Barometer.	Attached Therm'ter.	Therm'ter.	Wet Bulb Therm'ter.	Winds —from—	Clouds —to—	Strength of wind.	REMARKS. Storm began at 2 P. M. on the 20th, with slow drizzling rain—during the evening it rained fast, with thunder and lightning.
6	29.81	61	53½	51¾	NE.	S W	Brisk.	Dark clouds and rain.
7	29.82	60	53¾	52	"	"	"	"
8	29.80	58	54¼	52¼	"	"	Higher.	"
9	29.78	57	54½	52½	"	"	High.	"
10	29.79	58	55¼	53	"	"	"	Storm ended 10¾ A. M.—there had fallen
11	29.81	59	56¼	54	"	"	Less.	from 7 P. M. of the previous evening,
12	29.81	59½	57½	55	"	"	Brisk.	1.85 inches of rain.
1	29.82	59	59	56	"	"	Fresh.	Gray clouds overspread.
2	29.82	58	59¼	56½	"	"	Gentle.	"
3	29.83	60	60	57	"	"	Fresh.	"
4	29.83	60	60	57	"	"	"	"
5	29.83	60	58	55	"	"	Gentle.	"
6	29.84	60	57¼	54½	"	"	"	"
7	29.86	60	56¾	54	"	"	"	"
8	29.87	59½	57¼	54½	"	"	"	Some drizzly rain between 7½ and 8.
9	29.88	59½	56	53	"	"	"	Clouds partially breaking.
10	29.87	59½	54	51½	"	"	Fresh.	"
11	29.89	59	53¼	50½	"	"	"	Dark clouds dropping rain.
12	29.87	59	52½	50	"	"	"	"
1	29.88	59	52½	49¾	"	"	"	Dark clouds.
2	29.87	59	52½	49½	"	"	Brisk.	"
3	29.88	59	52	49	"	"	"	"
4	29.89	59	51½	49	"	"	"	"
5	29.92	60	51½	49	"	"	Gentle.	"
6	29.93	60	52½	49	"	"	Brisk.	Dropping rain at intervals.
7	29.95	60	52¾	49	"	"	"	"
8	29.94	59	58½	54½	"	"	Gentle.	Clouds breaking.
9	29.94	59	59¾	56¼	"	"	Fresh.	" overspread.
10	29.94	59	61½	57	"	"	Light.	"
11	29.95	59½	63½	59	"	"	"	"
12	29.95	60	64	57½	"	"	Gentle.	"
1	29.95	62	67	61½	"	"	"	Sun appears at intervals.
2	29.96	62	64	58	"	"	Light.	"
3	29.96	62	63½	59	"	"	"	Clouds again spread.
4	29.95	62	64¼	60¼	"	"	"	"
5	29.95	62	63	59	SE.	"	"	"
6	29.96	62	61	57½	"	"	"	"

METEOROLOGICAL OBSERVATIONS,

MADE AT THE INSTITUTE, FLUSHING, L. I., FOR THIRTY-SEVEN SUCCESSIVE HOURS, COMMENCING AT SIX A. M. OF THE TWENTY-FIRST OF SEPTEMBER, EIGHTEEN HUNDRED AND THIRTY-SIX, AND ENDING AT SIX P. M. OF THE FOLLOWING DAY.

(*Lat.* 40° 44′ 58″ N. *Long.* 73° 44′ 20″ W. *Height of barometer above low water mark of Flushing Bay,* 54 *feet.*)

Hour.	Barometer Corrected.	Attached Therm'ter.	External Therm'ter.	Wet Bulb Therm'ter.	Winds —from—	Clouds —to—	Strength of wind.	REMARKS.
6	30.194	67	62½	58	NE.	SE.	Light.	Bright clouds mostly spread.
7	30.209	67	63	58	"	"	Gentle.	Clouds a little darker.
8	30.164	67	64½	58½	"	"	"	"
9	30.165	67½	67½	59	"	"	Fresh.	Clouds lighter and more broken.
10	30.181	69	70	61	"	"	"	Sun appears at intervals.
11	30.182	69½	70	61	"	"	"	Clouds darkening.
12	30.183	70	71¼	62½	"	"	Gentle.	"
1	30.168	69	68½	60¼	"	"	"	"
2	30.156	68½	68	61	"	"	"	A few drops of rain.
3	30.135	68	67½	61	"	"	Very light.	Clouds overspread.
4	30.135	67½	67¼	60½	"	"	"	"
5	30.136	67	66	60½	"	"	"	"
6	30.128	65½	64½	60½	-	"	Calm.	Dark clouds and rain. } rained from 6 to 8 P. M. 12 inch's
7	30.151	65	63	60	-	-	"	"
8	30.156	67	61½	60	-	-	"	Clouds overspread.
9	30.162	67	61	59½	S.	Nd.	Light.	"
10	30.175	66	61	59½	"	NW.	Very light.	"
11	30.143	66	60½	59	"	"	Light.	"
12	30.143	66	61	59½	"	"	"	"
1	30.138	66	61¼	59½	"	"	"	"
2	30.144	66	61½	60	"	"	"	"
3	30.139	66	62	60	"	"	"	"
4	30.126	65	61½	60	"	"	"	"
5	30.108	65	61½	59¼	"	"	"	"
6	30.120	65	62	60	"	N.	Very light.	Clouds partially breaking.
7	30.120	65	66½	63	"	"	"	"
8	30.111	69	78	69	"	"	"	"
9	30.119	69	71	66	"	"	Light.	"
10	30.121	69	70	65	"	SE.	"	Clouds darkened suddenly about 9¼.
11	30.110	69	71½	65¼	"	E.	"	Overcast.
12	30.091	69	73½	66½	"	SE.	Gentle.	Dark clouds.
1	30.068	68	70¼	66¼	"	NE.	"	Rain. } Rained from 12¼ to 4¼ P. M. 20 inches.
2	30.033	68	69	65	"	"	Very light.	"
3	30.005	68	70	67	SE.	NW.	Gentle.	"
4	29.993	68	68	66	"	NE.	Very light.	"
5	29.994	67	67	65½	-	-	Calm.	Misty.
6	29.987	67	67	65½	-	-	"	"

The Barometer was made with extreme care for the purpose, by Pike of New-York, having a glass cistern with an adjustment for the neutral point, and can be read off to the .001 of an inch. Correction for capillarity, .025. In 34 observations made with this Instrument and the one used in June, there is a mean difference of .02 inches.

THE

MATHEMATICAL MISCELLANY.

NUMBER II.

JUNIOR DEPARTMENT.

This part of the Miscellany will be adapted to the ordinary mathematical attainments of youth in the college classes of our country. It will occasionally contain articles elucidating principles, and exhibiting methods, of arithmetical and analytical processes, better adapted to ordinary purposes than are generally found in our text books. In this way we shall endeavour to lend some aid to our brethren in the business of instruction, hoping to receive from them hints of the same kind in exchange, which we shall gladly publish.

The department will consist chiefly of questions calculated to interest the tyro in science, and thus perhaps be a further help to instructors in drawing out and encouraging the latent and unfolding talent which must so largely exist in our literary institutions. With proper co-operation in this object, which we earnestly solicit from the mathematical professors who patronise the Miscellany, we hope to supply a stimulus to industry, not afforded by the duties of the recitation room. The publication of a neat solution will come to be considered by the young aspirant to scientific distinction, as sufficient reward for the labour of preparing it.

ARTICLE I.

HINTS TO YOUNG STUDENTS.

1. Considering the great facilities for learning in the present day, in the multiplicity of books that are published on every subject, and the great number of schools, academies, and colleges of every grade, that are within the reach of almost every one, it is somewhat singular that there are not a greater number of able scholars. It is certain that reading alone will not make a thorough scholar. However well an author may be understood, and however much the subject may come within the grasp of ordinary comprehension, it would be difficult to master it completely without going somewhat beyond the point to which the author himself takes you. In fact the mind has in most instances to leap over many minor details, and grasping the subject at a point often detached from all its previous store of knowledge, it has then to proceed by connecting this with the ones previously established, and drawing consequences until another leap becomes necessary. This is especially the case in the acquirement of mathematical sciences; and it, perhaps, arises from the difficulty of adapting a regular course of collegiate instruction to this halting and irregular pace of the mind, that so few ripe scholars emanate from our places of learning. It is, perhaps, also, for this reason that, in minds of a certain order, solitary and unaided study produces greater effects than all the helps that libraries and lectures could give. We often say that if men like Simpson and Emerson made such astonishing progress in science by their own unassisted efforts, what would they not have done had they been assisted by the library of a college and the lectures of its professors? Whereas, the truth may be that the structure of their minds rendered the peculiar course they were obliged to adopt the very best adapted to their wants. And though this may only be the case in a few instances, it is sufficiently general to encourage those who have not the advantages of a collegiate education, and to such these "hints" are more especially addressed, to persevere in their efforts; not doubting that, though their road may be a painful one, it is as likely to lead them to the desired haven, as the more smooth and beaten one of the schools.

2. There are many things constantly taught in the recitation room, which could not with propriety be introduced into a text book; and these often consist not so much of amplifications, as of points connected with the minor details of the several subjects, or the manner of performing operations, which a text book can only say should be done, and which the unaided student must perform in the best manner he can until years of experience have taught him the least troublesome one. These hints will be principally directed to points of this kind, and will contain matter, some of which is scattered through books not within the reach of the general student, and some suggested during my own experience in teaching.

3. In Arithmetic and Algebra the processes for combining and transforming fractions are the most troublesome, and are, at the same time, susceptible of the greatest modifications. Accustom yourself from the very first to use the common signs of addition, subtraction and multiplication; they are of as much use in arithmetic as in algebra, and so far from making it more difficult as you would at first suppose, they tend greatly to render the operations both simple and brief. You will thus, at the same time, render yourself master of the first elements of the most brief and comprehensive language in the world—that of algebra. Instead of reading your fractions in the usual mode: say "the numerator divided by the denominator;" thus $\frac{2}{3}$ is read—two divided by three; you thus acquire a clearer idea of the meaning of the fraction in its combinations with others than you could by the most labored definition, for $6.\frac{2}{3}$, or 6 multiplied by $\frac{2}{3}$, is actually 6 multiplied by 2, and divided by 3.

4. In multiplying fractions, always first *represent* the operation by signs, because in that state it is more easy to reduce into lower terms than in any other, since the factors common to the two members can be immediately detected. Thus to multiply $\frac{2}{3}$, $\frac{3}{4}$ and $\frac{4}{5}$ together, you have

$$\frac{2}{3}.\frac{3}{4}.\frac{4}{5}=\frac{2.\,3.\,4}{3.\,4.\,5}=\frac{2}{5},$$

without any actual multiplication, since the factors 3 and 4 occurring at the same time in both numerator and denominator cancel each other, and need not be used. It is usual to draw a line across those factors that thus mutually destroy each other; the operation cannot be represented here for want of the proper type, but there can be no difficulty in performing it. In some cases a factor will only cancel another one in part, thus if you had the factor 3 in one member, and 6 in the other, you would draw a line across the 3 and the 6, but you would put the factor 2, which is not cancelled by the 3, over the 6, if it were in the numerator, or under it if it were in the denominator, and in any future operation on the fraction, the 2 is to be used instead of the 6; this is very obvious, because instead of 6 you might have written 2. 3, and then the 3 in the other member would cancel the 3, but leave the 2 uncancelled. Thus, although there may be no number common to both members, yet there may be factors of these members common to both, which can be detected and cancelled before the actual multiplication is done. One example will render all this clear:

$$\frac{3}{7}\times\frac{4}{9}\times\frac{5}{6}\times\frac{14}{15}\times\frac{9}{16}=\frac{3.\;4.\;5.\;14^{2}.\,9}{7.\;9.\;6_{2}.\,15_{3}.\,16_{4}}=\frac{1}{12}.$$

Here the manner in which the factors mutually destroy each other will be best shown by dividing the factors as they stand into their prime factors where it is necessary, and arranging them so that those which cancel may stand directly under each other, thus:

$$\frac{9\times3\times2\;.\;7\times5\times4}{9\times3\;.\;2\times7\times5.\,3\times4.\,4}=\frac{1}{3.4}=\frac{1}{12};$$

all the factors in the numerator are thus cancelled, and all those in the denominator, except 3 and 4.

5. A little practice will render you very expert at this kind of work, and enable you to do all questions in which fractions are concerned with a great deal of facility. The principal difficulty will be to know when one number is divisible by another, or to find the prime factors of a number. When the number is large, the following rules will enable you to tell whether a number is divisible by some of the smaller prime numbers without actually dividing it.

1. All even numbers are divisible by 2.

2. If the two last digits of a number be divisible by 4, the whole number is divisible by 4.

3. If the three last digits of a number be divisible by 8, the whole number is divisible by 8, &c.

4. If the sum of the digits which compose a number be divisible by 3, the number itself is divisible by 3.

5. If the last or unit's digit of a number be either 5 or 0, the number is divisible by 5 in both cases, and by 10 in the latter case.

6. If the sum of the digits that compose a number be divisible by 9, the number itself is divisible by 9; thus 576873 is divisible by 9, because $5+7+6+8+7+3=36$ is so.

7. If the sum of the digits in the odd places, (counting from the right or unit's place,) be equal to the sum of the digits in the even places, or if the difference of these sums be divisible by 11, the number itself is divisible by 11.

8. If a number be divisible by two different prime numbers, it is divisible by their product.

And in order to divide a number into its prime factors, which is often necessary in fractional operations, begin and divide it by 2 as often as you can, then by 3, 5, 7, 11, 13, &c., and so on by all the prime numbers thus to find the prime factors of 45864:

$$\begin{array}{r|r}
2 & 45864 \\ \hline
2 & 22932 \\ \hline
2 & 11466 \\ \hline
3 & 5733 \\ \hline
3 & 1911 \\ \hline
7 & 637 \\ \hline
7 & 91 \\ \hline
 & 13
\end{array}$$

Hence, $45864 = 2.\,2.\,2.\,3.\,3.\,7.\,7.\,13 = 2^3.\,3^2.\,7^2.\,13$.

If the number will divide by none of the prime numbers 2, 3, 5, 7, &c., until you get a quotient less than your divisor, then the number

itself is a prime number, or has no divisors. Thus, the number 23 is a prime number, because it will neither divide by 2, 3, nor 5, and when you divide it by 5, the quotient is less than 5. Generally we should say that a number is prime when it will divide by no prime number less than its square root.

6. The division of fractions ought always to bé reduced to multiplication, and indicated in that form. This is done by the simple consideration, that in dividing a number by 2, you take $\frac{1}{2}$ of it, or multiply it by $\frac{1}{2}$, in dividing by 3, you multiply by $\frac{1}{3}$, &c.; and generally, to divide by any number, you multiply by unity divided by that number, or the *reciprocal* of the number, bearing in mind that the reciprocal of a fraction is taken by simply inverting the fraction; so that to divide one fraction by another, you invert the divisor and multiply them together. This will be shown by the following example: If $\frac{2}{3}$ of a yard of cloth cost $\frac{5}{9}$ of a dollar, what will $\frac{3}{5}$ of a yard cost?

$\frac{2}{3}$ of a yard cost $\frac{5}{9}$ \$,
1 yard cost $\frac{5}{9}$ divided by $\frac{2}{3} = \frac{3}{2} . \frac{5}{9}$,

$$\frac{3}{5} \text{ yard cost } \frac{3}{2} . \frac{5}{9} . \frac{3}{5} = \frac{3.5.3}{2.9.5} = \frac{1}{2}\ \$.$$

In this example the multiplication of $\frac{3}{2}$ by $\frac{5}{9}$ is only indicated, not performed, because the number $\frac{3}{2} . \frac{5}{9}$ represents the price of a yard of cloth as well as its equal $\frac{5}{6}$; and by leaving it in the unreduced form, you can combine it better with the operations that are afterwards to be performed upon it, since in that state the factors of its two members are already indicated, and were you to actually multiply them, you would have in effect to decompose them again into these factors when you multiplied by the other fraction $\frac{3}{5}$, so as to reduce the result to its lowest terms. Indeed the reducing of a fraction to its lowest terms, in whatever manner it may be done, consists only in the dividing its two members into their factors, and cancelling those which are common to both; so that when you have a fraction already thus expressed, it is better to reduce in that form than in any other.

In general it is advisable to express the answer of a sum in terms of the numbers given in the question, combining them by the Algebraic signs, so as to represent one number, and then reduce the expression to its simplest form.

7. The number so expressed will often be not only a fraction, but one which has fractions in one or both of its members, such for instance as $\frac{3\frac{1}{7}}{5\frac{1}{2}}$. Mixed fractions like this are most easily reduced by multiplying both members by such a number as will divide by the denominators of the fractional parts; thus multiplying both members of the fraction $\frac{3\frac{1}{7}}{5\frac{1}{2}}$ by 14, or by 2. 7, it becomes $\frac{2.\ 22}{7.\ 11} = \frac{4}{7}$; and the fraction $\frac{4\frac{2}{10}}{3\frac{4}{15}}$ is reduced by multiplying its two members by 30, then

$$\frac{4\frac{2}{10}}{3\frac{4}{15}} = \frac{3.\ 49}{2.\ 49} = \frac{3}{2} = 1\tfrac{1}{2}.$$

(*To be Continued.*)

ARTICLE II.

QUESTIONS TO BE ANSWERED IN NUMBER III.

QUESTION I. BY ALFRED.

Given the equations

$$\begin{aligned} x^2 + xy + y^2 + xv + yv + v^2 &= 202 \\ x^2 + xy + y^2 + xz + yz + z^2 &= 394 \\ x^2 + xv + v^2 + xz + vz + z^2 &= 522 \\ y^2 + yv + v^2 + yz + vz + z^2 &= 586 \end{aligned}$$

to find v, x, y, z.

QUESTION II. BY ——.

Given the equation

$$\frac{l^1x + \frac{1}{2}}{lx} + \frac{3\,lx - \frac{1}{2}}{l^1x} = 1,$$

when l represents the common, and l^1 the Neperian logarithm of a number, to find x.

QUESTION III. BY ——.

Given the equation

$$a \sin x + b \cos x = c$$

to find x.

QUESTION IV.

(*From the Dublin Problems.*)

Express the sides of a plane triangle, as functions of the radius of the circumscribed circle, and the three angles.

QUESTION V. BY ——.

Three circles, whose radii are r_1, r_2, r_3 respectively, touch each other externally, prove that the area of the triangle, formed by joining their centres, is

$$\sqrt{r_1 r_2 r_3 (r_1 + r_2 + r_3)}$$

QUESTION VI.

By Mr. L. Van Bokkelen, Flushing Institute.

An inflexible wire is made to pass through a given plane surface, which can traverse freely along it, and the wire is then fixed horizontally in a direction perpendicular to the wind; what angle must the plane make with the wire so that the wind may drive it along the wire with the greatest velocity?

SENIOR DEPARTMENT.

ARTICLE VI.

SOLUTIONS TO QUESTIONS PROPOSED IN NUMBER I.

QUESTION I. BY P.

How many diagonals can there be drawn in a polygon of n sides.

FIRST SOLUTION, BY PROF. PEIRCE, HARVARD UNIVERSITY, CAMBRIDGE.

The number of diagonals is a function of n which may be expressed by $f(n)$. If the number of sides of the polygon is increased by unity, that is, if a new vertex is added, the number of diagonals is increased by those which are drawn to this new vertex, and also one of the former sides becomes a diagonal. But the number of diagonals which can be drawn to any one vertex, is equal to the number of all the other vertices minus two. Hence

$$f(n+1) = f(n) + (n-2) + 1\,;$$

or, as is easily obtained by development,

$$f(n+1) - f(n) = \tfrac{1}{2}\,[(n+1)^2 - n^2] - \tfrac{3}{2}\,[(n+1) - n].$$

using Δ as the symbol of finite differences, we have

$$\Delta . f(n) = \tfrac{1}{2}\,\Delta .\, n^2 - \tfrac{3}{2}\,\Delta .\, n\,;$$

the integral of which is

$$f(n) = \tfrac{1}{2}\,n^2 - \tfrac{3}{2}\,n + c.$$

The constant c is to be determined by some simple case, such as that of the triangle, in which there are no diagonals, or

$$f(3) = 0 = \tfrac{9}{2} - \tfrac{9}{2} + c = c.$$

Therefore $$f(n) = \frac{n\,(n-3)}{2}.$$

SECOND SOLUTION, BY MR. JAMES F. MACULLY, NEW-YORK.

The number of lines which join n points on a plane, two and two, include all the diagonals together with the n sides of the polygon.

Now it is evident that by two points one line is formed; by three $1+2$ lines, by four $1+2+3$ lines, &c.; the last point adding always as many lines as there are points preceding it, and therefore by n points

there will be formed $1 + 2 + 3 + \ldots + (n-1) = \frac{n^2 - n}{2}$ lines.

Hence the number of diagonals is $= \frac{n^2 - n}{2} - n = \frac{n^2 - 3n}{2}$.

Mr. Macully also favored us with another solution to this question.

THIRD SOLUTION, BY MR. GEO. R. PERKINS, CLINTON LIBERAL INSTITUTE, N. Y.

There being n vertices or corners to a polygon of n sides, and as lines may be drawn from any one of these points to every other point, the whole number of lines thus drawn will be equivalent to the number of combinations out of n things taken two at a time, that is to $\frac{n(n-1)}{1.\ 2}$. Now n of these lines must go to constitute the perimeter of the polygon, hence $\frac{n(n-1)}{1.\ 2} - n$ or $\frac{n(n-3)}{2}$ will be the number of diagonals in a polygon of n sides.

FOURTH SOLUTION, BY DR. STRONG, PROF. OF MATHEMATICS, RUTGERS COLLEGE, N. B., N. J.

The number of diagonals that can be drawn from each angle $= n - 3$, therefore the number drawn from all the angles $= n(n - 3)$, but each diagonal is common to two angles, therefore the number required $= \frac{n(n-3)}{2}$.

FIFTH SOLUTION, BY MR. N. VERNON, FREDERICK, MARYLAND.

By inspection, it will be perceived that a polygon of 4 sides has 2 diagonals, of 5 sides 5, of 6 sides 9, of 7 sides 14, of 8 sides 20, &c., a regular series. We have, therefore, only to determine the general term of this series, which is equal to $\frac{m}{2}(m+3)$. Now if we take n for the number of sides, $n - 3$ will equal m, and by substituting, we get $\frac{n}{2}(n-3)$ for the number of diagonals.

SIXTH SOLUTION, BY MR. O. ROOT, HAMILTON COLLEGE, CLINTON, N. Y.

Let n = number of sides of the polygon, then $(n-3)$ will be the number of diagonals which can be drawn from the first angle; $(n-3)$ will also express the number drawn from the second angle; $(n-4)$ the number drawn from the third; $(n-5)$ from the fourth, and so on: hence the whole number of diagonals will be expressed by $(n-3)$ + the sum of an arithmetical progression whose first term is $(n-3)$ and number of terms $(n-2)$, which is $= \frac{n}{2}(n-3)$.

Such, also, nearly, were the solutions by Messrs. Barton and Docharty.

QUESTION II. BY Δ.

Let x_1, x_2, x_3, be any three angles, prove that

1°. $\sin^2(x_1 - x_2) + \sin^2(x_1 - x_3) + \sin^2(x_2 - x_3)$
$+ 2\cos(x_1 - x_2)\cos(x_1 - x_3)\cos(x_2 - x_3) = 2.$

2°. $1 - \cos^2 x_1 \cos^2 x_2 - \cos^2 x_1 \cos^2 x_3 - \cos^2 x_2 \cos^2 x_3$
$+ 2\cos^2 x_1 \cos^2 x_2 \cos^2 x_3$
$- \sin^2 x_2 \sin^2 x_2 - \sin^2 x_1 \sin^2 x_3 - \sin^2 x_2 \sin^2 x_3$
$+ 2\sin^2 x_1 \sin^2 x_2 \sin^2 x_3 = 0.$

SOLUTION, BY MR. JAMES F. MACULLY.

1°. By a transformation well known, and much used in spherical trigonometry,

$$1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c$$
$$= 4\sin s \sin(s - a)\sin(s - b)\sin(s - c),$$
$$\text{where } s = \tfrac{1}{2}(a + b + c);$$

and by applying the formula $\cos^2 A = 1 - \sin^2 A$, this immediately becomes

$$\sin^2 a + \sin^2 b + \sin^2 c + 2\cos a \cos b \cos c$$
$$= 2 + 4\sin s \sin(s - b)\sin(s - b)\sin(s - c).$$

Now let $a = x_1 - x_2$, $b = x_1 - x_3$, $c = x_2 - x_3$; then will $s = x_1 - x_3$, $s - b = 0$ and $\sin(s - b) = 0$, and the formula will become the one in the question.

2°. By eliminating the cosines from this expression, it will be

$$1 - (1 - \sin^2 x_1)(1 - \sin^2 x_2) - (-1\sin^2 x_1)(1 - \sin^2 x_3)$$
$$- (1 - \sin^2 x_2)(1 - \sin^2 x_3)$$
$$+ 2(1 - \sin^2 x_1)(1 - \sin^2 x_2)(1 - \sin^2 x_3)$$
$$- \sin^2 x_1 \sin^2 x_2 - \sin^2 x_1 \sin^2 x_3$$
$$- \sin^2 x_2 \sin^2 x_3 + 2\sin^2 x_1 \sin^2 x_2 \sin^2 x_3.$$

Performing the multiplications indicated, we find the terms mutually destroy each other, and therefore the whole expression $= 0$.

Many of our correspondents proved the first property by expanding each term, and expressing them in the products of the sines and cosines of simple angles.

QUESTION III. BY. A. B.

Given the three equations.

$$y + z = 2\sqrt{(xy + xz - x^2)},$$
$$2y - z = 2\sqrt{(2xy - xz - x^2)},$$
$$3y - x = 5x - \sqrt{x};$$

to find x, y, and z.

FIRST SOLUTION, BY MR. GERARDUS B. DOCHARTY, FLUSHING, L. I.

If we square both members of the first equation, then transpose and extract the root, we shall have

$$2x - y - z = 0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and performing a like operation on the second equation

$$2x - 2y + z = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

By adding and subtracting equations (1) and (2) to and from each other, we have

$$4x = 3y, \text{ and } 2z = y;$$

therefore, $y = 2z$, and $x = \frac{3}{4}$, $y = \frac{3}{2}z$.

These being substituted in the third given equation,

$$6z - \tfrac{3}{2}z = 5z - \sqrt{z},$$

therefore $z = 2\sqrt{z}$, and $z = 4$; hence $y = 8$, and $x = 6$.

SECOND SOLUTION, BY ALFRED.

Square the two first eqations, then

$$y^2 + 2yz + z^2 = 4(xy + xz - x^2) \quad . \quad . \quad (1)$$
$$4y^2 - 8yz + z^2 = 4(2xy - xz - x^2) \quad . \quad . \quad (2)$$

Subtracting (1) from (2) we have

$$3y^2 - 6yz = 4xy - 8xz \quad . \quad . \quad . \quad . \quad (3)$$

or $\quad 3y(y - 2z) = 4x(y - 2z) \quad . \quad . \quad . \quad . \quad (4)$

therefore $\quad 3y = 4x$, and $x = \frac{3}{4}y$.

Substituting this value of x in either of the equations (1) or (2), we get

$$y^2 - 4yz + 4z^2 = 0 \quad . \quad . \quad . \quad (5)$$

and extracting the root $y - 2z = 0$, or $y = 2z$, $\therefore x = \frac{3}{4}y = \frac{3}{2}z$.

Now substitute these values of x and y in the third equation, and we have

$$\tfrac{9}{2}z = 5z - \sqrt{z}; \text{ hence } z = 4,\ x = 6, \text{ and } y = 8.$$

—— The difficulty, mentioned by *Alfred*, arose from his endeavouring to find both y and x in terms of z from equation (3) alone. This cannot be done, independently. The equation may be put in the form

$$(3y - 4x)(y - 2z) = 0$$

which is satisfied by making either $3y - 4x = 0$, or $y - 2z = 0$; and accordingly, as either of these assumptions are made, there would arise two distinct solutions, producing in general different final results, but all equally true. It only happens that, in the present case, either of them leads to the other, and therefore they both happen to be true together.

QUESTION IV. BY THE EDITOR.

Prove that in the two inequalities

$$x^2 + ax + b < 0,$$
$$x^2 + ax + b > 0;$$

the first has place when x is within the limits of the roots of the equation

$$x^2 + ax + b = 0,$$

and the second, when x is without the limits of those roots.

SOLUTION BY MR. P. BARTON, JUN., SCHENECTADY, N. Y.

If x_1 and x_2 represent the roots of the equation,

$$x^2 + ax + b = 0,$$

x_1 being the greater: we have from the theory of equations

$$x^2 + ax + b = (x - x_1)(x - x_2)$$

Now, if x be between the limits of x_1 and x_2, that is, if

$$x < x_1 \text{ and } > x_2, \text{ then will } x - x_1 < 0, \text{ and } x - x_2 > 0;$$

and therefore, $(x - x_1)(x - x_2) < 0,$

or, $x^2 + ax + b < 0.$

But if x be without the limits of x_1 and x_2, that is if $x > x_1$ or $x < x_2$ then will, in the first case, $x - x_1 > 0$, and $x - x_2 > 0$, and in the second case $x - x_1 < 0$, and $x - x_2 < 0$; in both cases they have the same sign, and therefore

$$(x - x_1)(x - x_2) > 0,$$

or, $x^2 + ax + b > 0.$

On the same principle were the elegant solutions by Dr. Strong, Prof. Peirce, and Mr. Perkins.

—— When x_1 and x_2 are imaginary, no real number can be between these limits, and therefore when $a^2 < 4b$, all real values of x render $x^2 + ax + b > 0$.

QUESTION V. BY MR. JAMES F. MACULLY.

To find such a positive value of x as will make

$$1 - 8x = \text{a square number},$$
$$x - 4x^2 + 4 = \text{a square number};$$

or prove that it is impossible.

FIRST SOLUTION, BY DR. T. STRONG.

Put $1 - 8x = a^2$, and $x - 4x^2 + 4 = b^2$,

then we have $x = \dfrac{1 - a^2}{8} = \dfrac{1 - \sqrt{65 - 16b^2}}{8},$

$$\therefore \sqrt{65 - 16b^2} = a^2,$$

put $b = 2 + y$, and we have

$$65 - 16(2 + y)^2 = 1 - 64y - 16y^2 = a^4;$$

assume $1 - 64y - 16y^2 = (my - 1)^2,$

then $y = \dfrac{m^2 - 64m}{m^2 + 16}$ and $a^2 = my - 1 = \dfrac{m^2 - 64m - 16}{m^2 + 16}.$

Assume $m^2 - 64m - 16 = (m-p)^2$, $\therefore m = \dfrac{p^2+16}{2p-64}$ . . (1);

$$\text{hence } m^2 + 16 = \frac{(p^2+16)^2 + 64(p-32)^2}{4(p-32)^2}.$$

put $p = 4q$, and we have

$$(q^2+1)^2 + 4(q-8)^2 = \text{a square};$$

let $q^2 + 1 = r^2 - s^2$, and $q - 8 = rs$, and it will be a square as required. Since $q^2 = r^2 - s^2 - 1$ and $q = 8 + rs$, we have

$$r^2 s^2 + 16rs + 64 = r^2 - s^2 - 1,$$

$$\text{or } (r^2+1)s^2 + 16rs = r^2 - 65;$$

$$\therefore s = \frac{-8r \pm \sqrt{r^4 - 65}}{r^2+1} \text{ and } q = 8 + rs = \frac{8 \pm r\sqrt{r^4-65}}{r^2+1};$$

$$\therefore p = 4q = \frac{32 \pm 4r\sqrt{r^4-65}}{r^2+1} \quad . \quad . \quad . \quad . \quad (a).$$

The equation $r^2 s^2 + 16sr + 64 = r^2 - s^2 - 1$ also gives

$$(1-s^2)r^2 - 16rs = 65 + s^4,$$

$$\therefore r = \frac{8s + \sqrt{65 - s^4}}{1-s^2},$$

by taking the sign $+$ before the radical,

$$\therefore q = 8 + rs = \frac{8 + s\sqrt{65-s^4}}{1-s^2} \text{ and } p = \frac{32 + 4s\sqrt{65-s^4}}{1-s^2} \ . \ (b).$$

Assume $r = 3$ then, by (a), we get $p = -\frac{8}{5}$ supposing the sign $-$ to be taken before the radical; then, by (1) we have $m = \dfrac{-29}{105}$ which gives

$$a^2 = \frac{m^2 - 64m - 16}{m+16} = \left(\frac{139}{421}\right)^2 = \frac{19321}{177241};$$

hence $x = \dfrac{1-a^2}{8} = \dfrac{19740}{177241}$ is a positive value of x which makes

$$1 - 8x = \left(\frac{139}{421}\right)^2, \text{ and } x - 4x^2 + 4 = \left(\frac{357208}{177241}\right)^2, \text{ as required.}$$

Again, by substituting in (b) $\dfrac{139}{421}$ for s, we shall get a value of p, which is greater than 32, $\therefore$ by (1), we shall have a positive value of m, which gives y positive and which substituted in $a^2 = \dfrac{m^2 - 64m - 16}{m^2+16}$ will give a new value of a, which will be less than unity, hence we obtain a new positive value of x, that will answer the question; then we may use this last value of a for s in (b), and thence find a new value of x, and so on to infinity.

Cor. Since $65 = a^4 + 16b^2 = \left(\frac{139}{421}\right)^4 + \left(\frac{1428832}{177241}\right)^2$, we see how to divide the number 65 into two squares, such that one of them shall be a fourth power, and less than unity; and it appears, from what has been done, that we may divide 65 into as many such numbers as we please.

SECOND SOLUTION, BY WM. LENHART, ESQ., YORK, PA.

Put $1 - 8x = n^2$, then $x = \frac{1 - n^2}{8}$, and by substitution,

$x - 4x^2 + 4 = \square$, becomes $\frac{65 - n^4}{16} = \square$, or $65 - n^4 = \square$.

As 65 is of the form $4n' + 1$ it may be assumed equal to $v'^2 + w^2$, therefore

$$65 - n^4 = v'^2 + w^2 - n^4 = \square = \left\{ t\,(v' - n^2) - w^2 \right\}^2$$
$$= t^2\,(v' - n^2)^2 - 2tw\,(v' - n^2) + w^2;$$

or, cancelling w^2 and dividing by $v' - n^2$,

$$v^1 + n^2 = t^2\,(v' - n^2) - 2tw.$$

$$\text{Hence } n^2 = \frac{(t^2 - 1)\,v' - 2tw}{t^4 + 1} = \square \quad . \quad . \quad . \quad (1),$$

and consequently, putting $v' = v^2$, and dividing the numerator by v^2,

$$t^2 + 1 \qquad = \square = A^2 \quad . \quad . \quad . \quad (2),$$

$$t^2 - 1 - \frac{2tw}{v^2} = \square = B^2 \quad . \quad . \quad . \quad (3).$$

Take the difference of (2) and (3), and

$$A^2 - B^2 = (A + B)\,(A - B) = \frac{2tw}{v^2} + 2 = \left(2t + \frac{2v^2}{w}\right)\frac{w}{v^2}.$$

Now put
$$A + B = 2t + \frac{2v^2}{w},$$
$$A - B = \frac{w}{v^2};$$

then $A = t + \frac{2v^4 + w^2}{2v^2\,w}$, and thence (2) becomes

$$t^2 + 1 = \left(t + \frac{2v^4 + w^2}{2v^2 w}\right)^2,$$

$$\text{and } t = \frac{-4v^8 - w^4}{4v^2\,w\,(2v^4 + w^2)},$$

which being substituted in (1) and properly reduced, there results

$$n = \frac{v\,(v^8 - 3\,w^4)}{(2v^4 + w^2)^2 + (2v^2\,w)^2},$$

in which v and w must be such as to make

$$v^4 + w^2 = 65 \text{ and } n < 1.$$

Now $\dfrac{ap^2 + 2bpq - aq^2}{p^2 + q^2}$ and $\dfrac{bq^2 + 2apq - bq^2}{p^2 + q^2}$ are well known to be the roots of two squares whose sum is equal to $a^2 + b^2$. Hence if $a = 4$, $b = 7$, then $a^2 + b^2 = 65$, and taking $p = 4$ and $q = 3$, the roots will be $\dfrac{196}{25}$ and $\dfrac{47}{25}$: so that by assuming $v = \dfrac{14}{5}$, $w = \dfrac{47}{25}$, we shall have $v^4 + w^2 = 65$, and find $n = \dfrac{20456100182}{32934617285}$, which being less than unity will of course render x positive, and such a number as to make

$$1 - 8x = \square \text{ and } x - 4x^2 + 4 = \square;$$

which was to be done.

THIRD SOLUTION, BY MR. LUCIAN W. CARYL, BUFFALO, N. Y.

Put $$1 - 8x = (1 + rx)^2 \quad . \quad . \quad . \quad . \quad (1)$$

$$\therefore x = \frac{-2r - 8}{r^2} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equation (2) shows that r must be less than -4 when x is positive. By substituting the value of x from equation (2) in $x - 4x^2 + 4$, we shall have

$$\frac{4r^4 - 2r^3 - 24r^2 - 128r - 256}{r^4} = \square.$$

Omitting the denominator, which is a square, and dividing the numerator by 4, which is also a square, we shall have

$$r^4 - \tfrac{1}{2}r^3 - 6r^2 - 32r - 64 = \square \quad . \quad . \quad (3)$$

It is easy to satisfy equation (3) since the first term of its first member, viz. r^4 is a square. To do this, put

$$r^4 - \tfrac{1}{2}r^3 - 6r^2 - 32r - 64 = (r^2 - \tfrac{1}{4}r - \tfrac{97}{32})^2$$

(*Young's Alg.* p. 309,) we shall have

$$-32r - 64 = \frac{97}{64}r + \frac{97^2}{32^2} \therefore r = \frac{-1153}{528}.$$

This value of r substituted in equation (2) gives x negative. We must therefore find another value of r. To do this, put in equation (3) $r = p - \dfrac{1153}{528}$, and we shall have

$$p^4 - \frac{4876}{528}p^3 + \frac{7216926}{528^2}p^2 - \frac{8037205276}{528^3}p + \frac{636541^2}{528^4} = \square \quad : \quad (4)$$

Since the first and last terms of the first member of equation (4) are squares, it is easy to satisfy it. To do this, put the first member of equation (4) equal to

$$\left(p^2+\frac{8037205276}{2\times 528\times 626541}p-\frac{636541}{528^2}\right)^2$$ (*Young's Alg. p.* 307,)

and we shall have

$$p=\frac{-32040959370424678 99}{9173643738302831 04}.$$

This value of p is less -3. Hence $r=p-\frac{1153}{528}$ is less than -5. This value of r substituted in equation (2) gives the value of x positive. There is no difficulty in making this substitution, and thus finding the value of x, except that which arises from the magnitude of the numbers we employ. We shall omit it.

We will now show how we may find other values of r than $-\frac{1153}{528}$, which will satisfy equation (3), and by being substituted in equation (2) will give us x, expressed in a smaller number of figures. To do this, we shall commence with the original equations

$$1-8x=\square,\ x-4x^2+4=\square;$$

and put
$$1-8x=y^2 \quad . \quad . \quad . \quad . \quad (5)$$

$$\therefore x=\frac{1-y^2}{8} \quad . \quad . \quad . \quad (6)$$

The value of x from equation (6), being substituted in $x-4x^2+4=\square$ gives us

$$\frac{65-y^4}{16}=\square,\text{ or } 65-y^4=\square \quad . \quad . \quad . \quad (7)$$

It is easy to perceive that $y=1$ or $y=2$ will satisfy equation (7). $y=1$ gives us from equation (6) $x=-\frac{3}{8}$. From equations (1) and (5) we have $1+rx=\pm y$. By substituting in this equation the value of $x=-\frac{3}{8}$ and the value of $y=2$, we have

$$1-\tfrac{3}{8}r=\pm 2 \ \therefore r=-\tfrac{8}{3}\text{ or } 8.$$

But $\frac{65-y^4}{16}=\frac{4r^4-2r^3-24r^2-128r-256}{r^4}$, since each member is equal to $x-4x^2+4$. By substituting the value of $y=2$ in the first member of this equation, and the values of $r=-\frac{8}{3}$ or $r=8$ in the second member, the numerical value of the first member will be equal to the numerical number value of the second member. But the numerical value of the first member will be a square. Therefore, the numerical value of the second will be a square also. But if $r=-\frac{8}{3}$ or $r=8$ will make $\frac{4r^4-2r^3-24r^2-128r-256}{r^4}=\square$, these values of r will also make $r^4-\frac{1}{2}r^3-6r^2-32r-64=\square$. Equation (3) will therefore be satisfied by taking $r=-\frac{8}{3}$ or 8. Let us therefore suppose in equation (3) $r=p-\frac{8}{3}$ and we shall have

$$p^4-\frac{67}{6}p^3+\frac{122}{3}p^2-\frac{2336}{27}p+\frac{56^2}{9^2}=\square \quad . \quad . \quad (8)$$

Put the first number of equation (8) $=\left(p^2-\frac{67}{12}p+\frac{56}{9}\right)^2$, (*Young's*

Alg. p. 307,) and we have

$$p = \frac{-1472}{255} \therefore r = p - \tfrac{8}{3} = \frac{-2152}{255};$$ substituting this value of r in equation (2), and we shall have

$$x = \frac{72165}{578888}$$

By substituting this value of x in equation (5), we shall have $y = \frac{14}{269}$. This value of y will satisfy equation (7). Hence (*Young's Alg. pp.* 311 and 312) we may find other values of y which will satisfy equation (7); and the corresponding values of x may be found from equation (6). We shall thus have a variety of different values for x. The question, therefore, is possible.

NOTE. The above method of finding values of r, viz. $-\frac{8}{3}$ or 8, which satisfy equation (3), is new, as far as I am acquainted, and might, probably, be usefully employed in many other cases.

—— Prof. Peirce's method of showing the possibility of the question, is highly ingenious.

QUESTION VI. BY THE EDITOR.

If a be a prime integer, find how many numbers less than a^n, are divisible by a, how many by a^2, how many by a^3, &c.

SOLUTION, BY MR. GEO. R. PERKINS, CLINTON LIBERAL INSTITUTE, N. Y.

All numbers divisible by a must be contained in the series

$$a,\ 2a,\ 3a,\ 4a,\ \&c.$$

in the present case this series must terminate at the term $(a^{n-1}-1)a$, because the numbers are to be less than a_n. In the same manner, all numbers less than a^n divisible by a^2, are contained in the series

$$a^2,\ 2a^2,\ 3a^2,\ 4a^2,\ \ldots\ (a^{n-2}-1)\,a^2:$$

those divisible by a^3, in the series

$$a^3,\ 2a^3,\ 3a^3, 4a^3,\ \ldots\ (a^{n-3}-1)\,a^3;$$

and so on. The number of terms in each of these series is equal to the coefficient of the last term; therefore, there are

$a^{n-1}-1$ numbers, less than a^n, divisible by a,
$a^{n-2}-1$ " " " a^2,
$a^{n-3}-1$ " " " a^3,
&c. &c.
$a^{n-m}-1$ " " " a^m.

—— Such, nearly, were the solutions of all our correspondents.

In the above expressions, the numbers divisible by a, include those divisible by a^2, by a^3, ... by a^{n-1}; those divisible by a^2, include the ones divisible by a^3, a^4, ... by a^{n-1}, and so on. It is evident therefore that there are

$a^{n-2}(a-1)$ numbers, less than a^n, divisible by a, once;
$a^{n-3}(a-1)$ " " " a, twice;
$a^{n-4}(a-1)$ " " " a, three times;
&c. &c.
$a^{n-m-1}(a-1)$ " " " a, m times;
and $a-1$ " " " a, $(n-1)$ times.

The subject is susceptible of great extension, and does not seem to have been submitted to analysis before.

QUESTION VII. BY MR. GERARDUS B. DOCHARTY.

Two circles touch each other internally; to find the sum of the areas of all the circles that shall touch each other, and also the two given circles.

FIRST SOLUTION, BY THE PROPOSER.

It is evident that there can be only two circles which shall touch each other, and also the two given circles: for simplicity, we shall suppose that the centre of one of them is on the straight line joining the centres of the given circles. Let the radii of the given and required circles be respectively R, r, r', r''; $R > r$; then if the line passing through the centres of R, r, r' be taken for the axis of x, that of y passing perpendicularly through the centre of r, and if we take 0, d for the co-ordinates of the centre of R; 0, x' for those of r', and y'', x'' for those of r'' we shall have

$$d = R - r \quad . \quad . \quad . \quad . \quad (1)$$
$$x' = R + d - r' = r + r' \quad . \quad . \quad . \quad (2)$$
$$y''^2 + x''^2 = (r + r'')^2 \quad . \quad . \quad . \quad (3)$$
$$y''^2 + (x'' - d)^2 = (R - r'')^2 \quad . \quad . \quad . \quad (4)$$
$$y''^2 + (x' - x'')^2 = (r' + r'')^2 \quad . \quad . \quad . \quad (5)$$

Equation (2) gives $r' = \frac{1}{2}(d + R - r) = R - r$ and $x' = R$, and by subtracting (3) from (4), and (3) from (5), substituting the values of d, r', and x';

$$(R - r)\,x'' = r''.(R + r) - r\,(R - r); \quad Rx'' = Rr - r''\,(R - 2r)$$

by eliminating x'' between these two equations, we get

$$r'' = \frac{Rr\,(R - r)}{R^2 - Rr + r^2},$$

from which the sum of their areas $= (r'^2 + r''^2) \times 3,14159$, &c. is immediately had.

SECOND SOLUTION, BY PROF. MARCUS CATLIN,* HAMILTON COLLEGE, N. Y.

It is evident that the centres of all circles touching the given ones internally, will be in the curve of an ellipse, whose focii are the centres of the given circles; hence its focal distance $= 2c = \text{R} - \text{R}'$ and its semitransverse $= a = \frac{1}{2}(\text{R} + \text{R}')$. Let r, r' be the radii of two such circles, which also touch each other, yx and $y'x'$, the co-ordinates of their centres, their origin being that vertex of the ellipse in contact with the touching circles. The distances of the centre of r from the two focii, are $\text{R}' + r$ and $\text{R} - r$, and therefore by a well known property of the ellipse

$$(\text{R}'+r)-(\text{R}-r)=\frac{2c\,(x-a)}{a}=\frac{\text{R}-\text{R}'}{\text{R}+\text{R}'}\,(2x-\text{R}-\text{R}') \quad . \quad (1).$$

from this equation $$r=\frac{\text{R}-\text{R}'}{\text{R}+\text{R}'}\,.\,x \quad . \quad . \quad . \quad . \quad . \quad (2).$$

This gives the radius of any circle touching the given ones in terms of the abscissa of its centre. Hence all the conditions of the question will be satisfied by the equations

$$\left.\begin{aligned}&(r+r')^2=(x'-x)^2+(y'-y)^2.\\ &r=\frac{\text{R}-\text{R}'}{\text{R}+\text{R}'}\,.\,x,\; r'=\frac{\text{R}-\text{R}'}{\text{R}+\text{R}'}\,.\,x'\\ &y^2=\frac{b^2}{a^2}(2ax-x^2),\; y'^2=\frac{b^2}{a^2}(2ax'-x'^2)\end{aligned}\right\} \quad . \quad (3).$$

Putting $e=\dfrac{\text{R}-\text{R}'}{\text{R}+\text{R}'}$, we find

$$x=\frac{r}{e},\; x'=\frac{r'}{e} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

and these, together with the 1st, 4th, and 5th of equations (3), give us

$$(r+r')^2=\left(\frac{r'-r}{e}\right)^2+\left[\frac{b}{a}\left(\frac{2ar}{e}-\frac{r^2}{e^2}\right)^{\frac{1}{2}}-\frac{b}{a}\left(\frac{2ar'}{e}-\frac{r'^2}{e^2}\right)^{\frac{1}{2}}\right]^2 (5).$$

But this is not sufficient to determine the sum of the areas of the given circles, and therefore there ought to be another condition given by the question.

Cor. Equation (2) shows, that the radius of any circle touching the given ones, is proportional to the abscissa of its centre. Consequently if the abscissa x increases in arithmetical progression, the radii of the corresponding circles will also increase in arithmetical progression; and hence the radius corresponding to the vertex of the conjugate axis of the ellipse equals half of the radius corresponding to the vertex of the transverse.

—— Several of our correspondents complain of the ambiguity in the enunciation of this question. And in order to accommodate those who have extended their researches to all the circles, each touching two of the others, that can be inscribed between the two given circles, we shall repropose it in this form for solution in Number III.

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* This gentleman's letter did not arrive until the copy was so far in the printer's hand.

QUESTION VIII. BY Q. Q.

Suppose the axis of the planet Venus to be inclined to the plane of her orbit at an angle of 15°, the time of rotation round that axis 23h. 21m., and her periodic time round the sun 224,7008 days; admitting also the planet to be spherical, her orbit a perfect circle, and her axis remaining parallel to itself: it is required to show the change of the seasons, the lengths of the day and night at the different parts of her surface in any time of her year, the diurnal change of the sun's declination, &c., and to compare these results with those exhibited in Wallace's "New Treatise on the Use of the Globes," pp. 276, 277.

FIRST SOLUTION, BY MR. O. ROOT.

In this question we have sin. dec. = sin $\left(\frac{360^\circ}{224,7008}\right)$ sin 75°, and cosine of hour angle from apparent noon = tan. dec. × tan. lat., which converted into time, allowing 23 hours 21 minutes for 360°, we shall have the length of the day and night. As the tropics of Venus are 15° from her poles, and her polar circles 15° from the equator, we can easily trace the change of seasons.

The error of Wallace arises from the small number of days in the year of Venus, according to his estimate.

SECOND SOLUTION, BY PROF. AVERY, HAMILTON COLLEGE, N. Y.

It appears on slight examination, that some of the remarkable conclusions to which Wallace has come with regard to temperature, are erroneous; also, the diurnal change of the sun's declination as reckoned from the equator is = 1° 36′ nearly, and not as Wallace says $36\frac{1}{4}$° Indeed it would seem that some of his conclusions are drawn on the supposition that the rotation of Venus is performed in something more than 24 of our days, and others on the supposition in the question, so that they are contradictory among themselves.

QUESTION IX. BY INVESTIGATOR.

Let there be three straight lines on a plane, whose equations are

$$y = a_1x + b_1,$$
$$y = a_2x + b_2,$$
$$y = a_3x + b_3,$$

prove that the three lines will have a common point of intersection if the three points whose co-ordinates are

$$b_1 \text{ and } ca_1,\ b_2 \text{ and } ca_2,\ b_3 \text{ and } ca_3,$$

be situated in the same straight line, c being a given line.

FIRST SOLUTION, BY MR. O. ROOT.

The three points being on a straight line, we have

$$\frac{b_1 - b_2}{b_2 - b_3} = \frac{a_1 - a_2}{a_2 - a_3}:$$

now if we make the three lines intersect in the same point, we shall get the same equation; for at the point of intersection of the first and second lines, $x = \frac{b_1 - b_2}{a_1 - a_2}$, and at that of the second and third $x = \frac{b_2 - b_3}{a_2 - a_3}$. If the three intersect in the same point, these values of x are equal, hence,

$$\frac{b_1 - b_2}{a_1 - a_2} = \frac{b_2 - b_3}{a_2 - a_3}, \text{ or } \frac{b_1 - b_2}{b_2 - b_3} = \frac{a_1 - a_2}{a_2 - a_3},$$

Therefore the proposition is manifest.

SECOND SOLUTION, BY MR. G. B. DOCHARTY.

If we equate either of the co-ordinates of the point of intersection of the first and second, with the corresponding co-ordinate of the intersection of the second and third lines, we shall find the condition necessary for the three lines to intersect in one point expressed by the equation

$$b_1 (a_2 - a_3) + b_2 (a_3 - a_1) + b_3 (a_1 - a_2) = 0 \quad . \; . \; . \; . \quad (1).$$

The equation of the line passing through the points b_1, ca_1, and b_2, ca_2 is

$$y - b_1 = \frac{b_1 - b_2}{a_1 - a_2} \cdot \frac{x - ca_1}{c},$$

and that of the line through the points b_1, ca_1, and b_3, ca_3 is

$$y - b_1 = \frac{b_1 - b_3}{a_1 - a_3} \left(\frac{x - ca_1}{c}, \right.$$

and if these lines coincide, or the three points are in the same straight line,

$$\frac{b_1 - b_2}{a_1 - a_2} = \frac{b_1 - b_3}{a_1 - a_3} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

which reduces to the same as equation (1), hence the proposition is true.

QUESTION X. BY Δ.

A point being given on the sphere, how must a second point be situated, at a given distance from it, so that the surface, included between the arcs of a great circle and a loxodromic curve, both passing through the two points, may be the greatest possible?

FIRST SOLUTION, BY DR. T. STRONG.

Let the radius of the sphere $= 1$, $a =$ the distance of the given point from the nearest pole, $\varphi =$ the angle included by a, and any arc of a great circle y, drawn from the same pole to the loxodromic curve, $c =$ the constant angle, or course, at which the curve cuts y, and $s =$ the surface bounded by a, y and the curve, then we have

$$ds = (1 - \cos y)\, d\varphi = 2 \sin^2 \tfrac{1}{2}y \times d\varphi \quad \ldots\ldots \quad (1),$$

and by the nature of the loxodromic curve

$$- \tan c\, dy = \sin y\, d\varphi \quad \ldots\ldots\ldots \quad (2)$$

$$\therefore d\varphi = - \tan c . \frac{dy}{\sin y} = \frac{- \tan c\, dy}{2 \sin \frac{1}{2}y \cos \frac{1}{2}y};$$

$$\therefore (1) \text{ becomes } ds = d\varphi + \frac{\cos y\, dy}{\sin y} . \tan c = - \frac{\sin \frac{1}{2}y\, dy}{\cos y} . \tan c:$$

and by taking the integral, we get

$$s = \varphi + \tan c \times \text{h. l. } \frac{\sin y}{\sin a} = 2 \tan c \times \text{h. l. } \frac{\cos \frac{1}{2}y}{\cos \frac{1}{2}a} \quad \ldots \quad (3)$$

supposing the integral to commence when $y = a$, also using h. l. to denote the hyperbolic logarithm. Since $d\varphi = - \tan c . \frac{dy}{\sin y}$, we get

$$\varphi = \tan c \times \text{h. l. } \frac{\tan \frac{1}{2}a}{\tan \frac{1}{2}y}, \therefore \tan c = \varphi + \text{h. l. } \frac{\tan \frac{1}{2}a}{\tan \frac{1}{2}y},$$

hence (3) becomes

$$s = \varphi \left[1 + \text{h. l. } \frac{\sin y}{\sin a} + \text{h. l. } \frac{\tan \frac{1}{2}a}{\tan \frac{1}{2}y}\right] = 2\varphi \left[\text{h. l. } \frac{\cos \frac{1}{2}y}{\cos \frac{1}{2}a} + \text{h. l. } \frac{\tan \frac{1}{2}a}{\tan \frac{1}{2}y}\right],$$

or we have $$s = \varphi \left[1 + \frac{\log. \sin y - \log. \sin a}{\log. \tan \frac{1}{2}a - \log. \tan \frac{1}{2}y}\right] \quad \ldots \quad (4).$$

Put $p = 3{,}14159$, &c., and let $x =$ the arc of a great circle joining the extremities of a and y, also, let φ', φ'' denote the angles of the spheric triangle thus formed, at the extremities of a and y respectively: put $s' =$ the area of the triangle, and we shall have

$$s' = \varphi + \varphi' + \varphi'' - p \quad \ldots\ldots\ldots\ldots \quad (5),$$

and when $s - s' =$ a *max.* or *min.* we have $ds = ds'$, hence we get by (4) and (5)

$$d\left(\varphi . \frac{\log. \sin y - \log. \sin a}{\log. \tan \frac{1}{2}a - \log. \tan. \frac{1}{2}y}\right) = d\varphi' + d\varphi'' = d\,(\varphi' + \varphi'') \quad . \quad (6);$$

also by spheric trig. we have

$$\tan \tfrac{1}{2}(\varphi' + \varphi'') = \cot \tfrac{1}{2}\varphi . \frac{\cos \frac{1}{2}(a - y)}{\cos \frac{1}{2}(a + y)} \quad \ldots \quad (7),$$

which will enable us to eliminate $d\,(\varphi' + \varphi'')$, as well as $\cos^2 \frac{1}{2}(\varphi' + \varphi'')$ from (6), we also have by spherics

$$\cos x = \cos y \cos x + \sin y \sin a \cos \varphi \quad \ldots\ldots \quad (8),$$

hence, $(\cos y \sin a \cos \varphi - \sin y \cos a)\, dy = \sin y \sin a \sin \varphi d\varphi$;

which will enable us to eliminate the differentials from (6), and we shall have an equation involving y, φ, and given quantities, which with (8) will give the values of y and φ as required.

SECOND SOLUTION, BY d.

Let the prime meridian pass through the given point, the origin being at the pole, which is at the distance y_2 from the point, then the second point, y_1, x_1, will be in a circle, whose pole is the first point, y_2 0, and its spherical radius the given distance between the points d; hence by equation (41), p. 37 of the Mathematical Miscellany,

$$\cos d = \cos y_2 \cos y_1 + \sin y_2 \sin y_1 \cos x_1 \quad \ldots\ldots \quad (1)$$

$$\therefore \cos x_1 = \frac{\cos d - \cos y_2 \cos y_1}{\sin y_2 \sin y_1},\ \sin x_1 = \frac{n}{\sin y_2 \sin y_1},$$

and
$$\frac{dx_1}{dy_1} = \frac{\cos d \cos y_1 - \cos y_2}{n \sin y_1} \quad \ldots\ldots\ldots \quad (2)$$

where $n = \sqrt{\{1 - \cos^2 y_1 - \cos^2 y_2 - \cos^2 d + 2 \cos y_1 \cos y_2 \cos d\}}$. Now, if s be the area of the triangle whose sides are y_1, y_2, and d, we have from known spherical principles

$$\cos \tfrac{1}{2}s = \frac{1 + \cos y_1 + \cos y_2 + \cos d}{4 \cos \frac{1}{2}y_1 \cos \frac{1}{2}y_2 \cos \frac{1}{2}d},\ \sin \tfrac{1}{2}s = \frac{n}{4 \cos \frac{1}{2}y_1 \cos \frac{1}{2}y_2 \cos \frac{1}{2}d},$$

and
$$\frac{ds}{dy_1} = \frac{\tan \frac{1}{2}y_1}{n}(1 + \cos y_1 - \cos y_2 - \cos d) \cdot \ \ldots \quad (3)$$

By equation (13), page 9, the equation of a loxodromic curve, passing through the point $y_2 0$, is

$$x = \tan \nu.\ \text{h. log.}\ (\tan \tfrac{1}{2}y_1 \cot \tfrac{1}{2}y) \quad \ldots\ldots\ldots \quad (4);$$

but the curve also passes through the point y_1, x_1, therefore

$$x_1 = \tan \nu\ .\ \text{h. log.}\ (\tan \tfrac{1}{2}y_2 \cot \tfrac{1}{2}y_1) \quad \ldots\ldots\ldots \quad (5).$$

If Σ be the area between y_2, y_1, and the curve, by equation (106), p. 47,

$$d\Sigma = (1 - \cos y)\, dx = -\tan \nu\ .\ dy \tan \tfrac{1}{2}y, \quad \ldots\ldots \quad (6),$$

and integrating this between $y = y_2$ and $y = y_1$, we get

$$\Sigma = 2 \tan \nu\ .\ \text{h. log.}\ \frac{\cos \frac{1}{2}y_1}{\cos \frac{1}{2}y_2},$$

or substituting the value of ν found by (5)

$$\Sigma = 2x_1\ .\ \frac{\text{h. log.}\ (\cos \frac{1}{2}y_1 \sec \frac{1}{2}y_2)}{(\text{h. log.}\ \cot \frac{1}{2}y_1 \tan \frac{1}{2}y_2)} \quad \ldots\ldots\ldots \quad (7).$$

Hence, by differentiating, we get

$$\frac{d\Sigma}{dy_1} = \frac{2dx_1}{dy_1} \cdot \frac{\text{h. log.}\ \dfrac{\cos. \frac{1}{2} y_1}{\cos. \frac{1}{2} y_2}}{\text{h. log.}\ \dfrac{\tan \frac{1}{2} y_2}{\tan \frac{1}{2} y_1}} + x_1 \frac{\text{h. log.}\ \dfrac{\sin y_1}{\sin y_2} + \cos y_1\ \text{h. log.} \dfrac{\tan \frac{1}{2} y_2}{\tan \frac{1}{2} y_1}}{\sin y_1\ .\ \left\{\text{h. log.}\ \dfrac{\tan \frac{1}{2} y_2}{\tan \frac{1}{2} y_1}\right\}^2} \quad (8)$$

Now the surface included between the great circle and the loxodromic is $= \Sigma - s$, and when this is a *max.* or *min.*

$$\frac{d\Sigma}{dy_1} - \frac{ds}{dy_1} = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9);$$

writing in (8), the values of $\frac{dx_1}{dy_1}$ and x_1 given in (1) and (2), and substituting (3) and (8) in (9), we have the final equation in y_1,

$$n \cos^{-1} \left\{ \frac{\cos d - \cos y_1 \cos y_2}{\sin y_1 \sin y_2} \right\} \times \left\{ \text{h. l.} \frac{\sin y_1}{\sin y_2} + \cos y_1 . \text{h. l.} \frac{\tan \frac{1}{2} y_2}{\tan \frac{1}{2} y_2} \right\}$$

$$+ 2 (\cos d \cos y_1 - \cos y_2) \times \text{h. log.} \frac{\cos \frac{1}{2} y_1}{\cos \frac{1}{2} y_2} \times \text{h. log.} \frac{\tan \frac{1}{2} y_2}{\tan \frac{1}{2} y_1}$$

$$- 2 \sin^2 \tfrac{1}{2} y_1 (1 + \cos y_1 - \cos y_2 - \cos d) \times \left\{ \text{h. l.} \frac{\tan \frac{1}{2} y_2}{\tan \frac{1}{2} y_1} \right\}^{2} = 0 \; (10).$$

$y_1 = y_2$ is one of the roots of this equation, and therefore the second point must be situated on a parallel of latitude passing through the first point.

QUESTION XI. BY MR. DAVID LANGDON, SCHENECTADY, N. Y.

A given sphere has a given rectangular perforation through its centre, what is the solidity of the remaining part?

FIRST SOLUTION, BY MR. D. LANGDON.

Put the length and breadth of the rectangular perforation $= 2a$ and $2b$ respectively. The perforation may be conceived to be generated by a plane, whose breadth is $2b$, moving parallel to itself from the distance a on one side of the centre of the sphere, to the distance a on the other side of it; the two sides of the plane being equidistant from the centre of the circle whose plane at any moment coincides with the moving plane, and the section of the sphere made by the plane is a zone of this circle, having equal chords at the distance $2b$ from each other. If r be the radius of this circle, x its distance from the centre, and R the radius of the sphere, so that $r = \sqrt{R^2 - x^2}$; the area of the variable generating zone will be

$$= 2r^2 \sin^{-1} \frac{b}{r} + 2b \sqrt{r^2 - b^2}$$

$$= 2(R^2 - x^2) \sin^{-1} \frac{b}{\sqrt{R^2 - x^2}} + 2b \sqrt{R^2 - b^2 - x^2}.$$

Hence if s be the content of the perforation, we have,

$$ds = 2dx (R^2 - x^2) \sin^{-1} \frac{b}{\sqrt{R^2 - x^2}} + 2b\,dx \sqrt{R^2 - b^2 - x^2};$$

and $s = \text{const.} + \frac{4}{3}bx\sqrt{R^2 - b^2 - x^2} + 2b(R^2 - \frac{1}{3}b^2)\sin^{-1}\frac{x}{\sqrt{R^2 - b^2}}$

$- 2x(R^2 - \frac{1}{3}x^2)\sin^{-1}\frac{b}{\sqrt{R^2 - x^2}} - \frac{4}{3}R^3\sin^{-1}\frac{bx}{\sqrt{(R^2 - b^2)(R^2 - x^2)}}$;

and between the limits $x = -a$ and $x = a$, this becomes

$$s = \frac{8}{3}ab\sqrt{R^2 - a^2 - b^2} + 4b(R^2 - \frac{1}{3}b^2)\sin^{-1}\frac{a}{\sqrt{R^2 - b^2}}$$

$$- 4a(R^2 - \frac{1}{3}a^2)\sin^{-1}\frac{b}{\sqrt{R^2 - a^2}} - \frac{8}{3}R^3\sin^{-1}\frac{ba}{\sqrt{(R^2 - b^2)(R^2 - a^2)}}.$$

This is the solidity of the perforation; that of the remaining solid will be $\frac{4}{3}R^3\pi - s$. The integrations would be quite as easy had we made the perforation in any other position, but the symmetry of the expressions, which is now somewhat remarkable, would be lost.

SECOND SOLUTION, BY PETRARCH, NEW-YORK.

Let the perforation be referred to three rectangular axes, intersecting in the centre, the axis of z coinciding with that of the perforation, the axis of y perpendicular to the side $2a$ of the perforation, and that of x perpendicular to the other side $2b$. We have

$$x^2 + y^2 + z^2 = r^2, \text{ and } z = \pm\sqrt{r^2 - y^2 - x^2}.$$

Now, if s be the solidity of the perforation, we have

$$d^3 s = dx\,dy\,dz;$$

integrating from $z = -\sqrt{r^2 - y^2 - x^2}$ to $z = +\sqrt{r^2 - y^2 - x^2}$,

$$d^2 s = 2dx\,dy\sqrt{r^2 - y^2 - x^2};$$

integrating again, from $y = -b$ to $y = b$,

$$ds = 2bdx\sqrt{r^2 - b^2 - x^2} + 2dx(r^2 - x^2)\tan^{-1}\frac{b}{\sqrt{r^2 - b^2 - x^2}}.$$

——(The third integration, from $x = -a$ to $x = a$, is precisely the same as that of Mr. Langdon.)

THIRD SOLUTION, BY PROF. AVERY.

Let r = radius of the sphere, a and b the length and breadth of the rectangular perforation, and d its semi-diagonal. If we subtract from the sphere the solidity of a rectangular prism, whose height $= 2\sqrt{r^2 - d^2}$, and the diagonal of whose end section $= 2d$, and to the remainder add the double of two second sections, whose heights $= a$ and b, breadths $= d - a$ and $d - b$, and lengths $= \frac{1}{2}\sqrt{r^2 - a^2} - \sqrt{r^2 - d^2}$ and $\frac{1}{2}\sqrt{r^2 - b^2}$

$-\sqrt{r^2-d^2}$, we shall obtain one half of the solidity of the remaining part of the sphere. The rule for calculating the second sections will be found in Hutton's Mensuration.

—— In this last manner, nearly, were the solutions by Messrs. Catlin, Docharty, and Root. Dr. Strong's method was very elegant.

QUESTION XII. BY P

Find a curve and its involute, such that the intercept of their common axis, between the ordinates to any two corresponding points of the curves, may be a constant line. Find, also, the areas and lengths of the two curves.

SOLUTION, BY PROF. BENJAMIN PEIRCE.

1. Let x and y be the co-ordinates of the curve,
x' and y' those of the involute;
we have, from all books on the Differential Calculus,

$$(x'-x)+(y'-y)\frac{dy'}{dx'}=0 \quad \ldots\ldots\ldots \quad (1),$$

$$1+\left(\frac{dy'}{dx'}\right)^2+(y'-y)\frac{d^2y'}{dx'^2}=0 \quad \ldots\ldots\ldots \quad (2)$$

Now, by the present hypothesis,

$$x'-x=\text{const.}=a \quad \ldots\ldots\ldots \quad (3),$$

let us also put $\dfrac{dy'}{dx'}=p'$, and these equations substituted in (1) and (2), give

$$\left.\begin{array}{r} a+(y'-y)\,p'=0 \\ 1+p'^2+(y'-y)\dfrac{dp'}{dx'}=0 \end{array}\right\} \quad \ldots \quad (4),$$

whence $$y'-y=-\frac{a}{p'},$$

and $$1+p'^2-\frac{a}{p'}.\frac{dp'}{dx'}=0.$$

Hence, $$dx'=\frac{adp'}{p'(1+p'^2)}=\tfrac{1}{2}a.\frac{d(p'^2)}{p'^2(1+p'^2)}$$
$$=\tfrac{1}{2}a\left(\frac{d(p'^2)}{p'^2}-\frac{d(p'^2)}{1+p'^2}\right);$$

the integral of which is, by a slight change,

$$\frac{2(x'+b)}{a}=\log.\frac{p'^2}{1+p'^2},$$

where b is the arbitrary constant quantity introduced by the integration; and, if e is the number whose Neperian logarithm is unity, we have

$$e^{\frac{2}{a}(x'+b)} = \frac{p'^2}{1+p'^2} \quad \ldots\ldots\ldots\ldots \quad (5),$$

whence putting, for the present,

$$\frac{2}{a}(x'+b) = 2n \quad \ldots\ldots\ldots\ldots \quad (6)$$

we have

$$e^{2n} = \frac{p'^2}{1+p'^2}$$

and

$$p'^2 = \frac{e^{2n}}{1-e^{2n}} \quad \ldots\ldots\ldots\ldots \quad (7).$$

Putting again

$$e^n = v \quad \ldots\ldots\ldots\ldots \quad (8)$$

we have $n = \log. v$, and by (6), $dn = \frac{dv}{v} = \frac{dx'}{a}$; whence (7) becomes

$$\left(\frac{dy'}{dx'}\right)^2 = \frac{v^2}{1-v^2}, \text{ or } \left(\frac{v}{a}\frac{dy'}{dv}\right)^2 = \frac{v^2}{1-v^2}$$

Hence

$$\frac{dy'}{a} = \frac{dv}{\sqrt{1-v^2}},$$

the integral of which is, introducing the arbitrary consonant c,

$$\frac{y'+c}{a} = \text{arc.} (\sin = v) \quad \ldots\ldots \quad (9),$$

and

$$\sin \frac{y'+c}{a} = v = e^{\frac{x'+b}{a}}$$

or,

$$\frac{x'+b}{a} = \log. \sin \frac{y'+c}{a} \quad \ldots\ldots \quad (10),$$

which is the equation of the involute.

2. We have from (9), $y' = a \cdot \text{arc.} (\sin = v) - c$,

and from (8),

$$dv = \frac{v\,dx'}{a};$$

whence $p' = \frac{dy'}{dx'} = \frac{v}{\sqrt{1-v^2}}$, which substituted in (4), give

$$y = y' + \frac{a}{p'} = \frac{a}{v}\sqrt{1-v^2} + a \,.\, \text{arc.} (\sin = v) - c \quad \ldots \quad (11).$$

But, by (3), $x' = x + a$, and by (8), $v = e^n = e^{\frac{x'+b}{a}} = e^{\frac{x+a+b}{a}}$; whence (11) becomes

$$y = a\sqrt{\left\{e^{-\frac{2}{a}(x+a+b)} - 1\right\}} + a \,.\, \text{arc.} (\sin = e^{\frac{x+a+b}{a}}) - c \quad (12)$$

which is the equation of the curve itself.

3. *To find the length of the involute.* Represent this length by s', and we have

$$\left(\frac{ds'}{dx'}\right)^2 = 1 + \left(\frac{dy'}{dx'}\right)^2 = 1 + p'^2 = 1 + \frac{v^2}{1-v^2} = \frac{1}{1-v^2},$$

and
$$ds' = \frac{adv}{v\sqrt{1-v^2}};$$

the integral of which is

$$s' = a \log.\left(\frac{1-\sqrt{1-v^2}}{v}\right) + const.$$

$$= a\log.\left(e^{-\frac{x'+b}{a}} - \sqrt{e^{-\frac{2}{a}(x'+b)} - 1}\right) + const.$$

4. *To find the length of the curve itself.* Represent this length by s, and also put $\frac{dy}{dx} = p$, and we have from (12),

$$p = \frac{-1}{v\sqrt{1-v^2}} + \frac{v}{\sqrt{1-v^2}} = -\frac{1-v^2}{v\sqrt{1-v^2}} = -\frac{\sqrt{1-v^2}}{v},$$

whence
$$\frac{ds^2}{dx^2} = 1 + \frac{1-v^2}{v^2} = \frac{1}{v^2},$$

and
$$ds = \frac{dx}{v} = e\,.\,e^{-\frac{x+b}{a}}\,.\,dx$$

$$\therefore s = const. - ae\,.\,e^{-\frac{x+b}{a}}.$$

5. *To find the area of the involute.* Represent this area by A', and we have

$$dA' = y'\,dx' = y'\,dy'\,.\,\text{cotan.}\left(\frac{y'+c}{a}\right),$$

the integral of which may be obtained by approximation.

6. *To find the area of the curve itself.* Represent this area by A, and we have

$$dA = ydx = \left(y' + \frac{a}{p'}\right)dx' = y'\,dx' + \frac{a\,dx'}{p'} = dA' + \frac{a^2dp'}{p'^2(1+p'^2)},$$

the integral of which is

$$A = A' + a^2\left[\text{arc.}\,(\tan = p') - \frac{1}{p'}\right],$$

$$= A' + a^2\left[\text{arc.}\left(\sin = e^{\frac{x+a+b}{a}}\right) - \sqrt{\left(e^{-\frac{2}{a}(x+a+b)} - 1\right)}\right].$$

Professor Catlin finds the length of the involute $= a.\ \mathrm{l.}\ \tan(45 + \frac{1}{2}z)$,
length of the evolute $= a(\sec.\ z - 1)$;
a being the constant intercept, and z the angle between the radius of curvature and the axis.

QUESTION XIII. BY P. J...

If from a given point in the axis of a parabola, perpendiculars be let fall upon the tangents of the curve, these right angles will be in a curve whose equation is $y^2(x-a) = x^2(b-x.)$ Now, if from the same point, perpendiculars be let fall upon the tangents of this second curve, it is required to find the locus of the right angles so formed—determine its inflexions and length, and, in the case where $b=0$, determine its area.

SOLUTION, BY THE PROPOSER.

Let $y'\ x'$, $y\ x$ and $\mathrm{Y}\mathrm{X}$ be the co-ordinates of contemporaneous points in the parabola, and the first and second tangent curves; the axis of the parabola being the axis of x, and a perpendicular to it through the given point the axis of y. If a be the distance from the vertex to the focus of the parabola, and b the distance from the vertex to the given point, the parabola's equation is

$$y'^2 = 4a(b - x') \quad \ldots\ldots\ldots\ldots \quad (1),$$

the equation of a tangent through $y'\ x'$

$$y - y' = \frac{dy'}{dx'}(x - x') \quad \ldots\ldots\ldots \quad (2),$$

and that of a perpendicular to it through the origin

$$y = -\frac{dx'}{dy'}.\ x \quad \ldots\ldots\ldots\ldots \quad (3).$$

From equation (2) and (3), we easily find

$$y = \frac{y' - x'.\ \frac{dy'}{dx'}}{\frac{dy'^2}{dx'^2} + 1}, \text{ and } x = \frac{\frac{dy'}{dx'}\left(x'.\ \frac{dy'}{dx'} - y'\right)}{\frac{dy'^2}{dx'^2} + 1} \quad \ldots \quad (4),$$

which, together with the equation of the generating curve, will enable us to find that of any curve generated in this manner. From equation (1) we have $\frac{dy'}{dx'} = \frac{-2a}{y'} = -\sqrt{\frac{a}{b-x'}} = \frac{-x}{y}$, from (3); hence $y' = \frac{2ay}{x}$, $x' = b - \frac{ay^2}{x^2}$, and substituting these in either of the equations (4), we find the equation of the first curve

$$y^2 = x^2.\ \frac{b-x}{x-a} \quad \ldots\ldots\ldots\ldots \quad (5),$$

$$\therefore \frac{dy}{dx} = \frac{-2x^2 + (b+3a)\,x - 2ab}{2\,(b-x)^{\frac{1}{2}}.\,(x-a)^{\frac{3}{2}}} \quad \ldots\ldots \quad (6),$$

and
$$\frac{d^2y}{dx^2} = \frac{(b-a)\,\{4ab - (b+3a)\,x\}}{4\,(b-x)^{\frac{3}{2}}.\,(x-a)^{\frac{5}{2}}} \quad \ldots\ldots \quad (7).$$

By the equations (4), we shall also have

$$\text{Y} = \frac{y - x.\dfrac{dy}{dx}}{\dfrac{dy^2}{dx^2}+1} = \frac{2x^2\,(b-x)^{\frac{1}{2}}\,(x-a)^{\frac{3}{2}}}{(b+3a)\,x^2 - 4a\,(b+a)\,x + 4a^2 b} \quad . \quad (8),$$

$$\text{X} = \frac{\dfrac{dy}{dx}\left(x.\dfrac{dy}{dx} - y\right)}{\dfrac{dy^2}{dx^2}+1} = \frac{x^2\,\{2x^2 - (b+3a)\,x + 2ab\}}{(b+3a)\,x^2 - 4a\,(b+a)\,x + 4a^2 b} \quad . . \quad (9).$$

Where it will be understood, that the radicals in (6), (7), and (8), can be either $+$ or $-$, consequently that the axis is a diameter of both curves. By eliminating x between these two equations we shall get the equation of the required curve; or if its polar equation be preferred, it may be found from the equations

$$\text{V}^2 = \text{Y}^2 + \text{X}^2 = \frac{\left(x.\dfrac{dy}{dx} - y\right)^2}{\dfrac{dy^2}{dx^2}+1} = \frac{(b-a)\,x^4}{(b+3a)x^2 - 4a(b+a)x + 4a^2 b} \; (10),$$

$$\tan\theta = \frac{\text{Y}}{\text{X}} = -\frac{dx}{dy} = \frac{2(b-x)^{\frac{1}{2}}.(x-a)^{\frac{3}{2}}}{2x^2 - (b+3a)x + 2ab} \quad \ldots\ldots \quad (11),$$

by the elimination of x; V being the radius vector, θ the angle it makes with the axis, and the given point the pole. The final rectilineal equation will be

$$\begin{aligned} &4(\text{Y}^2+\text{X}^2)^4 - \{4(b+3a)\text{X} + b^2 + 18ab - 27a^2\}\,(\text{Y}^2+\text{X}^2)^3 \\ &+ \{(b^2+30ab-15a^2)\text{X}^2 + 4ab(5b-9a)\text{X} + 4ab^3\}(\text{Y}^2+\text{X}^2)^2 \\ &- 4a(b-a)\,\{(5b-a)\text{X} + 2b^2\}\,\text{X}^2(\text{Y}^2+\text{X}^2) + 4ab(b-a)^2\text{X}^4 = 0 \;.\; (12); \end{aligned}$$

this in the case where $b=0$ (*fig.* 3) reduces to

$$4(\text{Y}^2+\text{X}^2)^3 - 12a\text{X}(\text{Y}^2+\text{X}^2)^2 + 3a^2(9\text{Y}^2+4\text{X}^2)(\text{Y}^2+\text{X}^2) - 4a^3\text{X}^3 = 0 \; (13);$$

and the polar equation of this curve is

$$\text{V} = a\cos\theta + \tfrac{3}{2}a\sqrt[3]{2}.\,\sin^{\frac{2}{3}}\theta\,(\sin^{\frac{4}{3}}\tfrac{1}{2}\theta - \cos^{\frac{4}{3}}\tfrac{1}{2}\theta).$$

But equations (8) and (9) are much more convenient, and from these we have

$$\left(\frac{d\text{Y}}{dx} = \frac{-x\,(x-a)^{\frac{1}{2}}}{(b-x)^{\frac{1}{2}}} \times\right) \frac{\{(b+3a)x - 4ab\}\,\{4x^3 - 3(b+3a)x^2 + 4a(2b+a)x - 4a^2 b\}}{\{(b+3a)x^2 - 4a(b+a)x + 4a^2 b\}^2} \;.\; (14).$$

$$\frac{d\text{x}}{dx}=\frac{x\{(b+3a)x-4ab\}\{4x^3-(b+11a)x^2+4a(b+2a)x-4a^2b\}}{\{(b+3a)x^2-4a(b+a)x+4a^2b\}^2} \quad (15),$$

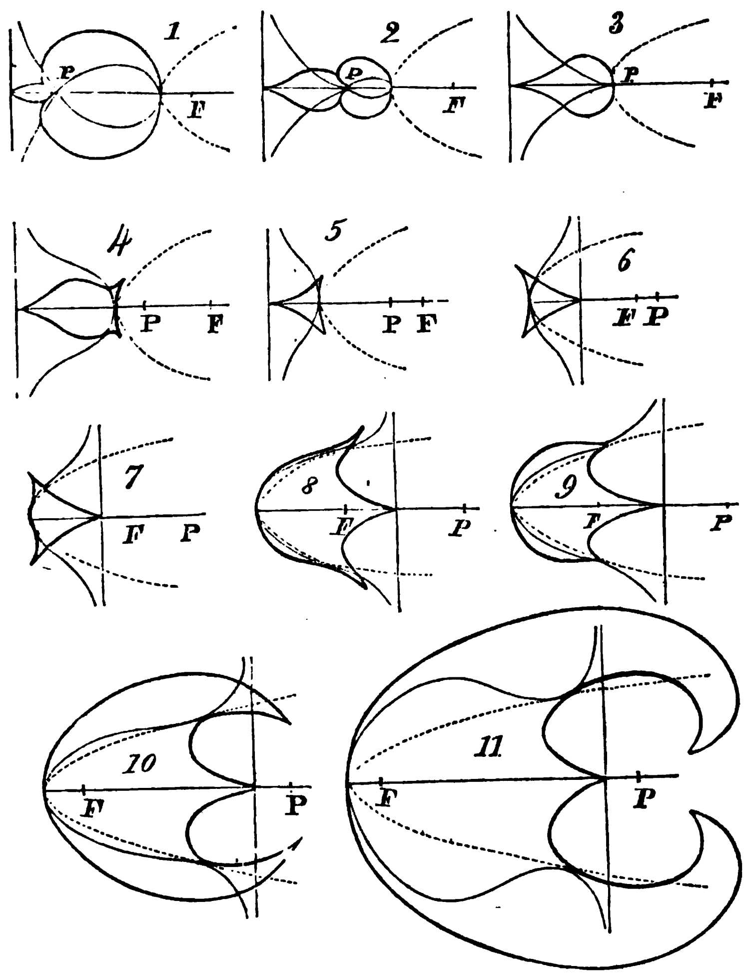
$$\frac{d\text{y}}{d\text{x}}=-\left(\frac{x-a}{b-x}\right)^{\frac{1}{2}}\cdot\frac{4x^3-3(b+3a)x^2+4a(2b+a)x-4a^2b}{4x^3-(b+11a)x^2+4a(b+2a)x-4a^2b} \quad (16),$$

$$\frac{d^2\text{y}}{d\text{x}^2}=\frac{1}{d\text{x}}.\,d\left(\frac{d\text{y}}{d\text{x}}\right)=\frac{(a-b)\{3(b+3a)x^2-8a(2b+a)x+12a^2b\}}{2x(x-a)^{\frac{1}{2}}(b-x)^{\frac{3}{2}}.\{(b+3a)x-4ab\}}\times$$
$$\frac{\{(b+3a)x^2-4a(b+a)x+4a^2b\}^3}{\{4x^3-(b+11a)x^2+4a(b+2a)x-4a^2b\}^3} \quad \cdot\cdot (17).$$

The limits of a solution will scarcely enable us to give even the results arrived at in investigating the properties of these beautiful curves. Equation (5) shows that x must be taken either at or between the limits of a and b; at the former limit there is always an assymptote to the first curve; when $b<0$, $x=0$ is within these limits, and the curve passes twice through the origin, and at this point $\frac{dy}{dx}=\pm\sqrt{\frac{-b}{a}}$, while $\frac{d\text{y}}{d\text{x}}=\mp\sqrt{\frac{a}{-b}}$, and $\frac{d^2\text{y}}{d\text{x}^2}=\infty$, *fig.* 1 *and* 2; thus in the second curve there are two points of *rebroussement*, having their apexes at the origin, and the two tangents at this point make a less angle with each other as b decreases, when $b=-\infty$ they coincide, and when $b=0$, they again coincide, being both perpendicular to the axis, (*fig.* 3). When $x=b$, $\frac{dy}{dx}=$, and $\frac{d\text{y}}{d\text{x}}=\infty$, or at this point both curves are always perpendicular to the axis, except in the cases $b=0$, when the first curve is the Cissoid, (*fig.* 3), having $\frac{dy}{dx}=0$, and $b=a$, when the first curve is a straight line, and the second a point. The most remarkable singular points in the curves are those indicated by the value $x=\frac{4ab}{b+3a}$, when a factor in $\frac{d^2y}{dx^2}$, $\frac{d\text{y}}{dx}$, $\frac{d\text{x}}{dx}$, and $\frac{d^2\text{y}}{d\text{x}^2}$ simultaneously vanish, showing a change of inflexion of both curves and a limit to the values of Y and X, and consequently a point of *rebroussement*, in the second curve. For this value we have

$$y=\frac{4b}{b+3a}.\sqrt{\frac{ab}{3}},\ \frac{dy}{dx}=\frac{b-9a}{3(b-a)}.\sqrt{\frac{b}{3a}},\ \frac{d^2y}{dx^2}=0,$$
$$\text{Y}=\frac{24ab\,(b-a)\,.\,\sqrt{3ab}}{(b+3a)^3},\ \text{X}=\frac{8ab^2\,(9a-b)}{(b+3a)^3},$$
$$\frac{d\text{Y}}{d\text{X}}=-\frac{5b^2-30ab\,.+9a^2}{b^2-30ab+45a^2}.\sqrt{\frac{3a}{b}},\ \frac{d^2\text{Y}}{d\text{X}^2}=\infty.$$

Hence the point does not come within the proper limits until $b>0$; as b increases towards a, (*figs.* 4, 5), the inflecting point approaches the assymptote, until the curve coincides with its assymptote when $b=a$; beyond this value the point recedes from the assymptote, as that line itself re-



cedes from the common vertex of the curves, (*figs.* 6. . . . 11). If ψ be the angle the tangent at this point makes with the axis, so that tan. $\psi = \frac{dy}{dx}$; ψ increases from 0°, when $b = 0$ passes through 90° when $b = a$, and through 180° when $b = 9a$, when the tangent is parallel to the axis, and it approaches to 270° as its limit as b continually increases.

The value of Y at this point is negative, beginning at 0 when $b = 0$, decreasing until $b = (9 - 6\sqrt{2})a$, it then increases passing through zero when $b = a$, and is greatest when $b = (9 + 6\sqrt{2})a$, it then decreases again, approaching to zero as its limit as b continually increases. X is $< b$ from

$b = 0$ to $b = a$, and is farthest from b when $b = .831a$, it is equal b, when $b = a$, it then becomes $> b$, attaining its greatest difference when $b = 1,2a$, it then continues to decrease passing through the magnitude b again, when $b = 1.3923a$, through $2a$ when $b = 3a$, through zero when $b = 9a$, and approaches to the limit of $-8a$ as b continually increases. If w be the angle made by the tangent of the second curve at the point of *rebroussement*, so that $\tan w = \frac{d\text{Y}}{d\text{X}} = -\frac{5b^2 - 30ab + 9a^2}{b^4 - 30ab + 45a}\sqrt{\frac{3a}{b}}$. Then w commences at 90°, when $b = 0$ and increases as b increases; $w = 180^\circ$ when $b = (3 - 6\sqrt{\tfrac{1}{5}})a$; and beginning there at 0°, it passes through 60° when $b = a$, 90° when $b = (15 - 6\sqrt{5})a$, and 180° when $b = (3 + 6\sqrt{5})a$; beginning here again at 0° it increases to 90° when $b = (15 + 6\sqrt{_5})a$ and approaches to 180° as its limit, while b continues to increase.

When $b > 2a$, there is a point in the curves having $x = 2a$, and there $y' = y = \text{Y}$, moreover $\frac{dy'}{dx'} = \frac{dy}{dx} = \frac{d\text{Y}}{d\text{X}}$ at that point, so that the three curves touch each other there. $2a > \frac{4ab}{b + 3a}$ when $b < 3a$, they are $=$ when $b = 3a$, and when $b > 3a$, $2a < \frac{4ab}{b + 3a}$; so that the inflection of the first curve and the *rebroussement* of the second occurs after they have been in contact when $b < 3a$, (*fig.* 8), at the point of contact when $b = 3a$ (*fig.* 9), and before the contact when $b > 3a$, (*figs.* 10, 11); and, in fact, X is always $< x$ when $x > 2a$. and $\text{X} > x$, when $x < 2a$.

By examining the other factors of $\frac{d^2\text{Y}}{d\text{X}^2}$, we shall find other singular points of the second curve. The factor $(b + 3a)x^2 - 4a(b + a)x + 4a^2b$ never changes its sign, through all the variations of b, between the limits of $x = a$, and $x = b$; but if we equate the factor $3(b + 3a)x^2 - 8a(2b + a)x + 12a^2b$ to zero, we shall have

$$x = \frac{4a(2b+a) + 2a\sqrt{(b-a)(7b-4a)}}{3(b+3a)}, \quad \ldots \ldots \quad (18),$$

and

$$x = \frac{4a(2b+a) - 2a\sqrt{(b-a)(7b-4a)}}{3(b+3a)} \quad \ldots \quad (19),$$

When $b < 0$, the value of x in (18) is a little less than a, producing a point of inflection in the second curve near the assymptote of the first, (*figs.* 1, 2, 3), while x in (19) is without the limits of a and b; between $b = 0$, and $b = \frac{4}{7}a$, they are both within the limits, and produce two points of inflexion, (*fig.* 4), when $b = 0$, the one is at $x = \frac{8}{9}a$, and the other at $x = 0$, and they approach each other as b increases until, when $b = \frac{4}{7}a$, they coincide. Between $b = \frac{4}{7}a$, and $b = a$, both (18) and (19) are imaginary, (*fig.* 5), and between $b = a$ and $b = \frac{4}{3}a$, the values of x in (18) and (19) are both beyond the limits of a and b, (*fig.* 6), in fact the latter is always so when $b > \frac{4}{3}a$, but when $b > \frac{4}{3}a$ the value of x in (18) is a little greater

than $\frac{4ab}{b+3a}$, producing a point of inflection immediately before that of *rebroussement* (*figs.* 7. . . . 11).* The tangent at this point is perpendicular to the axis when $b = 17\frac{11}{29}a$, in other cases its position is sufficiently indicated by the figures. It follows that the *rebroussement* is always of the first kind.

Let us put the factor $4x^3 - (b + 11a)x^2 + 4a(b + 2a)x - 4a^2b$ into the form $4(x - \alpha)(x - \beta)(x - \gamma)$. Between $b = -\infty$, and $b = 17\frac{11}{29}a$, β and γ are imaginary; when $b > 17\frac{11}{29}a$, they are re ' ; the point corresponding to $x = \beta$ occurs immediately before the point of inflexion, then $\frac{d\mathrm{x}}{dx} = 0$, $\frac{d\mathrm{Y}}{d\mathrm{x}} = \infty$, and $\frac{d^2\mathrm{Y}}{d\mathrm{x}^2} = \infty$, therefore the tangent is perpendicular to the axis and x is a *min.* The same is the case at the point where $x = \gamma$, which occurs between the points of inflection and *rebroussement*, while b is between $17\frac{11}{29}a$ and $(15 + 6\sqrt{5})a$, and when $b > (15 + 6\sqrt{5})a$, it occurs after the *rebroussement*, (*fig.* 11.) A third limit of x, and consequent inflexion occurs at the point where $x = \alpha$, which is within the proper limits when b is between $-\infty$ and 0, α being then a little less than 0, (*figs.* 1, 2); and when $b > \frac{4}{3}a$, the point where $x = \alpha$ recedes from the vertex between the points of inflexion and *rebroussement*, (*fig.* 7), passing through the latter point when $b = (15 - 6\sqrt{5})a$, and when $b > (15 - 6\sqrt{5})a$ it occurs after the *rebroussement*, (*figs.* 8. . . . 11). When $b > 0$, the origin of co-ordinates is a *conjugate* point belonging to both curves.

Rectification. If z be the length of the curve, we have from equations (14) and (15)

$$dz = \sqrt{d\mathrm{Y}^2 + d\mathrm{x}^2} = dx\sqrt{\frac{b-a}{b-x} + \frac{4a^2\,dx\sqrt{(b-a)(b-x)}}{(b+3a)x^2 - 4a(b+a)x + 4a^2b}} \quad . \quad . \quad (20).$$

Integrating this between $x = a$ and $x = b$, and doubling the result, we have for the length of the whole curve,

when $b < -3a$

$$z = 4(a-b) - \frac{2b}{p}\sqrt{a^2-ab}.\ \mathrm{h.\ log.}\ \frac{b+p}{b-p} + \frac{2b}{q}\sqrt{a^2-ab}.\ \mathrm{h.\ log.}\ \frac{b+q}{b-q},$$

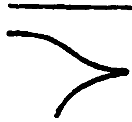
when $b > -3a$ and $< a$

$$z = 4(a-b) - \frac{2b}{p}\sqrt{a^2-ab}.\ \mathrm{h.\ l.}\ \frac{b+p}{b-p} + \frac{2b}{q\sqrt{-1}}\sqrt{a^2-ab}\left(\frac{\pi}{2} - \tan^{-1}\frac{b}{q\sqrt{-1}}\right);$$

where p and q are such that

$$p^2 + q^2 = 2(b-a)(2a+b),$$
$$p^2 - q^2 = 4a\sqrt{a^2-ab};$$

when $b = 0$



* This point, through the inadvertence of the engraver, is not exhibited in figs. 10 and 11; the inflection and rebroussement, when $b > 3a$, is shown by the figure in the margin.

$$z = \frac{8a}{\sqrt{3}}. \text{ h. log. } (2 + \sqrt{3}) - 4a = 2{,}082768a,$$

and when $b > a$

$$z = 4(b-a) + \frac{2a^2}{r}. \sqrt{\frac{b-a}{b+3a}}. \text{ h. log. } \frac{(b+r)^2+s^2}{(b-r)^2+s^2}$$
$$+ \frac{4a^2}{s} \sqrt{\frac{b-a}{b+3a}}, \tan^{-1} \frac{2bs}{r^2+s^2-b^2};$$

where r and s are such that

$$r^2 + s^2 = b\sqrt{(b-a)(b+3a)},$$
$$r^2 - s^2 = (b-a)(b+2a).$$

Area. The curve is always quadrable, either by circular arcs or logarithms, but the formulas would be too complicated for insertion here. Taking the case of $b = 0$, (*fig.* 3), and putting φ for the polar angle of the first curve so that $x = a \sin^2\varphi$, equations (10) and (11) give

$$v^2 = \frac{x^3}{4a-3x} = \frac{2a^2 \sin^6\varphi}{5+3\cos 2\varphi},$$

$$\tan\theta = \frac{2(a-x)^{\frac{3}{2}}}{(3a-2x)x^{\frac{1}{2}}} = \frac{2\cos^3\varphi}{\sin\varphi\,(2+\cos 2\varphi)},$$

$$\therefore d\theta = \frac{d\tan\theta}{1+\tan^2\theta} = \frac{-12\cos^2\varphi d\varphi}{5+3\cos 2\varphi}.$$

Hence the area between the curve, axis and radius vector v is

$$= \tfrac{1}{2}\int v^2 d\theta = -3a^2 \int \frac{\sin^4\varphi \sin^2 2\varphi d\varphi}{(5+3\cos 2\varphi)^2}$$

$$= a^2 \left[\frac{31}{24}\varphi - \tfrac{4}{3}\tan^{-1}(\tfrac{1}{2}\tan\varphi) - \tfrac{2}{9}\sin 2\varphi + \tfrac{1}{96}\sin 4\varphi - \tfrac{4}{9}.\frac{\sin 2\varphi}{5+3\cos 2\varphi}\right],$$

This taken twice between the limits $\varphi = 0$ and $\varphi = 90°$, gives the area of the whole curve $= \frac{1}{6}$ of the circle whose diameter is a, or $\frac{1}{18}$ of the space between the assymptote and curve of the generating Cissoid.

QUESTION XIV, BY S. S.

Find the sum of n terms of the series
$\cos\varphi \cos 2\varphi + 2\cos 2\varphi \cos 4\varphi + 3\cos 3\varphi . \cos 6\varphi +$, &c.

FIRST SOLUTION, BY THE PROPOSER.

For this series we have the general term

$$u_n = n\cos n\varphi \cos 2n\varphi$$

and $Su_n = const. + u_n + \Sigma.\, u_n$

$$= const. + n\cos n\varphi \cos 2n\varphi + \Sigma.\, n\cos n\varphi \cos 2n\varphi$$

$$= const. + n \cos n\varphi \cos 2n\varphi + (n-1)\,\Sigma.\cos n\varphi\, 2n\varphi$$
$$- \Sigma^2.\cos n\varphi \cos 2n\varphi \qquad (1),$$

using Taylor's formula for integrating by parts.

Now $\quad \Sigma.\cos n\varphi \cos 2n\varphi = \frac{1}{2}\Sigma.\cos 3n\varphi + \frac{1}{2}\Sigma.\cos n\varphi$

$$= \frac{\sin 3(n-\frac{1}{2})\varphi}{4\sin\frac{3}{2}\varphi} + \frac{\sin(n-\frac{1}{2})\varphi}{4\sin\frac{1}{2}\varphi},$$

and $\Sigma^2.\cos n\varphi\cos 2n\varphi = -\dfrac{\cos 3(n-1)\varphi}{8\sin^2\frac{3}{2}\varphi} - \dfrac{\cos(n-1)\varphi}{8\sin^2\frac{1}{2}\varphi}$,

and substituting these in (1),

$$Su_n = const. + n\cos n\varphi\cos 2n\varphi + \frac{(n-1)\sin 3(n-\frac{1}{2})\varphi}{4\sin\frac{3}{2}\varphi} + \frac{(n-1)\sin(n-\frac{1}{2})\varphi}{4\sin\frac{1}{2}\varphi}$$
$$+ \frac{\cos 3(n-1)\varphi}{8\sin^2\frac{3}{2}\varphi} + \frac{\cos(n-1)\varphi}{8\sin^2\frac{1}{2}\varphi} \quad \ldots\ldots \quad (2).$$

Let $n=1$, then

$$Su_1 = \cos\varphi\cos 2\varphi = const. + \cos\varphi\cos 2\varphi + \frac{1}{8\sin^2\frac{3}{2}\varphi} + \frac{1}{8\sin^2\frac{1}{2}\varphi},$$

$$\therefore const. = \frac{-1}{8\sin^2\frac{3}{2}\varphi} - \frac{1}{8\sin^2\frac{1}{2}\varphi},$$

and writing this in (2), and reducing

$$Su_n = n\cos n\varphi\cos 2n\varphi + \frac{(n-1)\sin 3(n-\frac{1}{2})\varphi}{4\sin\frac{3}{2}\varphi} + \frac{(n-1)\sin(n-\frac{1}{2})\varphi}{4\sin\frac{1}{2}\varphi}$$
$$- \frac{\sin^2\frac{3}{2}(n-1\varphi)}{4\sin^2\frac{3}{2}\varphi} - \frac{\sin^2\frac{1}{2}(n-1)\varphi}{4\sin^2\frac{1}{2}\varphi}$$
$$= \frac{n\sin(n+\frac{1}{2})\varphi(1+\cos\varphi+\cos(2n+1)\varphi)}{2\sin\frac{3}{2}\varphi} - \frac{\sin^2\frac{3}{2}n\varphi}{4\sin^2\frac{3}{2}\varphi} - \frac{\sin^2\frac{1}{2}n\varphi}{4\sin^2\frac{1}{2}\varphi},$$

which is the sum of n terms.

SECOND SOLUTION, BY PROF. AVERY, HAMILTON COLLEGE, N. Y.

$$\cos\varphi\cos 2\varphi + 2\cos 2\varphi\cos 4\varphi + 3\cos 3\varphi\cos 6\varphi + \&c.$$
$$= \tfrac{1}{2}(\cos\varphi + 2\cos 2\varphi + 3\cos 3\varphi + \&c.) + \tfrac{1}{2}(\cos 3\varphi + 2\cos 6\varphi + 3\cos 9\varphi + \&c.)$$
$$= \tfrac{1}{2}.\frac{ds}{d\varphi} + \tfrac{1}{6}.\frac{ds'}{d\varphi}.$$

Where, (see Lardner's Calculus, p. 519).

$$s = \sin\varphi + \sin 2\varphi + \sin 3\varphi + \ldots\ldots \sin n\varphi = \frac{\cos\frac{1}{2}\varphi - \cos(n+\frac{1}{2})\varphi}{2\sin\frac{1}{2}\varphi},$$

$$s' = \sin 3\varphi + \sin 6\varphi + \sin 9\varphi + \ldots\ldots \sin 3n\varphi = \frac{\cos\frac{3}{2}\varphi - \cos 3(n+\frac{1}{2})\varphi}{2\sin\frac{3}{2}\varphi};$$

$$\therefore \frac{ds}{d\varphi} = \frac{n\sin(n+\frac{1}{2})\varphi}{2\sin\frac{1}{2}\varphi} - \frac{\sin^2\frac{1}{2}n\varphi}{2\sin^2\frac{1}{2}\varphi},$$

$$\text{and } \frac{ds'}{d\varphi} = \frac{3n \sin 3(n+\frac{1}{2})\varphi}{2 \sin \frac{3}{2}\varphi} - \frac{3 \sin^2 \frac{3}{2}\varphi}{2 \sin^{2} \frac{3}{2}\varphi};$$

Hence, $\cos\varphi \cos 2\varphi + \cos 2 \cos 4\varphi + \ldots \, n \cos n\varphi \cos 2n\varphi$

$$= \frac{n \sin (n+\frac{1}{2})\varphi}{4 \sin \frac{1}{2}\varphi} + \frac{n \sin 3\,(n+\frac{1}{2})\varphi}{4 \sin \frac{3}{2}\varphi} - \frac{\sin^2 \frac{1}{2}n\varphi}{4 \sin^2 \frac{1}{2}\varphi} - \frac{\sin^2 \frac{3}{2}n\varphi}{4 \sin^2 \frac{3}{2}\varphi}.$$

—— We intended to insert Mr. Perkins' solution to this question; but the compositor found so much difficulty in representing the exponential terms, that we were obliged to abandon the attempt. Dr. Strong's second solution, unfortunately, did not arrive in time for insertion.

QUESTION XV. BY MR. JAMES F. MACULLY.

Three circles tangent to each other, are given on a plane; to find the greatest or least ellipse that touches all the three circles.

SOLUTION, BY Δ.

Let the radii of the three circles be r_1, r_2, r_3; a line through the centres of r_2 and r_3 the axis of x, and a common tangent to the same two circles the axis of y; the co-ordinates of the centre of r_1, m, n; those of r_2, 0, $-r_2$; and those of r_3, 0, r_3; the conditions of touching give

$$\left.\begin{aligned}(r_2+n)^2+m^2&=(r_1+r_2)^2\\(r_3-n)^2+m^2&=(r_1+r_3)_2\end{aligned}\right\}\quad \ldots\ldots (1);$$

these equations give

$$n=r_1.\frac{r_2-r_3}{r_2+r_3},\ m^2=\frac{4r_1\,r_2\,r_3(r_1+r_2+r_3)}{(r_2+r_3)^2}\quad \ldots\ldots (2),$$

in terms of the given radii.

Now let the co-ordinates of the centre of the ellipse be k, l; its eccentricity e; its two semi-axes a, b; the one making the angle φ, and the other $\frac{1}{2}\pi+\varphi$, with the axis of x; then if the three centres be referred to the axes of the ellipse as axes of co-ordinates, the centre of r_1 being y_1, x_1, that of r_2, y_2, x_2, and that of r_3, y_3, x_3, we shall have from the usual formulas for transformation,

$$\left.\begin{aligned}y_1&=(m-k)\cos\varphi-(n-l)\sin\varphi,\ x_1=(m-k)\sin\varphi+(n-l)\cos\varphi;\\y_2&=\ \ -k\cos\varphi+(r_2+l)\sin\varphi,\ x_2=\ \ -k\sin\varphi-(r_2+l)\cos\varphi;\\y_3&=\ \ -k\cos\varphi-(r_3-l)\sin\varphi,\ x_3=\ \ -k\sin\varphi+(r_3-l)\cos\varphi;\end{aligned}\right\}\ (3).$$

The equation of the ellipse referred to these new axes, is

$$a^2y^2+b^2x^2=a^2b^2\quad \ldots\ldots\ldots\ldots (4);$$

and a normal through the point yx of the ellipse and the centre y_1x_1 will have

$$y_1-y=\frac{a^2y}{b^2x}(x_1-x)\quad \ldots\ldots\ldots\ldots (5);$$

but the distance from the point of contact to the centre is r_1, or

$$(y_1-y)^2+(x_1-x)^2=r^{12}\quad \ldots\ldots\ldots\ldots (6)$$

Eliminating y and x between the three equations (4), (5), (6), a process which need not be repeated here, we get

$$4(p_1{}^2-3a^2 q_1)(q_1{}^2-3e^2 p_1 x_1{}^2)=(p_1 q_1-9a^2 e^2 x_1{}^2)^2 \quad (7),$$

where
$$\left.\begin{aligned} p_1 &= x_1{}^2+y_1{}^2+a^2(1+e^2)-r_1{}^2 \\ q_1 &= x_1{}^2+y_1{}^2+e^2(x_1{}^2+a^2-r_1{}^2 \end{aligned}\right\} \ldots\ldots (8)$$

In like manner, the contact of the ellipse with the circles r_2 and r_3 give the equations

$$4(p_2{}^2-3a^2 q_2)(q_2{}^2-3e^2 p_2 x_2{}^2)=(p_2 q_2-9a^2 e^2 x_2{}^2)^2 \quad . \quad (9),$$
$$4(p_3{}^2-3a^2 q_3)(q_3{}^2-3e^2 p_3 x_3{}^2)=(p_3 q_3-9a^2 e^2 x_3{}^2)^2 \quad . \quad (10,)$$

p_2, q_2; p_3, q_3; being the same functions of y_2, x_2, r_2; y_3, x_3, r_3 that p_1, q_1, are of y_1, x_1, r_1, in equations (8). If now the values of y_1, x_1, y_2, &c., given in equations (3), be substituted in equations (7). (9), 10, there will be three equations involving a, e, k, l, φ and known quantities. These equations are too complicated for insertion here; we can represent them by

$$u_1=0,\ u_2=0\ u_3=0 \quad \ldots\ldots\ldots\ldots \quad (11).$$

We have also the area of the ellipse $= ab\pi =$ a *max.* or *min.*

$$\therefore a^2 b^2 = a^4(1-e^2) = \text{a } max. \text{ or } min.$$

and differentiating, and dividing by $2a^3$,

$$2(1-e^2)\,da - aede = 0 \quad \ldots\ldots\ldots \quad (12).$$

Differentiate the three equations (11), multiplying them severally by the indeterminates λ, μ, ν, and add the results to (12); then equating the coefficients of the several differentials to zero, we shall have

$$\left.\begin{aligned}
&\lambda.\frac{du_1}{da}+\mu.\frac{du_2}{da}+\nu.\frac{du_3}{da}+2(1-e^2)=0,\\
&\lambda.\frac{du_1}{de}+\mu.\frac{du_2}{de}+\nu.\frac{du_3}{de}-ae=0,\\
&\lambda.\frac{du_1}{dk}+\mu.\frac{du_2}{dk}+\nu.\frac{du_3}{dk}=0,\\
&\lambda.\frac{du_1}{dl}+\mu.\frac{du_2}{dl}+\nu.\frac{du_3}{dl}=0,\\
&\lambda.\frac{du_2}{d\varphi}+\mu.\frac{du_2}{d\varphi}+\nu.\frac{du_3}{d\varphi}=0,
\end{aligned}\right\} \quad (13).$$

Eliminating the indeterminates λ, μ, ν there will be two equations, which, together with the three in (11), will determine the quantities a, e, k, l, φ, and consequently the magnitude and position of the ellipse.

QUESTION XVI. BY Δ.

If the quadrantal arc of a great circle revolve so that its extremities are always in two given great circles, to find the equation of the curve traced on the surface of the sphere by the centre of rotation of the revolving arc.

FIRST SOLUTION, BY PROF. B. PEIRCE.

The equation of the great circle whose quadrantal arc is the revolving one is, as given by Delta, in the Mathematical Miscellany,

$$\cot y \cot \omega + \cos(\varphi - x) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

The origin of co-ordinates we will suppose to be at the intersection of the given circles, and the *prime meridian* to bisect their angle, which we will represent by $2A$. We shall have then for the points of intersection with these circles

$$\left.\begin{array}{l}\cot y' \cot \omega + \cos(\varphi - A) = 0 \\ \cot y'' \cot \omega + \cos(\varphi + A) = 0\end{array}\right\} \quad . \quad . \quad . \quad . \quad (2),$$

also, since the intercepted portion is a quadrant we have

$$\cot y' \cot y'' + \cos 2A = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3);$$

the product of equations (2), after transposing their second terms, subtracted from (3) multiplied by $\cot^2 \omega$ is

$$\cot^2 \omega \cos 2A + \cos(\varphi + A) \cos(\varphi - A) = 0 \qquad (4);$$

the square of (1) after transposing its second member, is

$$\cot^2 \omega \cot^2 y - \cos^2(\varphi - x) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

which, multiplied by $\cos 2A$ and subtracted from (4) multiplied by $\cot^2 y$, leaves

$$\cot^2 y \cos(\varphi + A) \cos(\varphi - A) + \cos 2A \cos^2(\varphi - x) = 0 \qquad (6),$$

which is easily transformed into

$$\cot^2 y \cos 2\varphi + \cos 2A \cos 2(\varphi - x) = -\cos 2A \operatorname{cosec} 2y \quad . \quad (7).$$

But as the *centre of rotation* is at once upon the two consecutive positions of the quadrant, we may differentiate (7), supposing φ the only variable, and we obtain

$$\cot^2 y \sin 2\varphi + \cos 2A \sin 2(\varphi - x) = 0 \quad . \quad . \quad . \quad . \quad (8),$$

the square of which added to the square of (7) gives, by a slight reduction,

$$\cot^4 y + 2\cot^2 y \cos 2A \cos 2x + \cos^2 2A = \cos^2 2A \operatorname{cosec}^4 y,$$

which, by transposition and reduction, becomes,

$$\cot^2 y \sin^2 2A + 2\cos 2A \cos 2x = 2\cos^2 2A \quad . \quad . \quad . \quad . \quad (9),$$

which is the equation of the curve.

Cor. When the two circles are perpendicular, or $2A = 90°$, the equation becomes $\cot^2 y = 0$, or $y = 90°$, which is the equation of a great circle, perpendicular to both the given ones.

SECOND SOLUTION, BY PROF. MARCUS CATLIN, HAMILTON COLLEGE, N. Y.

Let one of the two given great circles be assumed for the prime meridian, the intersection of the two circles for the origin of co-ordinates. Let y_1, 0, be a point in the prime meridian, and y_2, a, be a point in the other circle; then the equation of a great circle passing through the two points $(y_1, 0)$ and (y_2, a) will be (Mathematical Miscellany, page 33, equation 12,)

$$\cot y \sin a - \cot y_1 \sin (a - x) + \cot y_2 \sin x = 0 \quad . \quad . \quad . \quad (1).$$

But, (*ibid.* p. 35, *eq.* (28)) since the distance between these points is a quadrant, we have the relation

$$\cot y_1 \cot y_2 + \cos a = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

By means of (2), eliminate y_2 from (1), then

$$\sin a \cot y \cot y_1 - \cot^2 y_1 \sin (a - x) + \cos a \sin x = 0 \qquad (3).$$

Now, if the point yx be the *centre of rotation* of the quadrant, it will remain fixed, while the arc takes its succeeding position, or y_1 varies by an indefinitely small quantity, therefore taking the differential of (3) with regard to y_1, we shall find

$$\cot y_1 = \frac{\sin a \cot y}{2 \sin (a - x)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Substitute (4) in (3) and the equation can be reduced to

$$\sin^2 a \cot^2 y + 4 \cos a \sin x \sin (a - x) = 0 \quad . \quad . \quad . \quad . \quad (5),$$

which is the equation of the locus required.

Cor. When $x = 0$, or $x = a$, then $y = 90°$, and x cannot be between the values 0 and a, unless $a > 90°$, therefore the locus is included between those two angles of the given circles which are greater than $90°$. And when $a = 90°$, $y = 90°$, that is the centre of rotation moves on the arc of a great circle whose pole is the origin of co-ordinates.

—— Dr. Strong, in a letter, which was received only in time to be noticed here, and which contains valuable remarks on many of the questions, in addition to the complete series of solutions previously received, finds that the centre of rotation divides the quadrantal arc so that, if θ be the part of it adjacent to the prime meridian,

$$\tan \theta = \frac{\cos y_1}{\cos y_2},$$

y_1 and y_2 being as in Prof. Catlin's solution. This beautiful property will enable us either to construct the locus or find its properties.

QUESTION XVII. BY INVESTIGATOR.

A plane intersects the axes of co-ordinates at the distances x', y' z', from the origin, so that $x'\,y' + x'\,z' + y'\,z = $ a given rectangle. To find the surface to which this plane is always a tangent.

FIRST SOLUTION, BY PROF. BENJAMIN PEIRCE.

Let x, y, z, be the co-ordinates of the surface,

x', y', z', those of its tangent plane;

we have for the equation of this plane

$$z' - z = \frac{dz}{dx}(x' - x) + \frac{dz}{dy}(y' - y) = p\,(x' - x) + q\,(y' - y),$$

and for the distances at which this plane intersects the axes,

$$z' = z - px - qy = \text{M},$$

$$y' = -\frac{1}{q}(z - px - qy) = -\frac{M}{q},$$

$$x' = -\frac{1}{p}(z - px - qy) = -\frac{M}{p},$$

whence, per question, $\qquad x'y' + x'z' + y'z' = A,$

$$\text{or, } M^2 \left(\frac{1}{pq} - \frac{1}{p} - \frac{1}{q}\right) = A;$$

an equation which I am unable to integrate, except in certain cases, of which the most general is

Case 1. Where $A = 0$; which may be satisfied, first by making

$$M = 0, \text{ or } z = px + qy,$$

of which the integral is

$z =$ homogeneous function of x and y of the first degree,

or the equation of the curve is *any homogeneous function of* x, y, $z = 0$.

Or, secondly, we may put

$$\frac{1}{pq} - \frac{1}{p} - \frac{1}{q} = 0, \text{ or } 1 = p + q, \text{ and } q = 1 - p.$$

Now, $\qquad dz = p\,dx + q\,dy = p\,(dx - dy) + dy;$

$$\therefore dz - dy = p\,(dx - dy);$$

the integral of which is

$$z - y = \text{function of } (x - y) = F\,(x - y)$$

$$\text{and } z = y + F\,(x - y).$$

Case 2. Where the given surface is that of a cylinder whose axis is parallel to z'; in which case we must strike out of the given equation the terms containing z', and it becomes

$$x'y' = A.$$

The equation of the tangent plane is

$$y' - y = \frac{dy}{dx}(x' - x) = p\,(x' - x);$$

and for the intersection with the axes,

$$y' = y - px, \; x' = -\frac{1}{p}(y - px);$$

therefore, $\qquad x'y' = -\frac{1}{p}(y - px)^2 = A;$

or, $\qquad x^2p^2 - (2yx - A)\,p + y^2 = 0.$

Let $yx = s$, or $y = \frac{s}{x}$, and $p = \frac{dy}{dx} = \frac{1}{x^2}\left(x\frac{ds}{dx} - s\right)$, which, substituted in the above equation, gives

$$\frac{ds^2}{dx^2} - \frac{4s}{x}\cdot\frac{ds}{dx} + \frac{A}{x}\cdot\frac{ds}{dx} + \frac{4s^2}{x^2} - \frac{As}{x^2} = 0.$$

A particular solution of which is $S = const. = \frac{1}{4}A$,

$$\text{or, } \quad xy = \tfrac{1}{4}A;$$

which is the equation of the rectangular hyperbola referred to its assymptotes, and which is, therefore, the base of the cylinder.

But to return to the preceding equation; we obtain from its solution

$$\frac{ds}{dx} - \frac{2s}{x} + \frac{\text{A}}{2x} = \pm \frac{\sqrt{\text{A}^2 - 4\text{A}s}}{2x}.$$

Put $\text{A} - 4s = v^2$, and it reduces to

$$-\frac{v\,dv}{dx} + \frac{v^2}{x} = \pm \frac{v\sqrt{\text{A}}}{x},$$

$$\text{or, } \frac{dv}{v \pm \sqrt{\text{A}}} = \frac{dx}{x};$$

whose integral, introducing the arbitrary constant c, is

$$v \pm \sqrt{\text{A}} = \text{c}x,$$

$$\text{or, } 4s = 4xy = \text{A} - v^2 = \pm 2\sqrt{\text{A}}.\ \text{c}x - \text{c}^2x^2$$

therefore, $$4y = \pm 2\text{c}\sqrt{\text{A}} - \text{c}^2 x,$$

which is the equation of two parallel straight lines.

SECOND SOLUTION, BY PROF. MARCUS CATLIN, HAMILTON COLLEGE, N. Y.

The equation of a plane passing through the three points $(x', 0, 0)$ $(0, y', 0)$, and $(0, 0, z')$, is $\frac{x}{x'} + \frac{y}{y'} + \frac{z}{z'} + 1 = 0 \cdot \cdot \cdot \cdot \cdot \cdot \cdot$ (1).

But also, $$x'y' + x'z' + y'z' = a \quad (2),$$
and eliminating x', we get

$$\frac{(y'+z')x}{a - y'z'} + \frac{y}{y'} + \frac{z}{z'} + 1 = 0 \quad (3).$$

If y', z' be now expressed in terms of the partial differentials of the surface to which the plane is tangent, (3) will be the differential equation of that surface; but a more interesting case is the particular surface formed by the intersections of the planes denoted by (3), by the variation of the intercepts y' and z'. Taking the partial differentials of that equation with respect to y' and z' we get

$$xy'^2 + 2xy'z' + ay - 2yy'z' - zy'^2 + ay' - 2z'y'^2 = 0 \quad (4),$$

$$xz'^2 + 2xy'z' + az - 2zy'z' - yz'^2 + az' - 2y'z'^2 = 0 \quad (5).$$

By virtue of (4) and (5) y' and z' may be eliminated from (3), and there will result the equation of the required surface.

QUESTION XVIII. BY RICHARD TINTO, ESQ. GREENVILLE, OHIO.

A sphere *given in magnitude and position*, is viewed in perspective from a given point; it is required to find the nature and position of a surface such, that in whatever position the picture be placed upon it, the image of the sphere may have the same given magnitude.

FIRST SOLUTION, BY THE PROPOSER.

The visual cone, which is one of revolution, has its vertex at the eye, and the image of the globe will be the section of that cone made by the

plane of the picture varying its position. Let the vertex of the cone be the origin of rectangular co-ordinates, the axis of the cone being the axis of z, and let 2α be the vertical angle of the cone, its equation will be

$$z = \cot\alpha \sqrt{y^2 + x^2} \quad . \quad . \quad . \quad (1).$$

Transfer the origin to a point in the axis of z, at the arbitrary distance k from the vertex, the axes of x and y being parallel to their former position; the new equation will be

$$z + k = \cot\alpha \sqrt{y^2 + x^2} \quad . \quad . \quad . \quad (2).$$

Preserving the same origin, let us make a second transformation to a system in which the plane of xy makes the angle θ with either of the former ones, and the intersection of the planes of xy in the second and third systems, which is taken for the new axis of y, makes the angle φ, with the second axis of y; so the equation of the plane of xy in the third system, when referred to the first, would be

$$z = (x\cos\varphi + y\sin\varphi)\tan\theta + k \quad . \quad . \quad . \quad (3).$$

The formulas for this transformation are given by all elementary writers on this subject; and if we make $z' = 0$, they become

$$x = x'\cos\theta\cos\varphi - y'\sin\varphi,$$
$$y = x'\cos\theta\sin\varphi + y'\cos\varphi,$$
$$z = x'\sin\theta.$$

Substituting these in equation (2), we shall get the equation of the section of the cone, made by the plane whose equations is (3); it is

$$x'\sin\theta + k = \cot\alpha\sqrt{x'^2\cos^2\theta + y'^2},$$

or, $y'^2\cos^2\alpha + x'^2(\cos^2\alpha - \sin^2\theta) - 2kx'\sin^2\alpha\sin\theta = k^2\sin^2\alpha$ (4), and transferring the origin to the centre of the section,

$$y'^2\cos^2\alpha + x'^2(\cos^2\alpha - \sin^2\theta) = \frac{k^2\sin^2\alpha\cos^2\alpha\cos^2\theta}{\cos^2\alpha - \sin^2\theta} \quad (5).$$

If we imagine (3) to be the plane of the picture, (4) will be the equation of the perspective image of the globe on that plane. The semiaxes of the elliptic image are given by equation (5); they are

$$\frac{k\sin\alpha\cos\alpha\cos\theta}{\cos^2\alpha - \sin^2\theta} \text{ and } \frac{k\sin\alpha\cos\theta}{\sqrt{\cos^2\alpha - \sin^2\theta}};$$

and if r be the radius of a circle which has an area equal to the given magnitude of the image,

$$\frac{k^2\sin^2\alpha\cos\alpha\cos^2\theta}{(\cos^2\alpha - \sin^2\theta)^{\frac{3}{2}}} = r^2; \therefore k = a\sec.\theta(\cos^2\alpha - \sin^2\theta)^{\frac{3}{4}},$$

where $a^2 = \dfrac{r^2}{\sin^2\alpha\cos\alpha}$. Substituting this in (3), we shall have the equation of a plane, referred to the first system of co-ordinates, cutting the cone in a section of the given area

$$z\cos\theta = (x\cos\varphi + y\sin\varphi)\sin\theta + a(\cos^2\alpha - \sin^2\theta)^{\frac{3}{4}} \quad . \quad (6).$$

This plane is a tangent to the required surface, and, therefore, if the point

of contact be x, y, z, these co-ordinates remain the same, while the plane takes its succeeding position, or θ and φ change by an indefinitely small quantity. Hence the partial differentials of (6) with respect to φ and θ, give

$$0 = -x\sin\varphi + y\cos\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$0 = z\sin\theta + (x\cos\varphi + y\sin\varphi)\cos\theta - \tfrac{3}{2}a\sin\theta\cos\theta(\cos^2\alpha - \sin^2\theta)^{-\frac14} \quad (8).$$

Equation (7) gives $\tan\varphi = \dfrac{y}{x}$, and writing this in (6) and (8) they become

$$z\cos\theta - \sin\theta\sqrt{y^2 + x^2} = a(\cos^2\alpha - \sin^2\theta)^{\frac34} \quad . \quad . \quad . \quad (9)$$

$$z\sin\theta + \cos\theta\sqrt{y^2 + x^2} = \tfrac{3}{2}a\sin\theta\cos\theta(\cos^2 a - \sin^2\theta)^{-\frac14}. \quad (10).$$

And by eliminating θ between these two equations, we obtain the equation of the required surface, the eliminated equation would evidently involve z and $\sqrt{y^2 + x^2}$, as the only variables, and therefore the surface is one of revolution round the axis of the cone. By writing x for z and y for $\sqrt{y^2 + x^2}$, these equations become

$$x\cos\theta - y\sin\theta = a(\cos^2\alpha - \sin^2\theta)^{\frac34} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$x\sin\theta + y\cos\theta = \tfrac{3}{2}a\sin\theta\cos\theta(\cos^2 a - \sin^2\theta)^{-\frac14} \quad . \quad . \quad (12)$$

which, by eliminating θ, give the equation of the curve, by the revolution of which round the axis of z the surface is generated.

The final equation is too complicated for insertion here, but the curve may be traced, and its position found from the following equations, easily deduced from (11) and (12):

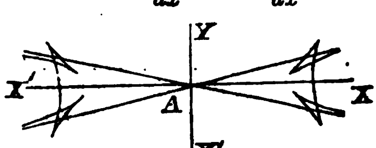
$$y = \frac{a\sin\theta(\sin^2\alpha + \frac12\cos^2\theta)}{(\cos^2\alpha - \sin^2\theta)^{\frac14}},\ x = \frac{a\cos\theta(\cos^2\alpha + \frac12\sin^2\theta)}{(\cos^2\alpha - \sin^2\theta)^{\frac14}} \quad . \ (13),$$

and putting, for brevity

$$\cos^2\alpha(\sin^2\alpha + \tfrac12) - (\tfrac34 + \cos^2\alpha)\sin^2\theta + \tfrac54\sin^4\theta = m \quad . \quad . \quad (14)$$

$$\frac{dy}{d\theta} = \frac{am\cos\theta}{(\cos^2\alpha - \sin^2\theta)^{\frac54}},\ \frac{dx}{d\theta} = \frac{am\sin\theta}{(\cos^2\alpha - \sin^2\theta)^{\frac54}},$$

$$\frac{dy}{dx} = \cot\theta,\ \frac{d^2y}{dx^2} = \frac{-(\cos^2\alpha - \sin^2\theta)^{\frac54}}{am\sin^3\theta}.$$



Hence the curve cuts the axis perpendicularly when $\theta = 0$, at the distance $x = a\cos^{\frac32}\alpha = r\cot\alpha$, and here the section of the cone is a circle; θ varies from 0° to $\frac12\pi - \alpha$, and at the latter point $y = \infty$, $x = \infty$ and $\dfrac{dy}{dx} = \tan\alpha$, or the cone is assymptotic to the surface. The curve is concave to the axis until $\cos 2\theta = \frac15\left(-2\cos 2\alpha + \sqrt{3(4\cos 4\alpha - 1)},\right)$ where $m = 0$, $\dfrac{dy}{d\theta} = \dfrac{dx}{d\theta} = 0$, and $\dfrac{d^2y}{dx^2} = \infty$,

and there is consequently a point of *rebroussement;* it is then convex until $\cos 2\theta = \frac{1}{8}\left(-2\cos 2\alpha - \sqrt{3}\,(4\cos 4\alpha - 1)\right)$, where the same indications show a second point of *rebroussement;* and it is afterwards concave to the axis. These singular points approach each other as α increases, and when $\alpha = \frac{1}{4}\cos^{-1}(\frac{1}{4})$ they unite in one point; for this and all greater values of α there is no inflection of the curve, it being always concave to the axis. When $\alpha < 15^\circ$, the curve cuts its assymptote at two points indicated by the values $\theta = \frac{1}{2}\sin^{-1}(2\sin 2\alpha)$ and $\theta = \frac{1}{2}\pi - \frac{1}{2}\sin^{-1}(2\sin 2\alpha)$, the former occurring before the first singular point, and the latter between the two singular points; when $\alpha = 15^\circ$ the first point of *rebroussement* is in the assymptote; and when $\alpha > 15^\circ$ the curve is wholly included between its assymptotes.

SECOND SOLUTION, BY DR. T. STRONG.

Supposing the eye and sphere to be given in position, the visual rays drawn as tangents to the sphere will form a cone of revolution, whose vertical angle is given; let t be the tangent of half that angle. Take the axis of the cone (supposed horizontal) for the axis of z, and a perpendicular to it through the vertex in the horizontal plane for the axis of x, also, a perpendicular to the horizontal plane through the same point for the axis of y; then the equation of the conical surface is

$$t^2 z^2 = x^2 + y^2 \quad \dots\dots\dots\dots \quad (1),$$

and the equation of the plane of the picture will be of the form

$$z = \text{A}x + \text{B}y + \text{C} \quad \dots\dots\dots\dots \quad (2).$$

Put s = the area of the picture, then by a well known formula

$$d^2\text{s} = dxdy\sqrt{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2};$$

but by (2), since the picture is a tangent to the surface,

$$\frac{dz}{dx} = \text{A}.\ \frac{dz}{dy} = \text{B} \quad \dots\dots\dots\dots \quad (3),$$

$$\therefore\ d^2\text{s} = dxdy\sqrt{1 + \text{A}^2 + \text{B}^2},$$

$$\text{and } d\text{s} = ydx\sqrt{1 + \text{A}^2 + \text{B}^2}, \quad \dots\dots \quad (4).$$

Let $d\text{s}' = ydx$; then s' is evidently the orthographic projection of the given area of the picture, on the plane of xy; from (1) and (2) we get the equation of this projection

$$t^2(\text{A}x + \text{B}y + \text{C})^2 = x^2 + y^2 \quad \dots\dots \quad (5).$$

Transform this equation to polar co-ordinates, $r\varphi$, the axis of x being the angular axis, and the vertex of the cone the pole, so that $x = r\cos\varphi$, $y = r\sin\varphi$; then the equation will give

$$r = \frac{\pm \text{C}t}{1 \mp (\text{A}\cos\varphi + \text{B}\sin\varphi)\,t} \quad \dots\dots \quad (6).$$

Putting in this $\text{A}t = e\cos\omega$, $\text{B}t = e\sin\omega$, so that $e = t\sqrt{\text{A}^2 + \text{B}^2}$, and $\tan\omega = \frac{\text{B}}{\text{A}}$, and it becomes

$$r = \frac{\pm ct}{1 \mp s\cos(\omega - \varphi)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7);$$

which shows that *if the surface of a right cone is cut by any plane, and the section is orthographically projected on any plane perpendicular to the axis of the cone, the projected curve is always a conic section, one of whose foci is at the point where the plane of projection meets the axis of the cone;* the same result also follows from equation (5), from which (7) is derived. In the present question the projected curve is necessarily an ellipse, whose semiparameter $= - ct$ and the ratio of the semiaxis to the eccentricity $= s$; hence its semiaxis $= \frac{-ct}{1 - s^2}$, and its semi-conjugate $= \frac{-ct}{\sqrt{1 - s^2}}$; therefore its area is

$$s' = \frac{pc^2t^2}{(1-s^2)^{\frac{3}{2}}} = \frac{pc^2t^2}{\{1 - (A^2 + B^2)t^2\}^{\frac{3}{2}}},$$

and by (4), $$s = s'\sqrt{1 + A^2 + B^2} = \frac{p\,c^2t^2\,(1 + A^2 + B^2)}{\{1 - (A^2 + B^2)\,t^2\}^{\frac{3}{2}}} \quad . \quad . \quad . \quad (8),$$

where $p = 3{,}14159$, &c., substituted for s its given value, and we shall obtain c in functions of A, B and given quantities. Let $c = \varphi$ (A, B) denote this function, which will reduce (2) to

$$z = Ax + By + \varphi(A, B) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

or, by (3), $$z = x.\frac{dz}{dx} + y.\frac{dz}{dy} + \varphi\left(\frac{dz}{dx}, \frac{dz}{dy}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

which is the partial differential equation of all the surfaces having the property in the question. But I suppose the proposer intended that the surface formed by the successive intersections of the tangent planes should be found; for this purpose take the differential of (9), considering A and B as alone variable, then put the co-efficients of dA and dB each $= 0$, and we shall have two equations from which A and B can be found in terms of x, y and given quantities, then substituting these values in (9), we shall get an equation which, when cleared of radicals, will be the equation of the required surface.

—— Dr. Strong also gave three other methods of determining the projected area, which want of room alone compels us to omit.

QUESTION XIX. BY PETRARCH, NEW-YORK.

A given semi-prolate-spheroid is placed with its base on the horizontal plane, and its axis vertical; with what velocity must a body be projected vertically along its interior surface, so that it may pass through the focus?

SOLUTION, BY PROF. BENJAMIN PEIRCE.

Let the co-ordinates of the point at which the body leaves the ellipse to move in a parabola be x, y, x being vertical; let the semiaxes of the ellipse be A, B, the distance of its focus from the centre of the spheroid,

which is the origin of co-ordinates be c, and the co-ordinates of the vertex of the parabola be x', y'; then the equation of the parabola will be

$$(y-y')^2 = 2p(x'-x) \quad \ldots\ldots\ldots\ldots (1),$$

and that of the ellipse which generates the spheroid

$$A^2y^2 + B^2x^2 = A^2B^2 \quad \ldots\ldots\ldots\ldots (2).$$

The ellipse and parabola must evidently have an osculation of the second degree, and the parabola must pass through the focus, the co-ordinates of which being 0 and c, (1) becomes for this point

$$y'^2 = 2p(x'-c),$$

by which dividing (1), we have

$$\left(\frac{y}{y'}-1\right)^2 = \frac{x'-x}{x'-c} \quad \ldots\ldots\ldots\ldots (3),$$

which may take the place of (1) as the equation of the parabola.

For an osculation of the first degree, the differentials of (2) and (3) must give the same values for $\frac{dy}{dx}$. These differentials are

$$\left.\begin{array}{c} A^2y\,dy + B^2x\,dx = 0 \\ \left(\frac{y}{y'}-1\right).\frac{dy}{y'} + \frac{dx}{x'-c} = 0 \end{array}\right\} \quad \ldots\ldots\ldots (4)$$

from which, eliminating dy and dx, we have

$$2B^2.\frac{x}{y'}\left(\frac{y}{y'}-1\right) - \frac{A^2y}{x'-c} = 0 \quad \ldots\ldots\ldots (5).$$

Again, for the osculation of the second degree, differentiate equations (4), then

$$A^2yd^2y + A^2dy^2 + B^2dx^2 = 0,$$

$$\left(\frac{y}{y'}-1\right)d^2y + \frac{dy^2}{y'} = 0;$$

and eliminating d^2y

$$-A^2dy^2 + B^2\left(\frac{y}{y'}-1\right)dx^2 = 0;$$

and eliminating dy and dx by means of (4) and reducing by (2),

$$y' = \frac{y^3}{B^2} \quad \ldots\ldots\ldots\ldots\ldots (6);$$

which, being substituted in (3) and (5), gives

$$\left(\frac{B^2}{y^2}-1\right)^2 = \frac{x'-x}{x'-c} \quad \ldots\ldots\ldots\ldots (7),$$

$$\frac{2B^4a}{y^3}\left(\frac{B^2}{y^2}-1\right) = \frac{A^2y}{x'-c} \quad \ldots\ldots\ldots\ldots (8).$$

Eliminating x' and y by means of (2), (7), and (8), we get finally

$$A^4 - 3A^2x^2 + 2cx^3 = 0 \ldots\ldots\ldots\ldots (9),$$

or, multiplying by $\frac{c^3}{x^3A^4}$,

$$\left(\frac{c}{x}\right)^3 - 3\left(\frac{c}{A}\right)^2.\left(\frac{c}{x}\right) + 2\left(\frac{c}{A}\right)^4 = 0 \quad \ldots\ldots (10)$$

which may be solved thus, by angular functions.

Let $$\frac{\text{C}}{\text{A}} = \cos n \quad \ldots\ldots\ldots\ldots \quad (11),$$

and we have for the three roots

$$\frac{\text{C}}{x} = -\frac{2\text{C}}{\text{A}}\cos\tfrac{1}{3}n,\ \frac{2\text{C}}{\text{A}}\cos(60^\circ + \tfrac{1}{3}n),\ \text{or}\ \frac{2\text{C}}{\text{A}}\cos(60^\circ - \tfrac{1}{3}n);$$

of which the first must be thrown aside, as giving a negative value of x, and the second, as giving x greater than A, for it gives

$$\frac{\text{C}}{x} < \frac{2\text{C}}{\text{A}}\cos 60^\circ < \frac{2\text{C}}{\text{A}}.\tfrac{1}{2} < \frac{\text{C}}{\text{A}}:$$

the last is therefore the root to be used in the problem, and it gives

$$x = \tfrac{1}{2}\text{A}\sec.(60^\circ - \tfrac{1}{3}n).$$

To find the velocity of projection. Let v be this velocity, v' the velocity at the point yx, φ the angle which the curve makes with the horizon at the point yx, and g the space through which a heavy body falls in one second, then by well known formulas

$$v'^2 = \frac{4g(x' - x)}{\sin^2\varphi},$$

$$v^2 = v'^2 + 4gx = \frac{4g(x' - x)}{\sin^2\varphi} + 4gx = \frac{2gxy^2}{\text{B}^2\sin^2\varphi} + 4gx \quad (12)$$

using the value of $x' - x$ found by dividing (7) by (8).

But $\dfrac{1}{\sin^2\varphi} = 1 + \left(\dfrac{dy}{dx}\right)^2$, and substituting the value of $\dfrac{dy}{dx}$ found by (4) and reducing by means of (2) and (11), we find

$$\frac{1}{\sin^2\varphi} = \frac{\text{A}^4\text{B}^2 - \text{B}^2\text{C}^2x^2}{\text{A}^4y^2} = \frac{\text{A}^2\text{B}^2 - \text{B}^2x^2\cos^2 n}{\text{A}^2y^2},$$

which substituted in (12) gives

$$v^2 = 2gx\left(3 - \frac{x^2\cos^2 n}{\text{A}^2}\right),$$

and substituting the value of x,

$$v^2 = \frac{\text{A}g}{\cos(60^\circ - \tfrac{1}{3}n)}.\left(3 - \frac{\cos^2 n}{4\cos^2(60^\circ - \tfrac{1}{3}n)}\right).$$

In the case of the sphere, C $= 0$, $\cos n = 0$, $n = 90^\circ$, and

$$v^2 = 3\text{A}g\sec.30^\circ = 2\text{A}g\sqrt{3}.$$

QUESTION XX. BY INVESTIGATOR.

A perfectly smooth plane is made to oscillate according to a given law, round one of its own lines placed in a given position as an axis; to find the circumstances of the motion of a body on this plane when acted upon by gravity.

SOLUTION, BY PROF. PEIRCE.

Let the origin of co-ordinates be *any point* in the given axis of rotation. Let the plane of xz be the vertical plane through the axis of rotation, and let the axis of z be vertical. The general equation of motion is

$$\frac{d^2x}{dt^2}\,\delta x + \frac{d^2y}{dt^2}\,\delta y + \left(\frac{d^2z}{dt^2} - g\right)\delta z = 0 \quad \cdots\cdots \quad (1).$$

Let A be the angle made by the axis of rotation with the axis of x. The axes of x and z may be changed into two others of x' and z' in the same plane, of which the axis of x' is the fixed axis of rotation, and we shall have

$$\left.\begin{aligned} x &= x'\cos A - z'\sin A \\ z &= x'\sin A + z'\cos A \end{aligned}\right\} \quad \cdots\cdots\cdots \quad (2).$$

Let B be the angle made by the axis of y with the *trace* of the given plane in the plane of $y\,z'$, and let r be the distance of the body from the axis x' of rotation, and we have

$$y = r\cos B, \qquad z' = r\sin B \quad \cdots\cdots\cdots\cdots \quad (3),$$

which substituted in (2) give

$$\left.\begin{aligned} x &= x'\cos A - r\sin A\sin B \\ z &= x'\sin A + r\cos A\sin B \end{aligned}\right\} \quad \cdots\cdots\cdots \quad (4),$$

and these, with the first of (3), substituted in (1), give the two following

$$\frac{d^2x'}{dt^2} - g\sin A = 0 \quad \cdots\cdots\cdots\cdots \quad (5),$$

$$\frac{d^2r}{dt} - \frac{dB^2}{dt^2}.\,r = g\cos A\sin B \quad \cdots\cdots\cdots \quad (6).$$

The integral of (5) is

$$x' = \tfrac{1}{2}gt^2\sin A + v't\cos a + x'' \quad \cdots\cdots\cdots \quad (7),$$

in which x'' is the *initial* distance of the body from the plane of yz', v' is the primitive velocity of the body, and a the angle which the primitive direction of motion makes with the axis of rotation. This equation shows that the motion of the body parallel to the axis of rotation is not affected by the rotation of the plane.

Case 1. Where the rotation of the plane is uniform. In this case we may put

$$B = nt + a' \text{ and } \frac{dB}{dt} = n \quad \cdots\cdots\cdots\cdots \quad (8)$$

and (6) becomes, $$\frac{d^2r}{dt^2} - n^2r = g\cos A\sin(nt + a') \quad \cdots\cdots \quad (9).$$

the integral of which is

$$r = c\,\varepsilon \quad + c'\,\varepsilon^{-nt} - \frac{g\cos A}{2n^2}.\sin(nt + a') \quad \cdots\cdots \quad (10),$$

and the values of the constants are determined by the following equations, in which r' is the original distance of the body from the axis of rotation,

$$\left.\begin{aligned} r' &= c + c' - \frac{g}{2n^2}.\cos A \sin a' \\ v' \sin a &= nc - nc' - \frac{g}{2n}.\cos A \cos a' \end{aligned}\right\} \quad \ldots (11).$$

Case 2. Where the axis of rotation is vertical and

$$A = 90^\circ, \cos A = 0 \quad \ldots\ldots\ldots\ldots (12),$$

then (6) becomes $$\frac{d^2 r}{dt^2} - \frac{dB^2}{dt^2}.\ r = 0 \quad \ldots\ldots\ldots (13).$$

Make $$r = \varepsilon^{\int v dt} \quad \ldots\ldots\ldots\ldots (14),$$

and (13) becomes $$dv + \left(v^2 - \frac{dB^2}{dt^2}\right) dt = 0 \quad \ldots\ldots\ldots (15);$$

which is identical with an equation treated by Lacroix, and which he proves to be integrable whenever we have

$$\frac{dB^2}{dt^2} = -\frac{4SQdQ}{dS} + Q^2 + S \quad \ldots\ldots\ldots (16),$$

in which S is an arbitrary function of t, and $Q = \frac{1}{4S}.\frac{dS}{dt} \quad \ldots (17).$
The integral of it is then

$$\tfrac{1}{2}\log.\frac{v + Q - \sqrt{S}}{v + Q + \sqrt{S}} + \tfrac{1}{2}.\int \frac{dS}{Q\sqrt{S}} = const. \quad \ldots\ldots (18).$$

It is also integrable whenever $\frac{dB^2}{dt^2} = -\frac{dR}{dt} + R^2 \quad \ldots\ldots\ldots (19),$
where R is an arbitrary function of t; and its integral is

$$\varepsilon^{2\int R dt} - (v+R)\int \varepsilon^{2\int R dt}\, dt = const. \quad \ldots\ldots\ldots (20.$$

It is also integrable in the case of Riccati, in which

$$\frac{dB^2}{dt^2} = at_m \quad \ldots\ldots\ldots\ldots\ldots (21),$$

where $m = \frac{-4i}{2i \pm 1}$, i being any whole number.

As a particular example, suppose in equation (19), $R = \frac{a''}{mt + c}$ (22),

whence, $$\frac{dB^2}{dt^2} = \frac{ma'' + a''^2}{(mt+c)^2} \quad \ldots\ldots\ldots\ldots (23)$$

$$B = \frac{1}{m}(ma'' + a''^2)^{\frac{1}{2}}.\log.(mt + c) \cdot \quad \ldots (24)$$

$$\varepsilon^{2\int R dt} = \varepsilon^{\frac{2a''}{m}\log.(mt+c)} = (mt + c)^{\frac{2a''}{m}} \quad \ldots\ldots\ldots (25),$$

and (20) becomes

$$(mt+c)^{\frac{2a''}{m}} - \left(v + \frac{a''}{mt+c}\right)\left(\frac{2a''}{m} + 1\right)^{-1}(mt+c)^{\frac{2a''}{m}+1} = const. = c \quad (26),$$

which gives $$v = \frac{(2-m)a'' + m}{m(mt + c)} - c\left(\frac{2a''}{m} + 1\right)(mt+c)^{-\frac{2a''}{m}-1} \quad (27),$$

and this becomes, when $a'' = -m = -1$, $v = \text{c}\,(t + c)$; whence

$$r = \text{B}\,\varepsilon^{\frac{1}{2}\text{c}\,(t+c)^2}.$$

As another example, let in equations (16, 17, 18,) $\text{s} = \frac{b^2}{t^4}$, whence $\text{Q} = -\frac{1}{t}$,

$\frac{d\text{B}^2}{dt^2} = \text{s} = \frac{b^2}{t}$; therefore $\text{B} = \text{A}' - \frac{b}{t}$, and (18) becomes

$$\tfrac{1}{2}\log.\frac{vt^2 - t - b}{vt^2 - t + b} - \frac{4b}{t} = const.$$

But to return to the general case, equation (6) is always integrable whenever (13) is, by the well known theorems of Lagrange. Thus let us suppose the complete integral of (13) to be

$$r = c_1 r_1 + c_2 r_2 \quad \ldots \ldots \quad (28),$$

c_1 and c_2 being arbitrary constants.

Suppose c_1 and c_2 to vary so as to satisfy equation (6), and we have

$$dr = c_1 dr_1 + c_2 dr_2 + r_1 dc_1 + r_2 dc_2 \quad \ldots \quad (29);$$

and we may suppose $\quad r_1 dc_1 + r_2 dc_2 = 0 \quad \ldots \ldots \quad (30),$

whence (29) becomes $\quad dr = c_1 dr_1 + c_2 dr_2 \quad \ldots \ldots \quad (31),$

and $\quad d^2 r = c_1 d^2 r_1 + c_2 d^2 r_2 + dc_1 dr_1 + dc_2 dr_2 \quad \ldots \quad (32);$

which substituted in (6), gives by means of (13),

$$dr_1 dc_1 + dr_2 dc_2 = g \cos \text{A} \sin \text{B} dt^2 \quad \ldots \ldots \quad (33).$$

Whence $\quad c_1 = \int \frac{g \cos \text{A} \sin \text{B} r_2\, dt^2}{r_2 dr_1 - r_1 dr_2}, \quad c_2 = -\int \frac{g \cos \text{A} \sin \text{B} r_1 dt^2}{r_2 dr_1 - r_1 dr_2} \quad (34)$

and $\quad r = r_1 \int \frac{g \cos \text{A} \sin \text{B} r_2 dt^2}{r_2 dr_1 - r_1 dr_2} - r_2 \int \frac{g \cos \text{A} \sin \text{B} r_1 dt^2}{r_2 dr_1 - r_1 dr_2} \quad \ldots \quad (35).$

Case of small perturbations. Suppose the value of B to differ, by some slight disturbing cause, from a value B$'$ for which the integral of (6) can be exactly obtained, and call this variation δB, so that $\text{B} = \text{B}' + \delta\text{B}$; let r' be the value of r corresponding to B$'$ and δr its perturbation, so that $r = r' + \delta r$; the squares and products of δr, δB and of their differentials being supposed so small that they may be neglected. In this case (6) becomes

$$\frac{d^2 r'}{dt^2} + \frac{d^2 \delta r}{dt^2} - \frac{d\text{B}'^2}{dt^2}.r' - 2r'.\frac{d\text{B}' d\delta\text{B}}{dt^2} - \frac{d\text{B}'^2}{dt^2}.\delta r = g\cos\text{A}(\sin\text{B}' + \cos\text{B}'\delta\text{B}).(36).$$

But by (6), $\quad \frac{d^2 r'}{dt^2} - \frac{d\text{B}'^2}{dt^2}.r' = g \cos \text{A} \sin \text{B}'$, hence (36) becomes

$$\frac{d^2 \delta r}{dt^2} - \frac{d\text{B}'^2}{dt^2}.\,\delta r = g \cos \text{A} \cos \text{B}'\delta\text{B} + 2r'.\frac{d\text{B}' d\delta\text{B}}{dt^2} \quad \ldots \quad (37),$$

which, by the preceding analysis can always be resolved.

Suppose now that, as in (8), $\text{B}' = nt + a'$, then (37) becomes

$$\frac{d^2 \delta r}{dt^2} - n^2 \delta r = g \cos \text{A} \cos (nt + a')\,\delta\text{B} + 2nr'.\frac{d\delta\text{B}}{dt^2} = \text{T} \quad . \quad (38),$$

where $r' = \text{c}\varepsilon^{nt} + \text{c}'\varepsilon^{-nt} - \frac{g}{2n^2}.\cos \text{A} \sin (nt + a') \quad \ldots \quad (39),$

and the integral of this is

$$\delta r = \frac{\varepsilon^{nt}}{2n}.\int \mathrm{T}\varepsilon^{-nt}\, dt - \frac{\varepsilon^{-nt}}{2n}.\int \mathrm{T}\varepsilon^{nt}\, dt \quad \cdot \cdot \cdot \cdot \cdot \quad (40).$$

Case of small oscillations. We obtain this case by making in (37), $\mathrm{B}' = const.$, which gives $\frac{d\cdot\delta r}{dt^2} = g \cos \mathrm{A} \cos \mathrm{B}' \delta \mathrm{B} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$ (41),

but also, $\frac{d^2 r'}{dt^2} = g \cos \mathrm{A} \sin \mathrm{B}'$,

$$\therefore r' = \tfrac{1}{2} g t^2 \cos \mathrm{A} \sin \mathrm{B}' + \mathrm{c}' t + \mathrm{c}'',$$

which is the value of r if the plane were fixed. The integral of (41) is

$$\delta r = \iint g \cos \mathrm{A} \cos \mathrm{B}' \delta \mathrm{B}.\, dt^2\,;$$

so that if $\delta \mathrm{B}$ is a very small periodical quantity represented by

$$\delta \mathrm{B} = \gamma.\, \sin(mt + a),$$

$$\delta r = -\frac{\gamma}{m^2}.\, g \cos \mathrm{A} \cos \mathrm{B} \sin(mt + a):$$

if therefore m is very small, δr will be large.

—— Had our limits permitted, we should have inserted the elegant solutions to this question by Dr. T. Strong and Mr. L. W. Caryl. Dr. Strong gives formulas adapted to the motion of the point on any moving surface, and as we propose to resume the subject in an early Number of the Miscellany, his valuable labours will not be lost.

Our correspondents will see, from the analysis of Professor Peirce, that in all cases where the equations are susceptible of integration for a horizontal axis, they can also be integrated under the same circumstances for an inclined one. Several gentlemen refer to a particular case solved in the Mathematical Diary, vol. i. p. 226.

List of Contributors, and of the Questions answered by each. The figures refer to the number of the Question, as marked in No. I. Art. V.

Prof. C. Avery, Hamilton College, Clinton, N. Y., Ans. all the Questions.
Alfred, Easton, Pa., answered Question 3.
A. B. Ans. 3.
P. Barton, Jun., Schenectady, N. Y., Ans. 1, 2, 3, 4.
Lucian W. Caryl, Buffalo, N. Y., Ans. 5. 20.
Prof. M. Catlin, Hamilton College, Clinton, N. Y., Ans. all the Questions.
Delta, Ans. 2. 10. 15, 16.
Gerardus B. Docharty, Flushing L. I., Ans. 1, 2, 3, 4, 5, 6, 7, 8, 9. 11.
Investigator, Ans. 9. 17. 20.
David Langdon, Schenectady, N. Y., Ans. 11.
Wm. Lenhart, York, Pa., Ans. 5.
James F. Macully, New-York, Ans. 1, 2, 3. 5. 9. 14, 15.
John Mann, Mannington, Silver Lake, Pa., Ans. 1. *(Received too late for insertion.)*
Geo. R. Perkins, Clinton Liberal Institute, N. Y., Ans. 1, 2, 3, 4. 6. 9. 10. 14 .20.
Prof. B. Peirce, Harvard University, Ans. 1, 2, 3, 4, 5, 6. 12. 16. 17. 19. 20.
Petrarch, New-York, Ans. 11. 19.

P. Ans. 1. 12, 13.
O. Root, Hamilton College, Clinton, N. Y. Ans. all the Questions.
Prof. T. Strong L. L. D., Rutgers' Col., N. B., Ans. all the Questions.
S. S. Ans. 14.
Richard Tinto, Greenville, Ohio, Ans. 18.
N. Vernon, Frederick, Md., Ans. 1. 4. 6.

*** In selecting from this valuable collection of solutions, we have been often obliged to exclude solutions which we should have been glad to publish, merely because they were left in an incomplete state by their author. We would respectfully remind our correspondents that solving a question, and telling how it might be solved, are two very distinct things. When a question does not involve some new principle in science, and such cannot always be expected in a work like the present, its solution can only be valuable, either to the writer or reader, in so far as it affords an exercise in analysis; and such a solution must be very unsatisfactory to the reader, when the most difficult part of the analysis is left unfinished.

In order to give sufficient time for a full consideration of the questions, as well as to afford us an equal interval of six months between the publication of the Numbers, there will be in future two sets of Questions published in advance.

New questions must be accompanied with their solutions.

All communications for Number III., which will be published on the first of May, 1837, must be post paid, addressed to the Editor, at the Institute, Flushing, L. I.; and must arrive before the first of March, 1837.

ERRATA.

Page 7, line 24, for "0079437," read "0179437;" this correction will make the diff. of long. $= 5^\circ\ 45'\ 54''$.

Page 9, line 6, the formula for meridional parts should read thus:

"log. merid. parts $= 3{,}8984895 +$ log. of l. cot. $\frac{1}{2}(90^\circ - l)$."

Page 31, equation (1), and throughout the page, for "cot. ω," read "tan. ω."

These errors have been kindly pointed out to us by the Hon. Mr. Justice Fletcher, Sherbrooke, L. C.

ARTICLE VII.

NEW QUESTIONS TO BE ANSWERED IN NUMBER III.

Solutions to these Questions must arrive before the first of March, 1837.

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(21). QUESTION I. BY C. C. OF CAMBRIDGE, MASS.

Let a, b, c be any three angles, prove that

$$1^\circ.\ 2\sin(a+b+c) = \cos a \sin(b+c) + \cos b \sin(a+c) + \cos c \sin(a+b) - 2\sin a \sin b \sin c.$$

$$2^\circ.\ 2\cos(a+b+c) = 2\cos a \cos b \cos c - \sin a \sin(b+c) - \sin b \sin(a+c) - \sin c \sin(a+b).$$

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(22). QUESTION II. BY MR. N. VERNON, FREDERICK, MD.

Divide a given plane triangle, into two equal parts, by a straight line of a given length; also into parts having any given ratio.

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(23). QUESTION III. BY ——.

If from any point, either within or without the plane of a given rectangle, straight lines be drawn to the angles of the rectangle; prove that the sum of the squares described on the lines drawn to two opposite angles is equal to the sum of the squares described on the lines drawn to the other two opposite angles.

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(24). QUESTION IV. BY MR. P. BARTON, JUN., SCHENECTADY.

The sum of the diameters of the bases of a conical frustum is 4, the excess of the altitude above the difference of the diameters is 24, and the distance of the centre of gravity from the less end is 17; what are its altitude and diameters.

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(25). QUESTION V. BY ——.

Given the roots of the equation

$$x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} + \dots\dots + u = 0;$$

to solve the two inequalities

$$x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} + \dots\dots + u > 0,$$
$$x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} + \dots\dots + u < 0.$$

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(26). QUESTION VI. BY ALFRED, OF EASTON, PA.

To find the n unknown quantities x, y, z, &c., there are given n equations, the first members of which are the sums of the squares and the products, two by two, of every $(n-1)$ of the numbers, and the second members are the known numbers a, b, c, &c., thus:

$$y^2 + z^2 + w^2 + \&c. \dots + yz + yw + zw + \&c. \dots = a,$$
$$x^2 + z^2 + w^2 + \&c. \dots + xz + xw + zw + \&c. \dots = b,$$
$$x^2 + y^2 + w^2 + \&c. \dots + xy + xw + yw + \&c. \dots = c,$$
$$\&c., \qquad \&c.$$

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(27). QUESTION VII. BY P.

To cut a given cone of revolution, by a plane passing through a given point in its surface, so that the area of the resulting elliptical section may be given or a *minimum*.

(28). QUESTION VIII, BY Δ.

If r_1, r_2 be two radius vectors of a parabola, and α the angle included between them, show that the distance from the focus to the vertex of the parabola is

$$= \frac{r_1 r_2 \sin^2 \frac{1}{2}\alpha}{r_1 + r_2 \pm 2\sqrt{r_1 r_2}. \cos \frac{1}{2}\alpha},$$

and tell the meaning of the ambiguous sign.

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(29). QUESTION IX. (COMMUNICATED BY MR. J. F. MACULLY.)

Find four affirmative numbers, such that the sum of the first and second, the sum of the second and third, the difference of the squares of the second and third, and their difference, may be four square numbers in continued proportion; the sum of the rectangles of every two of the last three together with the square of the first, a square; and the sum of the first, third, fourth, and twice the second a square.

⁂ This was published in the Belfast Almanac, but an erroneous solution was given to it.

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(30). QUESTION X. BY WM. LENHART, ESQ. YORK, PA.

It is required to find four integers such that the sum of every two of them may be a cube.

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(31). QUESTION XI. BY RICHARD TINTO, ESQ. GREENVILLE, OHIO.

Find the locus of the centre of a given sphere, so that its shadow on a given plane, made by a light fixed in a given position, may have a given magnitude.

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(32). QUESTION XII. BY P.

The surface of a polyedron is composed of a triangular, b quadrangular, c pentagonal, &c., planes; to find the number of diagonals that can be drawn in the polyedron.

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(33). QUESTION XIII. (*From the Ladies' Diary for* 1836.)

At two give points within a spherical shell (incapable of reflection) are placed two given unequal lights. It is required to assign the points in the interior surface which are respectively most and least enlightened, and the locus of the points where the light is of any specified intensity.

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(34). QUESTION XIV. BY INVESTIGATOR.

A given cone of revolution is attached, by its vertex and a point in the circumference of its base, to two fixed points in the same horizontal line, and then placed in the position of unstable equilibrium. If the equilibrium be suddenly disturbed, find when the pressures, in different directions, on the points of suspension of the system will be least, or when they will be entirely destroyed.

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(35). QUESTION XV. BY ——.

Two given circles touch each other internally; it is required to find the sum of the areas of all the circles that can be inscribed between

them, so that each one shall touch the two adjacent ones, and also the two given circles; the centre of one of the inscribed circles being given in position.

ARTICLE VIII.

NEW QUESTIONS TO BE ANSWERED IN NUMBER IV.

Solutions to these Questions must arrive before the first day of August, 1837.

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(36). QUESTION I. BY QUERIST.

It has been said that "in the ellipse all its circumscribing parallelograms are equal." Is this true?

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(37). QUESTION II. BY ——.

Show that if the bases of a number of different systems of logarithms are in geometrical progression, the logarithms of any given number, taken in these different systems successively will be in harmonical progression.

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(38). QUESTION III. (*From the Cambridge Problems.*)
Communicated by Prof. Avery.

In what time will a [redacted] principal double itself at a given rate of compound interest, when the interest is added every instant?

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(39). QUESTION IV. BY DR. STRONG.

Prove that lines, drawn through the points of trisection of a given line, and the points of trisection of the semicircumference of a circle described upon it as a diameter, pass through the vertex of an equilateral triangle described on the opposite side of the given line.

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(40). QUESTION V. BY MR. N. VERNON.

Given the radius, to determine the arc, when the lune formed by the arc, and the semi-circle described upon its chord, is the greatest possible.

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(41). QUESTION VI. BY PROF. CATLIN.

Required the greatest rectangle that can be inscribed in a given circular ring.

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(42). QUESTION VII. BY MR. O. ROOT.

Required the locus of all the points, so situated within a right angle, that the shortest line which can be made to pass through each of them and terminate in the constant sides of the right angle, shall be of a length.

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(43). QUESTION VIII. BY ——.

Having given two series of polygonal members, of the m^{th} and n^{th} orders respectively; to find those terms, when there are such, which are

common to both series. Or, to solve, when it is possible, the indeterminate equation.

$$(m-2)x^2-(m-4)x=(n-2)y^2-(n-4)y,$$

m, n, x, and y, being positive integers, of which m and n are given.

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(44). QUESTION IX. BY ——.

Having given a series of polygonal numbers, of the n^{th} order; to find two terms in that series, when there are such, whose sum and difference shall be equal to two other terms in the same series. Or, to solve, when it is possible, the two indeterminate equations

$$(n-2)x^2-(n-4)x+(n-2)y^2-(n-4)y=(n-2)z^2-(n-4)z,$$
$$(n-2)x^2-(n-4)x-(n-2)y^2+(n-4)y=(n-2)v^2-(n-4)v,$$

n, x, y, z, and v, being positive integers, of which n is given.

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(45). QUESTION X. BY P.

Let lines be drawn from a given point, meeting the tangents of any curve on the same plane, and making with them a constant angle α, the points of intersection will be found in another curve. Then if v, φ be the polar co-ordinates of any point in the first curve, v, θ, those of the corresponding point in the second curve, and s the length of that curve; the pole being in the given point, and the angular axis taken at pleasure; prove that

$$ds=\frac{vd\theta}{\sin\alpha}.$$

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(46). QUESTION XI. BY MR. JAMES F. MACULLY.

It is required to divide a given paraboloid of revolution into two equal parts, by a plane passing through a given line on its base.

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(47). QUESTION XII. BY PROF. CATLIN.

Two straight lines revolve in parallel planes with given velocities about given points in a common axis. Required the locus of the apparent intersection of these lines, when viewed from a given point.

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(48). QUESTION XIII. BY THE SAME GENTLEMAN.

A point oscillates with a given uniform motion between the centre of suspension and the centre of oscillation of a given pendulum. Required the locus of the oscillating point during a complete vibration of the pendulum.

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(49). QUESTION XIV. (COMMUNICATED BY DR. STRONG.)

The axes of a given cone and cylinder of revolution intersect at right angles; to find the portion of the solid common to both, the surface of the cone included by the cylinder, and the surface of the cylinder included by the cone.

*** This question was published in "Marratt's Scientific Journal," but no solution to it has yet been published.

(50). QUESTION XV. BY ——.

Having given the magnitude of two circles; it is required to place them in such a position upon a plane, that of any given number (n) of circles, having placed the first one in any assigned position in contact with the two given ones, the second in contact with the first and also the two given ones, the third in contact with the second and also the two given ones, &c.; the last, or n^{th}, shall not only have like contact with the last but one, and the two given ones, but shall also touch the first one. To find also the position and magnitude of these tangent circles.

ARTICLE IX.

A new demonstration of the LOGARITHMIC THEOREM, *and of the* BINOMIAL THEOREM *for negative and fractional exponents.*

BY PROF. MARCUS CATLIN, HAMILTON COLLEGE, N. Y.

Let $1+y=$ any number. Assume

$$(1+y)=\mathrm{P}^{a+by+cy^2+dy^3+\&c.} \quad \ldots\ldots \quad (1),$$

where $a+by+cy^2+dy^3+\&c.=$ the logarithm of $(1+y)$; a, b, c, &c., being independent of n in the following binomial

$$(1+y)^n=\mathrm{A}+\mathrm{B}y+\mathrm{C}y^2+\mathrm{D}y^3+\&c. \quad \ldots\ldots \quad (2).$$

Equations (1) and (2) give

$$(1+y)^n=\mathrm{P}^{na+nby+ncy^2+ndy^3+\&c.} \quad \ldots \quad (3),$$

$$\text{or, } \mathrm{A}+\mathrm{B}y+\mathrm{C}y^2+\mathrm{D}y^3+\&c.=\mathrm{P}^{na+nby+ncy^2\ ndy^3+\&c.} \quad (4),$$

$$\text{or, } 1+(\mathrm{A}-1+\mathrm{B}y+\mathrm{C}y^2+\mathrm{D}y^3+\&c.)=\mathrm{P}^{na+nby+ncy^2+ndy^3+\&c.} \quad (5).$$

But, by equation (1), the last expression may be written in this form

$$1+(\mathrm{A}-1+\mathrm{B}y+\mathrm{C}y^2+\mathrm{D}y^3+\&c.)$$
$$=\mathrm{P}^{a+b(\mathrm{A}-1+\mathrm{B}y+\mathrm{C}y^2+\&c.)+c(\mathrm{A}-1+\mathrm{B}y+\mathrm{C}y^2+\&c.)^2+\&c.} \quad (6).$$

By equating the second members of (5) and (6) we shall have

$$na+nby+ncy^2+ndy^3+\&c.=a+b(\mathrm{A}-1+\mathrm{B}y+\mathrm{C}y^2+\mathrm{D}y^3+\&c.)+c(\mathrm{A}$$
$$-1+\mathrm{B}y+\mathrm{C}y^2+\mathrm{D}y^3+\&c.)^2+\&c \quad (7).$$

Since a is independent of y, if we suppose $y=0$, equation (1) gives $a=0$; in a similar manner from (2) we find $\mathrm{A}=1$; hence equation (7) becomes

$$nby+ncy^2+ndy^3+\&c.=b(\mathrm{B}y+\mathrm{C}y^2+\mathrm{D}y^3+\&c.)+c(\mathrm{B}y+\mathrm{C}y^2+\mathrm{D}y^3+$$
$$\&c.)^2+d(\mathrm{B}y+\mathrm{C}y^2+\mathrm{D}y^3+\&c.)^3+\&c. \quad (8).$$

Equating the co-efficients of the several powers of y with zero, we find

$$\mathrm{B}b-nb=0, \qquad \therefore \mathrm{B}=n, \quad \ldots\ldots \quad (9),$$

$$\mathrm{B}^2c^2-nc+\mathrm{C}b=0, \quad \therefore -\mathrm{C}=\frac{n^2c^2-nc}{b}, \quad \ldots\ldots \quad (10),$$

$$nd-b\mathrm{D}-2\mathrm{BC}c+\mathrm{B}^3d=0, \quad \therefore \mathrm{D}=\frac{nd-2\mathrm{BC}c+\mathrm{B}^3d}{b} \quad \ldots\ldots \quad (11),$$

&c., &c.

Now since a, b, c, &c. are independent of n, by assuming n successive-

ly equal to 1, 2, 3, &c., and substituting the corresponding values of C, D, &c., derived from (2), the equations (10), (11), &c. will give, after the necessary reductions, $b=1$, $c=-\frac{1}{2}$, $d=\frac{1}{3}$, &c. Hence, by (1),

$$\log.(1+y)=y-\tfrac{1}{2}y^2+\tfrac{1}{3}y^3-\tfrac{1}{4}y^4+\&c.) \quad . \quad . \quad . \quad (12),$$

P, being the base of the system. If we put $P=P'^m$, equation (12) will become

$$\log.(1+y)=m\,(y-\tfrac{1}{2}y^2+\tfrac{1}{3}y^3-\tfrac{1}{4}y^4+\&c. \quad . \quad . \quad (13),$$

where P′ is the base, and m the modulus. Hence the logarithmic theorem is proved.

Again, substituting the values of b, c, d, &c. in equations (10), (11), &c. they become

$$C=\frac{n(n-1)}{1\,.\,2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14),$$

$$D=\frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15),$$

&c.

Substituting the values of A, B, C, D, &c. in (2), it becomes

$$(1+y)^n=1+ny+\frac{n\,(n-1)}{1\,.\,2}.\,y^2+\frac{n(n-1)\,(n-2)}{1\,.\,2\,.\,3}.\,y^3+\&c.\,.\,(16).$$

Hence the binomial theorem is proved for fractional and negative exponents. The theorem for positive integral exponents is, of course, assumed in this demonstration.

ARTICLE X.

A GENERAL INVESTIGATION,

With reference to the construction of a Table of Numbers, and the roots of two cubes of which they are composed, and practical illustrations of the equation

$$x^3+y^3=(x+y)\,.\,(x^2-xy+y^2)$$

when $x+y$ is a cube, or a multiple of a cube, or nine times a multiple of a cube; and also when x^2-xy+y^2 is a cube, or a multiple of a cube: together with the solutions of two general Problems, and their application to several Examples.

BY WILLIAM LENHART, ESQ., YORK, PENN.

ARTICLE (1). Those numbers composed of two cubes whose roots are integers, may be easily found by combining the different cubes in pairs, by means of their differences, and in various other ways.

(2). Suppose $x+y=a^3$, and $x>y$; then if a be an even number, we may assume

$$\left.\begin{array}{l} x=s+1,\ s+2,\ s+3,\ \&c. \\ y=s-1,\ s-2,\ s-3,\ \&c. \end{array}\right\} \text{ and thence have}$$

$x+y=2s=a^3$, or $s=\frac{1}{2}a^3$, and $x-y=2.\ 4.\ 6$, &c. Now by substitution in the general equation

$$x^3+y^3=(x+y).(x^2-xy+y^2)=(x+y)\{x(x-y)+y^2\}$$

and reduction, we shall find the following column of equations, namely:

$$\text{A}\left\{\begin{array}{l}\left(\frac{s+1}{a}\right)^3+\left(\frac{s-1}{a}\right)^3=2(s+1)+(s-1)^2=s^2+3 \\ \left(\frac{s+2}{a}\right)^3+\left(\frac{s-2}{a}\right)^3=4(s+2)+(s-2)^2=s^2+12 \\ \left(\frac{s+3}{a}\right)^3+\left(\frac{s-3}{a}\right)^3=6(s+3)+(s-3)^2=s^2+27 \\ \quad\&c., \qquad\qquad \&c., \qquad\qquad \&c.\end{array}\right.\begin{array}{l}9 \\ 15 \\ 21\end{array}$$

(3). When $a=2r$, it is evident that the cubes in the 2d, 4th, 6th equations in A, will not be prime to each other, and therefore, in this case, in making our numerical calculations, we use the 1st, 3d, 5th, 7th, &c. equations: that is, when a is of the form $2r$, we shall have the annexed column of equations with cubes prime to each other, namely:

$$\text{B.}\left\{\begin{array}{l}\left(\frac{s+1}{a}\right)^3+\left(\frac{s-1}{a}\right)^3=s^2+3 \\ \left(\frac{s+3}{a}\right)^3+\left(\frac{s-3}{a}\right)=s^2+27 \\ \left(\frac{s+5}{a}\right)^3+\left(\frac{s-5}{a}\right)=s^2+75 \\ \quad\&c. \qquad\qquad \&c.\end{array}\right.\begin{array}{l}24 \\ 48 \\ 72\end{array}$$

(4). When a is of the form $2rn$, and n is a prime number, then the cubes in the $\frac{1}{2}(n+1)$ equation in B, and those in every succeeding n^{th} equation, will not be prime to each other, and consequently may be rejected; but by doing so, the differences become irregular, which must not be. To avoid this, therefore, and that we may have regular differences, the equations in B must be divided into $n-1$ columns of equations, so that the difference of the roots of the cubes, or the terms in each column of cubes, may be $2n$. For example, suppose $n=3$, $r=1$; then $\frac{1}{2}(n+1)=2$. Hence, the cubes in the 2d equation in B, and those in every succeeding 3d equation will not be prime to each other, and consequently the $n-1$ columns of equations, having $2n=6$ for the difference of the roots of the cubes, will, according to the division, stand thus:

First Column C.		*Second Column C'.*	
$\frac{(s+1)^3}{a^3}+\frac{(s-1)^3}{a^3}=s^2+3,$	144	$\frac{(s+5)^3}{a^3}+\frac{(s-5)^3}{a^3}=s^2+75,$	288
$\frac{(s+7)^3}{a^3}+\frac{(s-7)^3}{a^3}=s^2+147,$	360	$\frac{(s+11)^3}{a^3}+\frac{(s-11)^3}{a^3}=s^2+363,$	504
$\frac{(s+13)^3}{a^3}+\frac{(s-13)^3}{a^3}=s^2+507,$	576	$\frac{(s+17)^3}{a^3}+\frac{(s-17)^3}{a^3}=s^2+867,$	720
&c.		&c.	&c.

(5). If we suppose a to be of the form $2^r nm$, and n, m, prime numbers, then, in B, the cubes in $\frac{1}{2}(n+1)$ and $\frac{1}{2}(m+1)$ equations, and those in the n^{th} and m^{th} equations respectively after these, will not be prime to each other; and we shall have $(n-1)\times(m-1)$ columns of equations, the difference of the roots of the cubes, or the terms in each column of cubes being $2nm$. If $n=3$, $m=5$, $r=1$, then $\frac{1}{2}(n+1)=2$, $\frac{1}{2}(m+1)=3$, $(n-1).(m-1)=8$, $2nm=30$. In this case, therefore, we shall have 8 columns of equations with regular differences, as in C. C′: thus rendering the numerical calculations, though very extensive and apparently heavy, quite simple and easy.

Note. It will be perceived that in each of the equations, the respective numbers on the right hand side are equal to three times the square of the corressponding numbers in the first column of cubes in C. C′. Thus, in the 2d equation in C. 147 is equal to $3\times(7)^2$, and in equation 3d in C′, $867=3\times(17)^2$. That the difference of the first two equations in the first column of equations is equal to the sum of $2nm$, terms of the series of differences of the equations in A: and, that the difference of the differences of the respective column of equations is equal to 6 times $(2nm)^2$. We therefore have three methods of commencing the columns of equations numerically, independent of the aid of cubes, which, when the numbers representing a are large, is a great consideration.

(6). Let us now suppose $x+y=a^3$, $x>y$, and a an odd number. We may now assume

$$x=s+1,\ s+2,\ s+3,\ \&c.$$
$$y=s\quad,\ s-1,\ s-2,\ \&c.$$

and thence have

$$x+y=2s+1=a^3, \text{ or } s=\tfrac{1}{2}(a^3-1), \text{ and } x-y=1.3.5.7, \&c.$$

and substituting these values in the equation

$$x^3+y^3=(x+y).(x(x-y)+y^2)$$

and reducing, as before, there will result

$$\text{D.}\quad\left\{\begin{aligned}
&\left(\frac{s+1}{a}\right)^3+\left(\frac{s}{a}\right)^3=1\,(s+1)+s^2=s^2+s+1\\
&\left(\frac{s+2}{a}\right)^3+\left(\frac{s-1}{a}\right)^3=3\,(s+2)+(s-1)^2=s^2+s+7\\
&\left(\frac{s+3}{a}\right)^3+\left(\frac{s-2}{a}\right)^3=5\,(s+3)+(s-2)^2=s^2+s+17\\
&\quad\&c.\qquad\qquad\qquad\&c.\qquad\qquad\qquad\&c.
\end{aligned}\right.$$

in which column of equations, it is plain that the cubes in the $\frac{1}{2}(a+1)$, equation, and those in every successive a^{th} equation, will be divisible by a^3, and consequently when a is taken 3. 5. 7, &c. will serve as a check upon the correctness of our numerical calculations. These remarks will also apply to the equations in B, when a is of the form $2^r n$, n being a prime, for then the cubes in the $\frac{1}{2}(n+1)$ equation, and also those in every successive n^{th} equation will reduce, and hence a similar check. Instead of dividing the equations in D into columns of equations to suit any particular value of a, we would advise, as more convenient, to use them as they

are, and to observe and use the check. There will, however, be a great saving of time and labour, in making a division of the equations in D, when a is divisible by 3, inasmuch as there will be but two columns, as in C. C′, thus:

First Column (E.)

$$\frac{(s+1)^3}{a^3}+\frac{(s)^3}{a^3}=s^2+s+1 \qquad {\scriptstyle 36}$$

$$\frac{(s+4)^3}{a^3}+\frac{(s-3)^3}{a^3}=s^2+s+37. \qquad {\scriptstyle 90}$$

$$\frac{(s+7)^3}{a^3}+\frac{(s-6)^3}{a^3}=s^2+s+127 \qquad {\scriptstyle 144}$$

&c.

Second Column (E′.)

$$\frac{(s+3)^3}{a^3}+\frac{(s-2)^3}{a^3}=s^2+s+19 \qquad {\scriptstyle 72}$$

$$\frac{(s+6)^3}{a^3}+\frac{(s-5)^3}{a^3}=s^2+s+91 \qquad {\scriptstyle 126}$$

$$\frac{(s+9)^3}{a^3}+\frac{(s-8)^3}{a^3}=s^2+s+217 \qquad {\scriptstyle 180}$$

&c. &c.

Note. The respective numbers on the right hand side of each of the equations in this article, it will be perceived, are equal to the difference of the cubes of the numbers, without their signs, in the corresponding columns of cubes, or equal to the product of the greater number by their sum, neglecting signs, together with the square of the lesser. The difference of the two first equations in the first column of equations is equal to the sum of a, terms of the series of differences of the equations in D. And the the difference of the differences of the respective columns of equations is 6 times a^2.

(7). Suppose $x+y=$ a multiple of a cube $=a'a^3$. Then, if a be odd, and a' be the double of a prime number, $x+y=a'a^3$ will be even, and $s=\frac{1}{2}a'a^3$ odd, and the 1st, 3d, 5th, &c., equations in A will therefore not be prime to each other; consequently, in this case, we shall have the following equations, namely:

$$\text{F.}\quad\left\{\begin{aligned}
&\left(\frac{s+2}{a}\right)^3+\left(\frac{s-2}{a}\right)^3=a'\left(4(s+2)+(s-2)^2\right)=a'(s^2+12). && {\scriptstyle 36a'}\\
&\left(\frac{s+4}{a}\right)^3+\left(\frac{s-4}{a}\right)^3=a'\left(8(s+4)+(s-4)^2\right)=a'(s^2+48) && {\scriptstyle 60a'}\\
&\left(\frac{s+6}{a}\right)^3+\left(\frac{s-6}{a}\right)^3=a'\left(12(s+6)+(s-6)^2\right)=a'(s^2+108) && {\scriptstyle 84a'}\\
&\left(\frac{s+8}{a}\right)^3+\left(\frac{s-8}{a}\right)^3=a'\left(16(s+8)+(s-8)^2\right)=a'(s^2+192) && {\scriptstyle 108a'}
\end{aligned}\right.$$

&c. &c. &c.

From which, when a is divisible by 3, we form, in the same manner as in Article (4), the two annexed columns of equations, namely:

First Column (G.)

$$\frac{(s+2)^3}{a^3}+\frac{(s-2)^3}{a^3}=a'(s^2+12) \qquad {\scriptstyle 180a'}$$

$$\frac{(s+8)^3}{a^3}+\frac{(s-8)^3}{a^3}=a'(s^2+192) \qquad {\scriptstyle 396a'}$$

$$\frac{(s+14)^3}{a^3}+\frac{(s-14)^3}{a^3}=a'(s^2+588) \qquad {\scriptstyle 612a'}$$

&c. &c.

Second Column (G′)

$$\frac{(s+4)^3}{a^3}+\frac{(s-4)^3}{a^3}=a'(s^2+48) \qquad {\scriptstyle 252a'}$$

$$\frac{(s+10)^3}{a^3}+\frac{(s-10)^3}{a^3}=a'(s^2+300) \qquad {\scriptstyle 468a'}$$

$$\frac{(s+16)^3}{a^3}+\frac{(s-16)^3}{a^3}=a'(s^2+768) \qquad {\scriptstyle 684a'}$$

&c. &c.

Note. In the other cases of this article, according to the form of a', or a, is, we use one or other of the preceding columns of equations, observing to multiply the right hand side of the first equation in each column of equations, and also the difference between the first and second equations by a', and then proceed regularly with the differences. Thus, if a' be of the form 2^r, or 4^r, and a not being divisible by 3, we use the equations in B; but, if a be divisible by 3, we then use those in C. C'; and so also do we it a' or a be divisible by 3. And again: if a' and a be odd numbers, not divisible by 3, we use the equations in D; but, if a' or a be odd numbers divisible by 3, we then use those in E. E'.

(8). When $x+y$ is equal to 9 or 9 times a multiple of a cube, viz. $9a'a^3$, it is easy to perceive that x^2-xy+y^2 is divisible by 3. Hence, in this case, according as is the form of a' or a, we use the equations in one or other of the foregoing articles, observing that the root of each of the cubes be divided by 3, and that the right hand side of each equation, and also the differences be multiplied by $\frac{1}{3}a'$

(9). Suppose $x^2-xy+y^2=m^3$, and x and y prime to each other; then from the general equation

$$x^3+y^3=(x+y).\ (x^2-xy+y^2)$$

we shall have $$\left(\frac{x}{m}\right)^3+\left(\frac{y}{m}\right)^3=x+y;$$

and in general

$$\text{I.}\left(\frac{nx+y}{m}\right)^3+\left(\frac{\overline{n+1}\,.\,x-ny}{m}\right)^3=(n^2+n+1)\,.\,(\overline{2n+1}\,.\,x-\overline{n-1}\,.\,y),$$

$$\text{II.}\left(\frac{x-\overline{n+1}.y}{m}\right)^3+\left(\frac{\overline{n+1}\,.\,x-ny}{m}\right)^3=(n^2+n+1)\,.\,(\overline{n+2}\,.\,x-\overline{2n+1}\,.y),$$

$$\text{or,}\left(\frac{\overline{n+1}\,.\,y-x}{m}\right)^3+\left(\frac{nx+y}{m}\right)^3=(n^2+n+1)\,.\,(\overline{n-1}\,.+\overline{n+2}\,.\,y).$$

$$\text{III.}\left(\frac{\overline{n+2}\,.\,x-\overline{2n+1}\,.\,y}{m}\right)^3+\left(\frac{\overline{2n+1}\,.\,x-\overline{n-1}\,.\,y}{m}\right)^3=(n^2+n+1)\times 9\times(\overline{n+1}\,.\,x-ny).$$

$$\text{IV.}\left(\frac{\overline{n-1}\,.\,x+\overline{n+2}\,.\,y}{m}\right)^3+\left(\frac{\overline{2n+1}\,.\,x-\overline{n-1}\,.\,y}{m}\right)^3=(n^2+n+1)\times 9\times(nx+y),$$

$$\text{or,}\left(\frac{\overline{1-n}\,.\,x-\overline{n+2}\,.\,y}{m}\right)^3+\left(\frac{\overline{n+2}\,.\,x-\overline{2n+1}\,.\,y}{m}\right)^3=(n^2+n+1)\times 9\times(x-\overline{n+1}\,.\,y).$$

Now, in order to make $x^2-xy+y^2=m^3$ put

$$x=v+w,\ y=v-w, \text{ and then } x^2-xy+y^2=v^2+3w^2=m^3,$$

which is a particular case of the well known general formula $ax^2+cy^2=$a cube, so ingeniously resolved by Euler, in his Algebra, Chap. xii. vol. ii. In the case of $v^2+3w^2=m^2$. he finds

$$v=p^3-9pq^2=p\,.\,\overline{p+3q}\,.\,\overline{p-3q},\qquad w=3p^2q-3q^3=3q\,.\,\overline{p+q}\,.\,\overline{p-q},$$

and thence $v^2+3w^2=(p^2+3q^2)^3=m^3$; or $m=p^2+3q^2$. We therefore have

$$^* x=v+w=p\,.\,\overline{p+3q}\,.\,\overline{p-3q}+3q\,.\,\overline{p+q}\,.\,\overline{p-q}.$$

$$y=v-w=p\,.\,\overline{p+3q}\,.\,\overline{p-3q}-3q\,.\,\overline{p+q}\,.\,\overline{p-q},$$

and with these values of x and y in terms of the indeterminates p and q, we shall here note down equation II. which, when $n=1$, becomes

$$\text{V.}\left(\frac{9q(p^2-q^2)+p(p^2-9q^2)}{3(p^2+3q^2)}\right)^3+\left(\frac{9q(p^2-q^2)-p(p^2-9q^2)}{3(p^2+3q^2)}\right)^3$$
$$=2q\,.\,\overline{p+q}\,\,.\,\,\overline{p-q},$$

because, when $p=3q\pm1$, the two cubes approach an equality, and because the right hand side of the equation, composed entirely of factors, presents a variety of speculations which we have attended to, but which, for the sake of brevity, we shall omit, leaving them to the skill and practice of the reader.

(10). If $am^3+a'x$ and $am^3+a'y$ represent any two numbers their sum will be

$$2am^3+a'(x+y) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and the sum of their cubes

$$2a^3m^9+3a^2a'm^6(x+y)+3aa'^2m^3(x^2+y^2)+a'^3(x^3+y^3) \quad . \quad (5).$$

Now, divide (5) by (4) and the quotient will be

$$a^2m^6+aa'm^3(x+y)+a'^2(x^2-xy+y^2),$$

which, by assuming $x^2-xy+y^2=m'm^3$, becomes $a^2m^6+aa'm^3(x+y)+a'm'm^3$, and thence

$$\left(\frac{am^3+a'x}{m}\right)^3+\left(\frac{am^3+a'y}{m}\right)^3=\Big(2am^3+a'(x+y)\Big)\times$$
$$\Big(a^2m^3+aa'(x+y)+a'^2m'\Big) \quad . \quad . \quad . \quad . \quad . \quad (6).$$

From this general equation, $a^2m^3+aa'(x+y)+a'^2m$, being a common factor as n^2+n+1 is in the equations I. II. III. IV. the reader will readily perceive that the following equations are deduced in the same manner as II. III. IV. have been from equation I. in Article (9), namely:

$$\left(\frac{a'(-x+y)}{m}\right)^3+\left(\frac{am^3+a'y}{m}\right)^3=(am^3+'a(-x+2y))\times$$
$$(a^2m^3+aa'(x+y)+a'^2m') \quad . \quad (7),$$

or, if $am^3+a'x$ be greater than $am^3+a'y$,

$$\left(\frac{a'(x-y)}{m}\right)^3+\left(\frac{am^3+a'x}{m}\right)^3=(am^3+a'(2x-y))\times$$
$$(a^2m^3+aa'(x+y)+a'^2m') \quad . \quad (8)$$

$$\left(\frac{am^3+a'(-x+2y)}{m}\right)^3+\left(\frac{2am^3+a'(x+y)}{m}\right)^3=9(am^3+a'y)$$
$$(a^2m^3+aa'(x+y)+a'^2m') \quad . \quad (9)$$

$$\left(\frac{am^3+a'(2x-y)}{m}\right)^3+\left(\frac{2am^3+a'(x+y)}{m}\right)^3=9(am^3+a'x)\times$$
$$(a^2m^3+aa'(x+y)+a'^2m') \quad . \quad (10),$$

* That x and y may be prime to each other it is evident, from these values, that p must be even and q odd, or q even and p odd, and that p must not be assumed equal to 3, or a multiple of 3.

or, if $(am^3+a'(-x+2y))>(2am^3+a'(x+y))$

$$\left(\frac{-am^3+a'(-2x+y)}{m}\right)^3+\left(\frac{am^3+a'(-x+2y)}{m}\right)^3=9(a'(-x+y))\times$$
$$(a^2m^3+aa'(x+y)+a'^2m') \quad . (11),$$

in all of which a and a' may be assumed any numbers positive or negative; but x and y must be such as to make $x^2-xy+y^2=m^3$, or $m'm^3$. It will always be a cube, when, as in Article (9),

$x=p\cdot\overline{p+3q}\cdot\overline{p-3q}+3q\cdot\overline{p+q}\cdot\overline{p-q}$, $y=p\cdot\overline{p+3q}\cdot\overline{p-3q}-3q\cdot\overline{p+q}\cdot\overline{p-q}$, and a multiple of a cube when we take for x, $(nx+y)$ and for y $(\overline{n+1}x-ny)$; or, indeed, any other of the numerators of the roots of the cubes in the equations I. II. III. IV.

(11). Although the equations in Article (10) appears to be very formidable, they are, nevertheless, regular and perfectly symmetrical, and a practical application of them soon becomes easy and familiar. We might show the system or regularity which pervades the whole of these equations by a general illustration of them, but, as a development of that kind would necessarily occupy much space, and, perhaps, be deemed rather irksome, we shall forbear, and only consider equation (6), which is the principal one: and as that shall be managed, the other equations may be managed also. If, therefore, we suppose $a=1$, and assume for a', ±1, ±2, ±3, &c., respectively, we shall find, by a regular substitution in (6) the following equations, viz.

$$\left(\frac{m^3\pm x}{m}\right)^3+\left(\frac{m^3\pm y}{m}\right)^3=(2m^3\pm(x+y)).(m^3\pm(x+y)+m'),$$
$$\left(\frac{m^3\pm 2x}{m}\right)^3+\left(\frac{m^3\pm 2y}{m}\right)^3=(2m^3\pm2(x+y)).(m^3\pm2(x+y)+4m'),$$
$$\left(\frac{m^3\pm 3x}{m}\right)^3+\left(\frac{m^3\pm 3y}{m}\right)^3=(2m^3\pm3(x+y)).(m^3\pm3(x+y)+9m'),$$
$$\left(\frac{m^3\pm 4x}{m}\right)^3+\left(\frac{m^3\pm 4y}{m}\right)^3=(2m^3\pm4(x+y)).(m^3\pm4(x+y)+16m'),$$

&c., &c., &c.

in which the system or law of progression is manifest, and may obviously be continued ad infinitum.

It may not be amiss to observe here, that when a is taken negative, the roots in the above equations form a descending series, and when $a'x$ or $a'y$ becomes greater than am^3 one of the roots will be positive, and the other negative, and thence we shall obtain numbers equal to the difference, instead of the sum of two cubes, and so shall continue to do until both $a'x$ and $a'y$ become greater than am^3, in which case, both roots becoming negative, the signs may be changed, and we shall have the roots

$$\frac{a'x-am^3}{m} \text{ and } \frac{a'y-am^3}{m}$$

and thence an ascending series of roots which will, in many cases, be in quite a different proportion from those produced by the descending series. This operation in practice is extremely curious as well as interesting, and the student will find, on trial, that it is well worthy of his attention.

(12). By my means of these formulas, we have calculated, and with much labour arranged in consecutive order, or in tabular form, a great variety of numbers, and the roots of two cubes of which they are composed; and by the aid of which we have been enabled to resolve, with comparative ease, many of the most abstruse problems belonging to the Diophantine or Indeterminate Analysis.

[Mr. Lenhart has inserted in his manuscript numerous examples of these tables; we regret that our limits, already too far encroached upon by this interesting paper, will only admit of our inserting the following one, as a specimen of the facility with which they may be calculated from Mr. Lenhart's formula.]

If we take $a=3$, in Article (6), then $s = 13$, and from the equations E and E′,

$$\begin{array}{llll}
(\tfrac{14}{3})^3 + (\tfrac{13}{3})^3 = 183, & & (\tfrac{16}{3})^3 + (\tfrac{11}{3})^3 = 201, & \\
 & 36 & & 72 \\
(\tfrac{17}{3})^3 + (\tfrac{10}{3})^3 = 219, & & (\tfrac{19}{3})^3 + (\tfrac{8}{3})^3 = 273, & \\
 & 90 & & 126 \\
(\tfrac{20}{3})^3 + (\tfrac{7}{3})^3 = 309, & & (\tfrac{22}{3})^3 + (\tfrac{5}{3})^3 = 399, & \\
 & 144 & & 180 \\
(\tfrac{23}{3})^3 + (\tfrac{4}{3})^3 = 453, & & (\tfrac{25}{3})^3 + (\tfrac{2}{3})^3 = 579. & \\
 & 198 & & \\
(\tfrac{26}{3})^3 + (\tfrac{1}{3})^3 = 651, & & &
\end{array}$$

(13). If, in Article (9), we suppose $p=4$ and $q=1$, we shall find $x=73$, and $y=-17$; and assuming $n=0, 1, 2, 3,$ &c., we should find so many series of numbers and their component cubes from I. II. III. IV. severally. If we take $n=18$, which has the peculiar property of making the factor n^2+n+1 of these formulas a cube, we shall find from equation I,

$$\left(\frac{1297}{19}\right)^3 + \left(\frac{1693}{19}\right)^3 = 343 \times 2990,$$

or, dividing by $343=(7)^3$,

$$\left(\frac{1297}{133}\right)^3 + \left(\frac{1693}{133}\right)^3 = 2990,$$

and so from II, III, and IV, in the same way.

Again, suppose $p=11$, $q=2$; then $x=1637$, $y=233$, and $m=133$; and taking $n=0$, equation II. gives us

$$\left(\frac{1404}{133}\right)^3 + \left(\frac{1637}{133}\right)^2 = 3041 \cdot \cdot \cdot \cdot \cdot \cdot \quad (12).$$

And if $x=361$ and $y=359$, then $x+y=720=8\times 9\times 10=9a'a^3$; cr $a'=10$ and $a=2$. Hence, according to Article (8), and the first equation in c, Article (4), we shall have

$$\left(\frac{s+1}{3a}\right)^3 + \left(\frac{s-1}{3a}\right)^3 = \frac{a'}{3}(s^2+3)$$

that is $$\left(\frac{361}{6}\right)^3 + \left(\frac{359}{6}\right)^3 = \frac{10}{3}\left((360)^2+3\right) = 432010 \ . \quad . \ (13);$$

and with (12) and (13), as will be presently shown, we find numbers to answer the conditions of Questions VIII. (251) and IX. (252) proposed in No. XIII. of the Mathematical Diary.

We shall now proceed to the resolution of the following general Problems, and to make an application of their solutions to a few examples.

Problem I. To divide a given number (A) into three cubes.

Solution. Let a^3x^3, b^3x^3 and c^3x^3 represent the 3 required cubes, then $a^3x^3 + b^3x^3 + c^3x^3 = A$, and consequently $x^3 = \frac{A}{a^3+b^3+c^3}$; or supposing t to be one of the tabular numbers, and putting $a^3 + b^3 = t$, $x^3 = \frac{A}{t+c^3}$, from which, when $A = t + c^3$, we have $x = 1$, and when $r^3A = t + c^3$, then $x = \frac{1}{r}$. We may also suppose $c = p - s$, in which case $x^3 = \frac{A}{t+c^3}$ becomes $x^3 = \frac{A}{t + p^3 - 3p^2s + 3ps^2 - s^3}$: or, putting $r^3A = t - s^3$ it will be changed to $x^3 = \frac{A}{p^3 + 3p^2s + ps^2 + r^3A}$, which being divided above and below by A, gives $x^3 = 1 + \left(\frac{p^3}{A} - \frac{3p^2s}{A} + \frac{3ps^2}{A} + r^3\right)$ that is, $\frac{p^3}{A} - \frac{3p^2s}{A} + \frac{3ps^2}{A} + r^3 = \text{cube} = \left(r + \frac{ps^2}{r^2A}\right)^3$ by assumption. This equation reduced gives $p = \frac{3r^3As}{r^3A - s^3}$. Then $c = p - s = s\left(\frac{2r^3A + s^3}{r^3A - s^3}\right)$ and $x = \frac{1}{r}\left(\frac{r^3A - s^3}{r^3A + 2s^3}\right)$. Consequently the three required roots will be $ax = \frac{a}{r}\left(\frac{r^3A - s^3}{r^3A + 2s^3}\right)$; $bx = \frac{b}{r}\left(\frac{r^3A - s^3}{r^3A + 2s^3}\right)$; $cx = \frac{s}{r}\left(\frac{2r^3A + s^3}{r^3A + 2s^3}\right)$.

Note. It is manifest trom these results that s^3 must be less than r^3A, for if otherwise, the roots ax, bx, become either $\frac{0}{3rs^2}$ or negative, which cannot be admitted. And from the equations

$$r^3A - c^3 = t, \qquad r^3A + s^3 = t,$$

we deduce the two following Rules for finding, in numbers, three cubes to correspond with all values of (A); and thence may readily be perceived the easy manner in which any number of cubes may be found to answer the same purpose.

Rule I. Multiply the given number (A) by a cube (r^3), and from the product deduct a series of cubes, prime to (r^3), until you find a remainder that shall be equal to (t) one of the the tabular numbers composed of two cubes. Substitute the two cubes in place of the remainder, transpose the deducting cube that made the remainder, divide by the multiple cube (r^3), and the result will be three cubes equal to the given number (A).

Remark. From this rule it appears evident that to divide (A) into three cubes, each greater or less than a possible number (N), the deducting cube must be greater or less than (r^3N^3), and each of the two cubes

which compose the remainder or tabular number (t), after being divided by (r^3) must also be greater or less than (N).

Rule II. Multiply the given number (A) by a cube r^3, and to the product add a series of cubes, prime to (r^3), until you find a sum that shall be equal to (t), a tabular number composed of two cubes the roots of which, viz. (a) and (b), set down. Next, substitute the root (r) of the multiple cube, the root of the cube, which by its addition to (r^3A) made the tabular number (t), and also the roots (a) and (b) in the formulas

$$\frac{a}{r}\left(\frac{r^3\text{A}-s^3}{r^3\text{A}+2s^3}\right);\ \frac{b}{r}\left(\frac{r^3\text{A}-s^3}{r^3\text{A}+2s^3}\right);\ \frac{s}{r}\left(\frac{2r^3\text{A}+s^3}{r^3\text{A}+2s^3}\right);$$

and you will have the roots of three cubes that shall be equal to the given number (A).

Examples under Rule I.

Example I. Assume A $=1$; then $1\times(2)^3-(1)^3=7=(\frac{4}{3})^3+(\frac{5}{3})^3$ by table. Consequently $1=(\frac{3}{6})^3+(\frac{4}{6})^3+(\frac{5}{6})^3$. Or, $1\times(3)^3-(2)^3=19=(\frac{1}{3})^3+(\frac{8}{3})^3$. Hence $1=(\frac{1}{9})^3+(\frac{6}{9})^3+(\frac{8}{9})^3$.

Ex. II. Suppose A $=2$; then $2\times(10)^3-(7)^3=1657=(\frac{1}{6})^3+(\frac{71}{6})^3$ by table. Consequently $2=(\frac{1}{60})^3+(\frac{42}{60})^3+(\frac{71}{60})^3$. Again, $2\times(61)^3-(28)^3=432010$, which, by our table or equation (13), is equal to $(\frac{359}{6})^3+(\frac{361}{6})^3$. Hence, $2=(\frac{168}{366})^3+(\frac{359}{366})^3+(\frac{361}{366})^3$.

Now, as each of these roots is less than unity, by subtracting the cube of each numerator from the cube of the common denominator, we shall find

$$1982015, \quad 2759617, \quad 44286264;$$

which are three positive integers to answer Question VIII. (251) in Number XIII. of the Diary, which requires "*three positive integers, such, that their sum and the sum of every two of them may be cubes.*"

Examples under Rule II.

Example I. Assume A $=1$; then, taking $r=2$ and $s=1$, we find the tabular number $t=9$, which gives $a=2$ and $b=1$, and these values substituted in the general formulas in Rule II produce the roots $\frac{7}{20}$, $\frac{14}{20}$, $\frac{17}{20}$.

Example II. Assuming A $=2$, $r=3$, $s=2$, we find $t=62$, which, by table, gives $a=\frac{7}{3}$, $b=\frac{11}{3}$. Hence the roots $\frac{161}{315}$, $\frac{253}{315}$, $\frac{348}{315}$.

Example III. If we assume A $=4$, $r=9$, and $s=5$, then $r^3\text{A}+s^3=t=3041$, which, by our table, or equation (12), gives $a=\frac{1404}{133}$ and $\frac{1637}{133}$. Hence, substituting the values of A, r, s, a and b, in the general formulas in Rule II. we find the roots

$$\frac{1404}{1197}\times\frac{2791}{3166};\quad \frac{1637}{1197}\times\frac{2791}{3166};\quad \frac{665}{1197}\times\frac{5957}{3166};$$

and therefore

$$4=\left(\frac{3918564}{3789702}\right)^3+\left(\frac{3961405}{3789702}\right)^3+\left(\frac{4568867}{3789702}\right)^3.$$

And, as each of these roots is greater than unity, by subtracting unity from the cube of each, we shall obtain the following fractions, namely:

$$\frac{5743015291812773736}{54427098504275016408},\ \frac{7738158893915488717}{54427098504275016408},\ \frac{40945924318546753955}{54427098504275016408}.$$

to answer question IX. (252), in Number XIII of the Mathematical Diary, which requires "*to divide unity into three such positive parts, that if each part be increased by unity, the sums shall be three rational cubes.*"

Problem II. To divide $n\pm1$ into n cubes, each greater or less than unity: n being supposed greater than 2.

Solution (1.) When $n=3$, we multiply $n\pm1$ by m^3, and from the product deduct a series of cubes prime to, and greater or less than m^3, until we find a remainder R, that shall be equal to a tabular number composed of two cubes r^3, r'^3, whose roots divided by m, shall each be greater or less than unity. For instance, suppose the deducting cube to be $(m\pm q)^3$ then $m^3(n\pm1)-(m\pm q)^3=R=r^3+r'^3$ or, transposing $(m\pm q)^3$ and dividing by m^3, $n\pm1=\left(\frac{r}{m}\right)^3+\left(\frac{r'}{m}\right)^3+\left(\frac{m\pm q}{m}\right)^3$ each of which, by hypothesis, is $>$ or $<$ than unity.

(2.) We may also have the equation $m^3(n\pm1)+s^3=R=r^3+r'^3$, ($s$ and m being prime to each other) and substituting m, $n\pm1$, s, r, and r' in place of r, A, s, a, and b respectively in the formulas of Rule II, we shall have three cubes equal to $n\pm1$. But the suppositions must be such as to make them greater or less than unity, which can only be ascertained by trial. The methods of these two articles have been verified in examples II. and III. under Rules I. and II. in the solution to Problem I.

(3). When n is greater than 3, and an even number, we proceed thus: Suppose A, B, C, D, &c. to be a series of tabular numbers composed of two cubes, viz. $a^3+a'^3$, $b^3+b'^3$, $c^3+c'^3$, &c., whose roots are such, that, when divided by m, each shall be greater or less than unity. Then, when $n=4$, for example, from $m^3(n\pm1)$ deduct A, B, C, &c. in succession until you find a remainder R, equal to a tabular number composed of r^3, r'^3 whose roots divided by m, shall also be $>$ or $<$ than unity. Now, if we suppose $m^3(n\pm1)-A=R=r^3+r'^3$ we shall find

$$n\pm1=\left(\frac{a}{m}\right)^3+\left(\frac{a'}{m}\right)^3+\left(\frac{r}{m}\right)^3+\left(\frac{r'}{m}\right)^3 \quad . \; . \; . \quad (14);$$

each of which, by hypothesis, is $>$ or $<$ than unity.

(4). When $n=6$ we shall have a combination of remainders, namely:

$$m^3(n\pm1)-A-B;\ m^3(n\pm1)-A-C;\ m^3(n\pm1)-B-C,\ \&c.$$

and as many divisions of $n\pm1$ may be effected as there are remainders equal to tabular numbers composed of two cubes having the requisite properties. If we suppose

$$m^3(n\pm1)-A-B=R=r^3+r'^3,$$

then $$m^3(n\pm1)=A+B+R=a^3+a'^3+b^3+b'^3+r^3+r'^3,$$ or,

$$n\pm1=\left(\frac{a}{m}\right)^3+\left(\frac{a'}{m}\right)^3+\left(\frac{b}{m}\right)^3+\left(\frac{b'}{m}\right)^3+\left(\frac{r}{m}\right)^3+\left(\frac{r'}{m}\right)^3 (15),$$

each greater or less than unity.

(5). Or, we may suppose a cube $>$ or $<$ than m^3 to be added to one of the tabular numbers A, B, or C, &c., and this sum deducted from $m^3(n\pm1)$ will leave a remainder R$'$, which, by Rule I or II, may be di-

vided into three cubes having the proper requisites. As an illustration let us add $(m \pm q)^3$ to $A=a^3+a'^3$ and deduct the sum from $m^3(n\pm1)$; then

$$m^3(n\pm1)-(A+(m\pm q)^3)=R'.$$

Now, if $R'\times p'^3-(pm\pm q')^3=R=r^3+r'^3$ we shall have

$$R'=\left(\frac{r}{p}\right)^3+\left(\frac{r'}{p}\right)^3+\left(\frac{pm\pm q'}{p}\right)^3=m^3(n\pm1)-\left(A+(m\pm q)^3\right)$$

which, being reduced and divided by m^3, gives

$$n\pm1=\left(\frac{a}{m}\right)^3+\left(\frac{a'}{m}\right)^3+\left(\frac{m\pm q}{m}\right)^3+\left(\frac{r}{pm}\right)^3+\left(\frac{r'}{pm}\right)^3$$
$$+\left(\frac{pm\pm q'}{pm}\right)^3 \ldots \quad (16)$$

each greater or less than unity.

(6). Or, if R' should be divisible by a cube x^3, that is, should $R'=x'x^3$, we then multiply x' by p^3, and deduct cubes prime to p^3 and $>$ or $<$ than $\left(\frac{pm}{x}\right)^3$ until we find a remainder $R=r^3+r'^3$, $\frac{rx}{pm}$ and $\frac{r'x}{pm}$ being each $>$ or $<$ than unity. Thus, suppose $x'\times p^3-\left(\frac{pm\pm q'}{x}\right)^3=R=r^3+r'^3$,

then $$x'=\left(\frac{r}{p}\right)^3+\left(\frac{r'}{p}\right)^3+\left(\frac{pm\pm q'}{px}\right)^3, \quad \text{or,}$$

$$R'=x'x^3=\left(\frac{rx}{p}\right)^3+\left(\frac{r'x}{p}\right)^3+\left(\frac{pm\pm q'}{p}\right)^3=m^3(n\pm1)-\left(A+(m\pm q)^3\right).$$

Hence there results

$$n\pm1=\left(\frac{a}{m}\right)^3+\left(\frac{a'}{m}\right)^3+\left(\frac{m\pm q}{m}\right)^3+\left(\frac{rx}{pm}\right)^3+\left(\frac{r'x}{pm}\right)^3$$
$$+\left(\frac{pm\pm q'}{pm}\right)^3 \quad (17)$$

each greater or less than unity.

(7). Again. We may also suppose $m^3(n\pm1)$ to be divided into two parts s and s' suitably near an equality, and then by Rule I or II divide them into three cubes, such that their roots divided by m, may each be $>$ or $<$ than unity. Let us suppose

$$s\times p^3-(pm\pm q)^3=t=s^3+s'^3,$$
$$s'\times p'^3-(p'm\pm q')^3=t'=r^3+r'^3,$$

then $s=\left(\frac{s}{p}\right)^3+\left(\frac{s'}{p}\right)^3+\left(\frac{pm\pm q}{p}\right)^3$ and $s'=\left(\frac{r}{p'}\right)^3+\left(\frac{r'}{p'}\right)^3$
$+\left(\frac{p'm\pm q'}{p'}\right)^3$, and $n\pm1=\frac{s}{m^3}+\frac{s'}{m^3}$, becomes

$$n\pm1=\left(\frac{s}{pm}\right)^3+\left(\frac{s'}{pm}\right)^3+\left(\frac{pm\pm q}{pm}\right)^3+\left(\frac{r}{p'm}\right)^3+\left(\frac{r'}{p'm}\right)^3$$
$$+\left(\frac{p'm\pm q'}{p'm}\right)^3 \quad (18)$$

each greater or less than unity.

And in the same manner, as in the foregoing articles, divisions may be effected when $n = 8.\ 10.\ 12.$ &c.

(8). When n is an odd number greater than 3, we proceed thus: In the case of $n=5$, we may subtract from $m^3(n\pm1)$ cubes prime to, and $>$ or $<$ than m^3, and from the remainder deduct successively A, B, C, &c., until we find a remainder $R=r^3+r'^3$, r and r' having the necessary properties. Thus, we may have

$$m^3(n\pm1)-(m\pm q)^3-A=R=r^3+r'^3,$$

and thence by reduction and division

$$n\pm1=\left(\frac{a}{m}\right)^3+\left(\frac{a'}{m}\right)^3+\left(\frac{m\pm q}{m}\right)^3+\left(\frac{r}{m}\right)^3+\left(\frac{r'}{m}\right)^3 \quad . \quad . \quad (19)$$

each greater or less than unity.

(9). Or, we may take from $m^3(n\pm1)$ the tabular numbers A, B, C, &c. in succession, and proceed with the remainder R′, as in Article (5), and thus find

$$n\pm1=\left(\frac{a}{m}\right)^3+\left(\frac{a'}{m}\right)^3+\left(\frac{r}{pm}\right)^3+\left(\frac{r'}{pm}\right)^3+\left(\frac{pm\pm q'}{pm}\right)^3 \quad . \quad (20)$$

each greater or less than unity.

(10). Or, if R′ should be divisible by a cube, in that case we proceed as in Article (6), and shall find

$$n\pm1=\left(\frac{a}{m}\right)^3+\left(\frac{a'}{m}\right)^3+\left(\frac{rx}{pm}\right)^3+\left(\frac{r'x}{pm}\right)^3+\left(\frac{pm\pm q'}{pm}\right)^3 \quad . \quad (21)$$

each greater or less than unity.

And in a similar manner we effect divisions when $n = 7.\ 9.\ 11$, &c.

(11). There are several other methods of accomplishing the different divisions that are exceedingly curious, and which contain artifices of the neatest kind; but those, which we have here recorded, appear to us to be the easiest and most convenient in practice, and seem also to embrace all that is required by the problem. And by the aid of our table of numbers composed of two cubes, and the articles contained in this solution, we are enabled to solve the two following neat and abstruse questions, namely:

I. To divide unity into n positive parts, such that if unity be *diminished* by each part, the *n remainders* shall be rational cubes.

II. To divide unity into n positive parts, such that if unity be *increased* by each part, the *n sums* shall be rational cubes.

APPLICATIONS.

Example I. Divide 5 into four cubes, each greater than unity.

Application of Article (3). Here $n = 4.\ n+1 = 5$, and assuming $m=27$ we have $m^3(n+1)=98415$. Now, if from our table of numbers composed of two cubes, we take $A = 43290 = (\frac{191}{7})^3 + (\frac{198}{7})^3 = a^3+a'^3$, we shall have $m^3(n+1) - A = R = 55125 = 441 \times (5)^3$. But $441 = (\frac{11}{2})^3 + (\frac{13}{2})^3$, therefore $R = (\frac{55}{2})^3 + (\frac{65}{2})^3 = r^3 + r'^3$. Hence, substituting these values in (14), and reducing to the same common denominator

$$5=\left(\frac{382}{378}\right)^3+\left(\frac{385}{378}\right)^3+\left(\frac{398}{378}\right)^3+\left(\frac{455}{378}\right)^3$$

each of which is greater than unity.

Example II. Divide 3 into four cubes, each less than unity.

Application of Article (3). Here $n=4$, $n-1=3$. Assume $m=27$, then $m^3(n-1)=59049$. By table $A = 31400 = (24)^3 + (26)^3 = a^3 + a'^3$, therefore, $m^3(n-1) - A = R = 27649 = (\frac{287}{12})^3 + (\frac{289}{12})^3 = r^3 + r'^3$. Hence, from (14) we have

$$3 = \left(\frac{287}{324}\right)^3 + \left(\frac{288}{324}\right)^3 + \left(\frac{289}{324}\right)^3 + \left(\frac{312}{324}\right)^3$$

each less than unity.

Example III. Divide 6 into five cubes, each greater than unity..

Application of Article (8). Here $n=5$, $n+1=6$. Suppose $m=23$, $q=3$ and $A = 27769 = (\frac{277}{12})^3 + (\frac{299}{12})^3 = a^3 + a'^3$, so shall $m^3(n+1) - (m+q)^3 - A = 27657 = R = (\frac{25}{4})^3 + (\frac{27}{4})^3 = r^3 + r'^3$, and by a substitution in (19) and reducing &c.

$$6 = \left(\frac{277}{276}\right)^3 + \left(\frac{285}{276}\right)^3 + \left(\frac{291}{276}\right)^3 + \left(\frac{299}{276}\right)^3 + \left(\frac{312}{276}\right)^3$$

which are each greater than unity.

OTHERWISE.

Application of Article (9). Assume $m=23$, and take from the table $A = 27748 = (\frac{132}{6})^3 + (\frac{142}{6})^3 = a^3 + a'^3$, then $m^3(n+1) = 73002$, $m^3(n+1) - A = 45254 = R'$. Now, take $p=1$, $q'=2$, then $pm+q'=25$, and, as in Article (5), $R' \times p^3 - (pm+q')^3 = R = 29629 = (\frac{163}{7})^3 + (\frac{180}{7})^3 = r^3 + r'^3$. We therefore obtain from (20) the following cubes, each > than unity

$$6 = \left(\frac{973}{966}\right)^3 + \left(\frac{978}{966}\right)^3 + \left(\frac{1043}{966}\right)^3 + \left(\frac{1050}{966}\right)^3 + \left(\frac{1080}{966}\right)^3.$$

OTHERWISE.

Application of Article (10). As before, assume $m=23$, and take $A = 31466 = (\frac{112}{5})^3 + (\frac{121}{5})^3 = a^3 + a'^3$, then $m^3(n+1) - A = 41536 = 649 \times (4)^3 = R' = x'x^3$, therefore $x' = 649$ and $x=4$. Now take $p=1$, $q'=1$, then, as in Article (6), $x' \times p^3 - \left(\frac{pm+q'}{x}\right)^3 = R = 433 = (\frac{35}{6})^3 + (\frac{37}{6})^3 = r^3 + r'^3$. Hence, substituting in (21), reducing and arranging, we find

$$6 = \left(\frac{350}{345}\right)^3 + \left(\frac{357}{345}\right)^3 + \left(\frac{360}{345}\right)^3 + \left(\frac{370}{345}\right)^3 + \left(\frac{393}{345}\right)^3$$

which are also each > than unity.

Example IV. Divide 4 into five cubes, each less than unity.

Application of Article (8). Here $n=5$, $n-1=4$. If we assume $m=27$, $\cancel{q=3,}$ and $A = 33201 = (25)^3 + (26)^3 = a^3 + a'^3$, we shall have $m^3(n-1) - (m-q)^3 - A = R = 31707 = (\frac{503}{21})^3 + (\frac{550}{21})^3 = r^3 + r'^3$, and by a substitution of the respective values in (19), and a proper reduction, find

$$4 = \left(\frac{503}{567}\right)^3 + \left(\frac{504}{567}\right)^3 + \left(\frac{525}{567}\right)^3 + \left(\frac{546}{567}\right)^3 + \left(\frac{550}{567}\right)^3,$$

which are each less than unity.

Example V. Divide 7 into six cubes, each greater than unity.

Application of Article (7). Here $n=6$, $n+1=7$. Assume $m=8$, then $m^3(n+1)=3584$. Now take $s'=1737$, then $s'=1847$, and supposing $p=3$ and $q=1$, we shall have $pm+q=25$, and $s\times p^3-(pm+q)^3=t=31274$ $=(\frac{123}{5})^3+(\frac{127}{5})^3=s^3+s'^3$. Again: suppose $p'=4$ and $q'=1$, then $p'm+q=33$, and $s'\times p'^3-(p'm+q')^3=t'=82271=(\frac{101}{3})^3+(\frac{106}{3})^3=r^3+r'^3$. Hence, substituting in (18), we shall find

$$7=\left(\frac{123}{120}\right)^3+\left(\frac{127}{120}\right)^3+\left(\frac{25}{24}\right)^3+\left(\frac{101}{96}\right)^3+\left(\frac{106}{96}\right)^3+\left(\frac{33}{32}\right)^3,$$

or, reducing to the same denominator and arranging

$$7=\left(\frac{492}{480}\right)^3+\left(\frac{495}{480}\right)^3+\left(\frac{500}{480}\right)^3+\left(\frac{505}{480}\right)^3+\left(\frac{508}{480}\right)^3+\left(\frac{530}{480}\right)^3,$$

each of which is greater than unity.

And by similar modes of application we have readily found

$$5=\left(\frac{542}{615}\right)^3+\left(\frac{543}{615}\right)^3+\left(\frac{582}{615}\right)^3+\left(\frac{583}{615}\right)^3+\left(\frac{605}{615}\right)^3+\left(\frac{610}{615}\right)^3,$$

each cube being less than unity.

$$8=\left(\frac{482}{480}\right)^3+\left(\frac{487}{480}\right)^3+\left(\frac{490}{480}\right)^3+\left(\frac{500}{480}\right)^3+\left(\frac{513}{480}\right)^3+$$
$$\left(\frac{518}{480}\right)^3+\left(\frac{520}{480}\right)^3,$$ each of which is greater than unity.

$$6=\left(\frac{429}{525}\right)^3+\left(\frac{497}{525}\right)^3+\left(\frac{504}{525}\right)^3+\left(\frac{510}{525}\right)^3+\left(\frac{511}{525}\right)^3+$$
$$\left(\frac{513}{525}\right)^3+\left(\frac{516}{525}\right)^3,$$ which are each less than unity.

And it is obvious that, in the same manner precisely, a variety of divisions may be effected when $n=8, 9, 10,$ &c.

WM. LENHART.

York, Penn., July, 1836.

NEW BOOKS.

An Elementary Treatise on Spherical Geometry, by Professor Benjamin Peirce, Harvard University, Cambridge.

Elements of Analytical Geometry; embracing the equations of the Point, the Straight Line, the Conic Sections, and Surfaces of the First and Second Order; by Professor Davies, of West Point. Published by Wiley and Long, New-York.

A Treatise on the Differential and Integral Calculus, by the same Author, is also about to be published by the same house.

The New Edition of "*Application de l'Analyse a la Géométrie,*" by Monge, is said to be at last published by Bachelier.

THE

NOTICE TO SUBSCRIBERS.

To the names of the gentlemen associated for the support of the Mathematical Miscellany, noticed in the last number, the following have been since added on the same terms:—

Prof. F. N. Benedict, University of Vermont, ... $10 per annum.
Profs. Caldwell and McClintock, Dickinson College, Pa., ... 5 "
J. H. McElfresh, Esq., Frederick, Md., ... 5 "

In addition to this we have also received, to aid us in the publication of the work, from

Dr. Strong, Rutgers' College, N. J., ... $15 00
Rev. J. Penney, D. D., Hamilton College, N. Y., ... 15 00

and several gentlemen subscribed for two copies.

they were before the operation. Hence has arisen the common method of selecting for the denominator of the transformed fractions, the continued product of all the given denominators, which is necessarily divisible by each of them. But as this number is often very large, and the consequent reductions tedious, it becomes an object to find the *least* common denominator of two or more fractions; that is the least number by which all their denominators will divide without a remainder, or the least common multiple of their denominators.

9. The least common multiple of two or more numbers is the continued product of the highest powers of all the prime factors which enter

into them; and among many methods for finding this number, that which best agrees with the spirit of these "hints" is the following:

Decompose each number into its prime factors, by art 5., and selecting the highest power of each of these prime numbers, neglecting all the lower powers, multiply them together and the result will be the least common multiple of the given numbers. Thus if one of the numbers divides by 2 three times, or has for one of its factors 2^3, while the other only divides by 2 twice, the 2^2 is neglected, and 2^3 is taken for a factor of the required multiple.

10. Example.—Let it it be required to transform the fractions

$$\frac{289}{45864}, \frac{2521}{3780}, \frac{9209}{22050}, \frac{5477}{54600}$$

to equivalent fractions, having a common denominator.

By decomposing the denominators we find

$$\begin{aligned}
45864 &= 2.2.2.3.3.7.7.13 = 2^3.3^2.7^2.13, \\
3780 &= 2.2.3.3.3.5.7 \quad = 2^2.3^3.5.7, \\
22050 &= 2.3.3.5.5.7.7 \quad = 2.3^2.5^2.7^2, \\
54600 &= 2.2.2.3.5.5.7.13 = 2^3.3.5^2.7.13,
\end{aligned}$$

Hence the least common multiple of the denominators is

$$2^3.3^3.5^2.7^2.13 = 3439800,$$

which is the least common denominator of the given fractions.

The advantage of this method is seen more immediately in finding the equivalent fractions, for having already the factors of the denominators of the given fractions and the required ones, the transforming multipliers are had at once by inspection.

$$\frac{289}{45864} = \frac{289}{2^3.3^2.7^2.13} \times \frac{3.5^2}{3.5^2} = \frac{21675}{2^3.3^3.5^2.7^2.13},$$

$$\frac{2521}{3780} = \frac{2521}{2^2.3^3.5.7} \times \frac{2.5.7.13}{2.5.7.13} = \frac{2294110}{2^3.3^3.5^2.7^2.13},$$

$$\frac{9209}{22050} = \frac{9209}{2.3^2.5^2.7^2} \times \frac{2^2.3.13}{2^2.3.13} = \frac{1436604}{2^3.3^3.5^2.7^2.13},$$

$$\frac{5477}{54600} = \frac{5477}{2^3.3.5^2.7.13} \times \frac{3^2.7}{3^2.7} = \frac{345051}{2^3.3^3.5^2.7^2.13},$$

and the sum of these four fractions is

$$\frac{21675 + 2294110 + 1436604 + 345051}{2^3.3^3.5^2.7^2.13} = \frac{4097440}{2^3.3^3.5^2.7^2.13}$$

$$= \frac{2^5.5.25609}{2^3.3^3.5^2.7^2\,13} = \frac{2^2.25609}{3^3.5.7^2.13} = \frac{102436}{85995} = 1\frac{16441}{85995}.$$

Although to one unaccustomed to the use of the signs of multiplication, &c., these operations may appear difficult, yet, if you be prepared by previous practice in multiplying fractions, you will not only soon become familiar with the different steps, but with the principles on which they are founded, which cannot fail to present themselves to your notice.

11. In Decimal Fractions, the chief difficulties of young students arise from a misapprehension of their nature and use. They are simply fractions whose denominators are 10 or some power of 10; thus $\frac{27}{100}$, is a decimal fraction, and the usual mode of writing it, namely ,27, is a

part of the same conventional system, which distinguishes our peculiar scheme of notation. In a mixed decimal fraction therefore, like 33,333, the scale of relation among the several digits is precisely the same as if the comma were removed, each figure having ten times the value of that to the right of it; but the comma makes the absolute value of each figure one thousand times less than it otherwise would be, for the unit figure is the one to the left of the decimal point, and the number is actually $33\frac{333}{1000}$. You will thus see the reason of the rules for placing the decimal point in a product or quotient; for $,5 \times ,03 = \frac{5}{10} \times \frac{3}{100} = \frac{15}{1000} =$ $,015$*, and $,75 \div ,5 = \frac{75}{100} \div \frac{5}{10} = \frac{75}{100} \times \frac{10}{5} = \frac{15}{10} = 1,5$.†

In using decimals it must be borne in mind that, in general, they are only the approximate values of the fractions they are meant to represent, and therefore that the results obtained from operating upon them will only be approximations. Thus by adding a decimal which is true only to the fifth decimal place, to another which is true to the seventh decimal place your sum will only be true to the fifth place of decimals, and so on.

12. In multiplying a number, as 36,782594, which is true to the sixth decimal figure by 7, or any number between 1 and 10, your product can be depended on only to the fifth decimal figure; in multiplying it by 70 or any number between 10 and 100, your product can be depended on only to the fourth decimal figure; in multiplying it by 700, or any number between 100 and 1000, your product can be depended on only to the third decimal figure, and so on. When your multiplier is less than unity the approximation may increase. Thus, when the multiplier is ,7, or any number between 1 and ,1, you may depend on the sixth decimal figure; when it is ,07 or any number between ,1 and ,01, you may depend on the seventh decimal figure, and so on. It is evident therefore, that for a mixed multiplier, like 73,28053, where several products are to be added together, the degree of the approximation must be measured by that which the left hand figure alone would give; that is the whole product can be depended on only to the fourth decimal figure. A similar course of reasoning will enable you to determine the degree of certainty in a product arising from any approximating decimal multiplied by any true one. When both factors are approximations, by calling one of them the multiplicand, you can find the number of decimal figures which can

* Generally, two decimal fractions, either simple or mixed, may be represented by $\frac{a}{10^m}$ and $\frac{b}{10^n}$, where m and n are the number of *decimal* figures they contain, or the number of figures to the right of the decimal points. Now $\frac{a}{10^m} \times \frac{b}{10^n} = \frac{ab}{10^{m+n}}$; that is the product will have $m + n$ decimal figures, or as many as both of the factors, the numbers being multiplied as if they were integers.

† Generally, $\frac{a}{10^m} \div \frac{b}{10^n} = \frac{a}{10^m} \times \frac{10^n}{b} = \frac{\frac{a}{b}}{10^{m-n}}$; that is, after dividing the numbers as if they were integers, there must be $m-n$ decimal figures in the quotient, or as many as the dividend has more than the divisor.

depended on as if the other factor or multiplier were true; then calling the latter factor the multiplicand and the former the multiplier, find in the same manner as before the number of figures which can be depended on:—the least of the two numbers so found is to be regarded as the true one, unless they happen to be the same.*

13. It is always advisable, however, to perform the calculations for one or two more figures than can be actually depended on as certain, for otherwise errors will arise from adding the different products together;—and thus when many operations are required to be performed on a series of numbers, it is necessary to express the approximations for these numbers, two or three figures nearer the truth than you actually require in the final result.

Example.—Multiply 36,782594 by 73,28053, the former being true to the sixth and the latter to the fifth decimal figure.

The product of 36,782594 by 73,2 &c., is true only to the fourth decimal figure, and the product of 73,28053 by 36, 7 &c., is true only to the third decimal figure, and therefore we can only depend on the result to the third place of decimals. Keeping therefore the fourth figure in all the products the work may stand thus:

36,782594				
73,28053				
2574,7815	multiplying by	70	and omitting	80
110,3477	"	3	"	82
7,3565	"	,2	"	188
2,9426	"	,08	"	0752
,0183	"	,0005	"	912970
,0011	"	,00003	"	0347782
2695,4477	product.			

* In general, two approximating decimals may be represented by $\frac{a}{10^m}$ and $\frac{b}{10^n}$, where m and n are the number of decimal figures expressed. The errors, or numbers by which these differ from the true ones can then be represented by $\frac{x}{10^{m+1}}$ and $\frac{y}{10^{n+1}}$, so that the true numbers will be $\frac{a}{10^m}+\frac{x}{10^{m+1}}$ and $\frac{b}{10^n}+\frac{y}{10^{n+1}}$. The product of these two numbers will be.

$$\frac{ab}{10^{m+n}}+\frac{ay+bx}{10^{m+n+1}}+\frac{xy}{10^{m+n+2}}.$$

Neglecting the final quantity, as very small in comparison with the others, the error in the product will be $\frac{ay+bx}{10^{m+n+1}}$. Now x and y can never be greater than 5, for if x were greater than 5, we should make the last figure in a one greater, and the quantity x would then be in defect and should be written with a negative sign; hence the error in the product cannot be greater than $\frac{5a+5b}{10^{m+n+1}}$ or $\frac{a+b}{2\,.\,10^{m+n}}$. Hence also the simple rule: Subtract the number of figures in the half sum of the two factors, taken as if

But it will be seen that the rejected figures amount to more than ,00028; to provide against this circumstance, it is usual when the first of the rejected figures is 5, or more than 5, to make the figure first put down in the product one greater than it otherwise would be; in this manner the products by 70, by 3, and by ,0005 would become severally 2574,7816, 110,3478 and ,0184, and the complete product would be 2695,448.

14. The results arrived at in these articles may be included in a Rule for multiplying decimals, so mechanical in its operation that the youngest student can perform the calculation.

Place the unit figure of the multiplier under that figure of the multiplicand which has the same value as the last figure you want in the product; that is if you want to preserve four decimal figures in your products, place the unit figure of the multiplier under the fourth decimal figure of the multiplicand, &c., and place the remaining figures of the multiplier in an inverted order. Then multiply by each figure of the multiplier, placing the products directly under each other, beginning each multiplication with that figure of the multiplicand which stands above the figure you are multiplying by, and minding to increase the first figure of the product as it would have been had the operation been carried on to one or two figures more to the right; thus, if the product of the next figure to the right of the one you begin with be between 5 and 15 increase your first figure by 1, if it be between 15 and 25 increase your first figure by 2, &c. The sum of the products thus found is the true one required.

Example. Multiply 1,732080536 by 71,648273.

By the Rule in Note to Art. 12, there will be 9 figures in the half sum of the two numbers, and there are 15 decimal figures in both numbers, therefore there will be $15 - 9 = 6$ decimal figures true in the product; hence if we preserve 7 decimal figures in the partial products we must place the unit figure of the multiplier under the seventh decimal figure of the multiplicand and the work will stand thus:

```
1,732080536
 372846,17
-----------
1212456375
  17320805
  10392483
    692832
    138566
      3464
      1212
        52
-----------
124,1005789
```

they were whole numbers, from the number of decimal figures in both factors, and the remainder will be the number of decimal figures in the product that may always be depended on as true.

By attending, however, to the form of the error, it will be seen that there will often be more than these actually true, and by taking the fractions so that the approximations shall be one in excess and the other in defect, the error in the product may be much lessened.

And the product, true to the sixth decimal figure, is 124,100579.

15. In dividing decimal fractions the extent of the approximation for the first figure in the quotient can be determined as in multiplication, and then for every successive figure in the quotient, one figure may be omitted in the divisor. For instance in dividing 31,5160435 by ,7630842456,

```
,7630842456)31,5160435(41,30087
             305233698
             ---------
               9926737
               7630842
               -------
               2295895
               2289253
               -------
                  6642
                  6105
                  ----
                   537
                   534
                   ---
                     3
```

The first figure in the quotient is evidently 4 and its value is 40; now 40 multiplied by ,7630842456 will be true to the eighth place of decimals, but as the dividend is only expressed to the seventh place, we reject the two last figures of the divisor, adding 2 to the first figure, however, as if they had actually been multipled; thus the divisor used for finding the first figure 4, is ,76308424, the one used for finding the second figure 1, is ,7630842, the one used for finding the third figure 3, is ,763084, &c., rejecting one figure in the divisor for every quotient figure, and if one figure is marked by a point or otherwise every time you multiply, there can no confusion arise from the process.

16. In finding the approximate root of a number, there is in general a new divisor to be found for every successive figure in the root; but after this operation has been performed three or four times, it will be found that three, four, or more of the first figures in two successive divisors will be the same, and therefore they may be used as the divisor to find several successive figures in the root, the number of which may be determined as in the last article. Thus to find the approximate square root of 17.

```
            17)4,1231056256
               16
               ---
     81       )100
                81
               ----
     822      )1900
               1644
               -----
     8243     )25600
               24729
               ------
     82461    )87100
               82461
               --------
      8246205)4639000
      . . . . .  412310
                 ------
                  51590
                  49477
                  -----
                   2113
                   1649
                   ----
                    464
                    412
                    ---
                     52
                     49
                     --
                      3
```

The next complete divisor would have been 82462106, and the figures 82462 would be common to this and all succeeding divisors, and as these are sufficient to find five figures in the root, the succeeding part of the operation may be performed as in division, unless the approximation is required to extend beyond the tenth place of decimals, and in that case more true divisors must be found.

This contraction is of still greater advantage in approximating to the third and higher roots of numbers, where the numerical calculations for finding the divisors soon become extremely laborious, as the process will be sufficiently obvious from the example already given, I will here leave the subject.

ARTICLE IV.

SOLUTIONS TO THE QUESTIONS PROPOSED IN NUMBER II.

QUESTION I. BY ALFRED.

Given the equations,

$$\begin{aligned} x^2 + xy + y^2 + xv + yv + v^2 &= 202, \\ x^2 + xy + y^2 + xz + yz + z^2 &= 394, \\ x^2 + xv + v^2 + xz + vz + z^2 &= 522, \\ y^2 + yv + v^2 + yz + vz + z^2 &= 586, \end{aligned}$$

to find v, x, y, z.

FIRST SOLUTION, *by Mr. George K. Birely, Frederick College, Maryland.*

Let $s = x + y + v + z$; then by subtracting each of the three first equations from the fourth one, we shall have

$$s(z - x) = 384, \text{ or } z = x + \frac{384}{s}, \quad \ldots \ldots \quad (1.)$$

$$s(v - x) = 192, \text{ or } v = x + \frac{192}{s}, \quad \ldots \ldots \quad (2.)$$

$$s(y - x) = \ \ 64, \text{ or } y = x + \frac{64}{s}, \quad \ldots \ldots \quad (3.)$$

and by adding these three equations together, with the identical equation $x = x$ we find

$$s = 4x + \frac{640}{s}, \quad \ldots \ldots \quad (4.)$$

$$\therefore x = \frac{s}{4} - \frac{160}{s}, \quad \ldots \ldots \quad (5.)$$

By substituting this value of x in equations (1), (2), and (3),

$$y = \frac{s}{4} - \frac{96}{s}, v = \frac{s}{4} + \frac{32}{s}, z = \frac{s}{4} + \frac{224}{s} \quad \ldots \ldots \quad (6.)$$

Equations (5) and (6) substituted in the fourth of the given equations, we shall have

$$s^4 - 1136s^2 + 114688 = 0 \quad \ldots \ldots \quad (7.)$$

$$\therefore s^2 = 1024 \text{ or } 112, \text{ and } s = 32 \text{ or } 4\sqrt{7} \quad \ldots \ldots \quad (8.)$$

Taking $s = 32$, and writing it in (5) and (6) we find $x = 3$, $y = 5$, $v = 9$, and $z = 15$.

SECOND SOLUTION, *by Mr. Solomon Graves, Clinton Lib. Institute, New-York.*

Subtracting each equation separately from the last, we get,

$$z^2 - x^2 + y(z - x) + v(z - x) = 384, \text{ or } x + y + z + v = \frac{384}{z-x} \quad (1.)$$

$$v^2 - x^2 + y(v - x) + z(v - x) = 192, \text{ or } x + y + z + v = \frac{192}{v-x} \quad (2.)$$

$$y^2 - x^2 + v(y - x) + z(y - x) = 64, \text{ or } x + y + z + v = \frac{64}{y-z} \quad (3.)$$

and by equating the second members of these three equations we get

$$\frac{384}{z-x}=\frac{64}{y-x}, \text{ whence } z=6y-5x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

$$\frac{192}{v-x}=\frac{64}{y-x}, \text{ whence } v=3y-2x \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

substituting these values of z and v in the first and second given equations, we get

$$13y^2-10xy+\ 3x^2=202 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

$$43y^2-58xy+21x^2=394 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

Now put $y=nx$, then these equations give

$$x^2=\frac{202}{13n^2-10n+3}=\frac{394}{43n^2-58n+21} \quad . \quad . \quad (8.)$$

$$\therefore 99n^2-216n+85=0, \text{ and } n=\frac{5}{3} \text{ or } \frac{17}{33} \quad . \quad . \quad (9.)$$

If $n=\frac{5}{3}$, $y=\frac{5}{3}x$, and this substituted in equation (6) gives $x^2=9$, and $x=3$; therefore $y=\frac{5}{3}x=5$. Substituting these values of x and y in equations (4) and (5) we find $z=15$ and $v=9$. . . . . . .

——The proposer's solution was much like the second one. Neat solutions were also received from Messrs. Barton, Biddle, and Montgomery.

QUESTION II., *by* ——.

Given the equation

$$\frac{l'x+\frac{1}{2}}{lx}+\frac{3lx-\frac{2}{3}}{l'x}=1$$

where l represents the common and l' the Naperian logarith. of a number, to find x.

SOLUTION, *by Alfred, of Athens, Ohio.*

Let m = the modulus of the common system of logarithms, then will $l'x=\frac{lx}{m}$, and substituting this in the given equation

$$\frac{lx+\frac{1}{2}m}{mlx}+\frac{3mlx-\frac{2}{3}m}{lx}=1,$$

multiply by mlx,

$$lx+\tfrac{1}{2}m+3m^2lx-\tfrac{2}{3}m^2=mlx,$$

$$\text{and } (6-6m+18m^2)lx=4m^2-3m,$$

$$\therefore lx=\frac{4m^2-3m}{6(3m^2-m+1)}=-,0807802;$$

and making the decimal positive, to agree with the tables

$$lx=-1+,9192198, \text{ and } x=,8302708$$

——Solutions were also received from Messrs. Barton, Birely, Bowden, and Montgomery.

Having the numbers m and $\frac{1}{m}$ already calculated in most treatises on logarithms, it is better to find lx from the formula

$$lx = \frac{4m - 3 \cdot \frac{1}{m}}{6(3m - 1 + \frac{1}{m})}.$$

and using m and $\frac{1}{m}$ calculated to 20 places of figures we shall find

$$bx = -,08078.02290.72556.50592.3$$
$$= -1 + ,91921.97709.27443.49407.7$$
$$\text{and } x = ,83027.08125.22980.49236.$$

QUESTION III., *by* ——.

Given the equation

$$a \sin. x + b \cos x = c$$

to find x.

FIRST SOLUTION, *by Mr. T. B. Biddle, Institute at Flushing.*

Square the given equation, then

$$a^2 \sin^2 x + 2ab \sin x \cos x + b^2 \cos^2 x = c^2.$$

and subtracting this from the identical equation

$$a^2 + b^2 = a^2 + b^2.$$

we have $a^2(1 - \sin^2 x) - 2ab \sin x \cos x + b^2(1 - \cos^2 x) = a^2 + b^2 - c^2$, that is, $a^2 \cos^2 x - 2ab \sin x \cos x + b^2 \sin^2 x = a^2 + b^2 - c^2$; and extracting the square root,

$$a \cos x - b \sin x = \pm \sqrt{a^2 + b^2 - c^2} \quad . \quad . \quad . \quad . \quad (1.)$$

Subtract this equation multiplied by b, from the given equation multiplied by a, then

$$(a^2 + b^2) \sin x = ac \mp b\sqrt{a^2 + b^2 - c^2}$$

$$\text{or, } \sin x = \frac{ac \mp b\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2}, \quad . \quad . \quad (2.)$$

Or if we add equation (1) multiplied by a, to the given equation multiplied by b, we find

$$(a^2 + b^2) \cos x = bc \pm a\sqrt{a^2 + b^2 - c^2}$$

$$\text{or } \cos x = \frac{bc \pm a\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2}. \quad . \quad . \quad (3.)$$

In order to adapt these results to logarithmic calculation, let $a = d \sin \varphi$, $b = d \cos \varphi$, and $c = d \cos \theta$; that is, find the auxiliary angles φ and θ, such that

$$\text{Cot } \varphi = \frac{b}{a}, \text{ and } \frac{\cos \theta}{\sin \varphi} = \frac{c}{a}, \text{ or } \cos \theta = \frac{c \sin \varphi}{a} \quad (4.)$$

we should also have $d = \sqrt{a^2 + b^2}$. but it is not needed in the calculation. Hence $a^2 + b^2 - c^2 = d^2 - d^2 \cos^2 \theta = d^2 \sin^2 \theta$, and equation (2) becomes

$$\sin x = \sin \varphi \cos \theta \mp \cos \varphi \sin \theta = \sin (\varphi \mp \theta)$$
$$\therefore x = \varphi \mp \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

It is evident that if c were the hypothenuse of a right angled triangle, of which a and b are the two legs, we should have $a^2 + b^2 - c^2 = 0$, and $\sin x = \frac{a}{c}$; or x is that angle of the triangle opposite the side a.

SECOND SOLUTION, *by Mr. S. Graves.*

For $\cos x$ in the given equation, write its value $\sqrt{1 - \sin^2 x}$ and transpose

$$b\sqrt{1-\sin^2 x} = c - a\sin x.$$

squaring, $b^2 - b^2\sin^2 x = c^2 - 2ac\sin x + a^2\sin^2 x$

or $(a^2 + b^2)\sin^2 x - 2ac\sin x = b^2 - c^2$.

from which $\sin x = \dfrac{ac \pm b\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2}$.

which look out in the tables, and we have x.

——Good solutions were also sent by Alfred, and Messrs. Barton, Birely, and Montgomery.

QUESTION IV. (*From the Dublin Problems*).

Express the sides of a plane triangle, as functions of the radius of the circumscribed circle, and the three angles.

FIRST SOLUTION, *by Mr. J. J. Bowden, Institute at Flushing.*

Call the radius R, the sides a, b, c, and the angles respectively opposite them A, B, C. The angle included between the radii drawn to the angles B and C is an angle at the centre, and it subtends the same arc as the inscribed angle A, it is therefore $= 2$A. Thus we have an isosceles triangle whose sides are R, R and a, and its vertical angle 2A; therefore

$$\begin{aligned} a^2 &= R^2 + R^2 - 2R^2\cos 2A \\ &= 2R^2(1 - \cos 2A) \\ &= 4R^2\sin^2 A \end{aligned}$$

Therefore $\quad a = 2R\sin A$

Similarly $\quad b = 2R\sin B$, and $c = 2R\sin C$.

Also the area of the triangle $= \frac{1}{2}ab\sin C = 2R^2\sin A\sin B\sin C$.

SECOND SOLUTION, *by Mr. P. Barton, Jun., Orange County, Franklin, Mass.*

Let a, b, c, represent the natural sines of the three angles, and r the radius of the circumscribing circle, then it is manifest from the nature of the circle that the three sides will be

$$2ar,\ 2br,\ 2cr.$$

——The solutions of Alfred, and Messrs. Birely, Graves, and Montgomery, were also well worthy of insertion.

QUESTION V. *by* ——

Three circles, whose radii are r_1, r_2, r_3, respectively touch each other externally, prove that the area of the triangle, formed by joining their centres, is

$$\sqrt{r_1\, r_2\, r_3\, (r_1 + r_2 + r_3)}$$

FIRST SOLUTION, *by Mr. P. Barton, Jun.*

The three sides of the triangle are $r_1 + r_2$, $r_1 + r_3$, and $r_2 + r_3$; their half sum is $r_1 + r_2 + r_3$: this half sum diminished by each side, gives the remainders r_1 r_2, r_3; hence, by a well known rule, the area is $\sqrt{(r_1 + r_2 + r_3)r_1\, r_2\, r_3}$.

SECOND SOLUTION, *by Mr. Geo. K. Birely.*

It is obvious that the sides of the triangle will be $r_1 + r_2$, $r_2 + r_3$, $r_3 + r_1$; but the area of a triangle is equal to the square root of half the sum of the sides multiplied by the several differences between this half sum and the sides. Now the half sum is $r_1 + r_2 + r_3$, and the three differences are r_1, r_2, r_3; therefore the area is $= \sqrt{r_1 r_2 r_3(r_1 + r_2 + r_3)}$.

——Nearly similar to this were the solutions by Alfred, and by Messrs. Biddle, Bowden, Graves and Montgomery.

QUESTION VI., *by Mr. L. Van Bokkelen.*

An inflexible wire is made to pass through a given plane surface, which can traverse freely along it, and the wire is then fixed horizontally in a direction perpendicular to the wind; what angle must the plane make with the wire so that the wind may drive it along the wire with the greatest velocity?

FIRST SOLUTION, *by Mr. Thomas C. Montgomery, Institute at Flushing.*

Let s be the surface of the given plane, x the angle it makes with the wire, and f the force of a column of air blowing in a perpendicular direction on an unit of the surface, which would be a function of its density and the velocity of the wind at the time. Now the wind that acts upon the plane surface in its oblique position is that which would fall on the projection of this surface on a plane perpendicular to the direction of the wind, or parallel to the wire; and the area of this projection is evidently $s \cos x$, hence the force of the wind that acts on the plane in the direction of the wind is $= fs \cos x$.

This force is equivalent to a force $fs \cos^2 x$ acting in a direction perpendicular to the plane and tending to drive it in a direction parallel to itself; and this can be resolved into two forces, one of which, $fs \cos^3 x$, acts perpendicular to the wire and is destroyed by its reaction, and the other $fs \cos^2 x \sin x$ acts in the direction of the wire, and is to be a *maximum.*

$$\therefore \cos^2 x \sin x = \text{a } max.$$

Equating its differential to zero, we find $\sin^2 x = \frac{1}{3}$, or $\cos 2x = \frac{1}{3}$, therefore $x = 35° \ 15' \ 52''$.

SECOND SOLUTION, *by Alfred.*

Let $x =$ sine of the angle made by the plane with the wire: then its cosine $= \sqrt{1 - x^2} =$ sine of the angle made by the plane with the direction of the wind: the perpendicular force of the wind on the plane is represented by $1 - x^2 =$ square of the sine of incidence. Resolving this force into two others, one perpendicular, the other parallel to the wire, the latter force (which is only effectual to move the plane) will be represented by $x(1 - x^2) = x - x^3 = a\ maximum$, therefore $dx - 3x^2 dx = 0$, or $1 - 3x^2 = 0 \therefore x = \sqrt{\frac{1}{3}} =$ sine of $35° \ 16'$.

——Nearly thus was the solution of Mr. Graves.

THIRD SOLUTION, *by Mr. P. Ketchum, Hamilton College, Clinton, New-York.*

By a resolution of the wind's force, its pressure in a direction perpendicular to its course, is found to depend upon the sin. cos. of the required angle, while its quantity depends upon the cos of the same angle; then

$\sin \cos^2 = \text{a } max.$, or $\cos^3 - 2 \sin^2 \cos = 0$, and $\sin = \sqrt{\frac{1}{3}}$; hence the angle $= 35° \ 16'$.

——Mr. Ketchum also solved the preceding five questions, but his letter did not arrive until their solutions had been copied for the press.

Messrs. Barton and Birely also sent solutions.

ARTICLE V.

QUESTIONS TO BE ANSWERED IN NUMBER IV.

Their solutions must arrive before August 1st, 1837.

(7.) QUESTION I. (*Communicated by Mr. Lenhart.*)

Given $\begin{cases} xy = x^2 - y^2 \\ x^2 + y^2 = x^3 - y^3 \end{cases}$ to determine x and y by a pure quadratic.

(8.) QUESTION II., *by* ——.

Find the angle x, from the equation

$$\frac{1 + a\cos(x + \theta)}{\sin(x + \varphi)} = \frac{1 + a\cos\theta}{\sin\varphi},$$

(9.) QUESTION III., *by Mr. George K. Birely.*

Through a given point in a right line given in position, it is required to describe a circle, having its centre on the same line, which shall touch a circle, given in position and magnitude.

(10.) QUESTION IV. (*From the Dublin Problems.*)

Express the sides and area of a plane triangle, as functions of the radius of the inscribed circle and the three angles.

(11.) QUESTION V. BY *Mr. P. Ketchum.*

Required the sides of a trapezoid in which the oblique sides are equal, the sum of the parallel sides is 10, two-thirds of the difference of the parallel sides is equal to their perpendicular distance, and the distance of the centre of gravity from the longer of the parallel sides is equal to $1\frac{4}{9}$.

(12.) QUESTION VI. BY ——.

The semi-axes of two ellipses are 2,1 and 5,3. It is required to place them with their transverse axes on the same straight line, so that they may intersect each other at right angles.

SENIOR DEPARTMENT.

ARTICLE XI.

SOLUTIONS TO QUESTIONS PROPOSED IN ARTICLE VII.

QUESTION I. BY C. C. OF CAMBRIDGE, MASS.

Let a, b, c, be any three angles, prove that

$$1^\circ\ 2\sin(a + b + c) = \cos a \sin(b + c) + \cos b \sin(a + c) + \cos c \sin(a + b) - 2\sin a \sin b \sin c.$$

$$2^\circ\ 2\cos(a + b + c) = 2\cos a \cos b \cos c - \sin a \sin(b + c), - \sin b \sin(a + c) - \sin c \sin(a + b).$$

FIRST SOLUTION, *by Dr. T. Strong, Professor of Mathematics in Rutgers' College, New-Brunswick, N. J.*

$\sin(a+b+c) = \cos a \sin(b+c) + \cos(b+c)\sin a.$
$\quad = \cos a \sin(b+c) + \cos b \cos c \sin a - \sin a \sin b \sin c,$
and $\sin(a+b+c) = \cos b \sin(a+c) + \sin b \cos a \cos c - \sin a \sin b \sin c$;
$\therefore$ by addition we have
$2\sin(a+b+c) = \cos a \sin(b+c) + \cos b \sin(a+c) + \cos c \sin(a+b)$
$\quad - 2 \sin a \sin b \sin c,$
put $p = 3{,}14159$, &c., then change a, b, c, into $a + \frac{1}{2}p$, $b + \frac{1}{2}p$, $c + \frac{1}{2}p$, and the above result becomes
$2\cos(a+b+c) = 2\cos a \cos b \cos c - \sin a \sin(b+c) - \sin b \sin(a+c)$
$\quad - \sin c \sin(a+b).$

SECOND SOLUTION, *by Alfred, of Athens, Ohio.*

1°. $\sin(a+b+c) = \sin a \cos b \cos c + \sin b \cos a \cos c + \sin c \cos a \cos b - \sin a \sin b \sin c$, which is a well known formula; if now we add this to itself and arrange the terms we get

$$\begin{aligned}2\sin(a+b+c) &= (\sin a \cos b + \sin b \cos a)\cos c \\ &+ (\sin a \cos c + \sin c \cos a)\cos b \\ &+ (\sin b \cos c + \sin c \cos b)\cos a - 2\sin a \sin b \sin c \\ &= \sin(a+b)\cos c + \sin(a+c)\cos b + \sin(b+c)\cos a \\ &\quad - 2\sin a \sin b \sin c.\end{aligned}$$

2°. $\cos(a+b+c) = \cos a \cos b \cos c - \sin a \sin b \cos c - \sin a \sin c \cos b - \sin b \sin c \cos a,$

and by treating this in the same manner, we get

$$2\cos(a+b+c) = 2\cos a \cos b \cos c - \sin a \sin(b+c) - \sin b \sin(a+c) - \sin c \sin(a+b).$$

QUESTION II., BY MR. N. VERNON, FREDERICK, MD.

Divide a given plane triangle, into two equal parts, by a straight line of a given length; also into parts having any given ratio.

FIRST SOLUTION, *by Mr. O. Root, Hamilton College, Clinton, N. Y.*

Let b and c represent the sides of the given triangle, A their included angle, opposite to which is the dividing line a; if x and y represent the distances from A to the intersections of this line with b and c, and n the ratio of the part cut off to the whole triangle, then we shall have the following equations,

$xy = nbc$, and $2xy = 2nbc$ . . . . (1.)
$x^2 + y^2 - 2xy \cos A = a^2$, and $x^2 + y^2 = a^2 + 2nbc \cos A$ (2.)

By adding and subtracting (1) and (2), writing for $1 + \cos A$ and $1 - \cos A$ their equivalents, $2\cos^2 \frac{1}{2}A$ and $2 \sin^2 \frac{1}{2}A$, and extracting the roots

$$x + y = \sqrt{a^2 + 4nbc \cos^2 \tfrac{1}{2}A},$$
$$x - y = \sqrt{a^2 - 4nbc \sin^2 \tfrac{1}{2}A}.$$

where x and y are had by addition and subtraction,
When the triangle is equally divided, $n = \frac{1}{2}$.

SECOND SOLUTION, *by Mr. Vernon, the Proposer.*

Let ABC (Fig. 1.) be the given triangle; bisect AB in D and through D draw CE equal to the given line; make the triangle CEF = CDB, and on CE describe a segment of a circle to contain an angle = CBD; through

F and parallel CE draw FG cutting the segment in G, join CG and EG; make BH = GC, and BI = EG, and then HI will be the required line.

By construction the triangle CEG = CEF = CDB = $\frac{1}{2}$ CAB, and it has the angle CGE = CBA, consequently the triangle HBI = CGE = $\frac{1}{2}$ CAB.

If the given line CE be less than CD drawn from the greatest angle to the opposite side, or greater than the line drawn from the least angle bisecting the opposite side, the question becomes impossible. Also if GC exceed the height of the segment CGE it is impossible; and when GC equals the height, the triangle becomes isosceles.

In the same manner, the triangle may be divided into any given proportion, by first dividing one of the sides, as AB in the given ratio, and then proceeding as before.

——Professors Catlin and Peirce construct the triangle from knowing its base, area, and vertical angle, in the usual manner.

QUESTION III., BY ——.

If from any point, either within or without the plane of a given rectangle, straight lines be drawn to the angles of the rectangle: prove that the sum of the squares described on the lines drawn to two opposite angles is equal to the sum of the squares described on the lines drawn to the other two opposite angles.

FIRST SOLUTION, *by Dr. Strong.*

Imagine the diagonals drawn, then they will be equal and will bisect each other; also suppose right lines to be drawn from the given point to the extremities of the diagonals and to the point of bisection; then the lines drawn to the extremities of the diagonal and the diagonal form a triangle whose base, the diagonal, is bisected by the line drawn from the given point to the point of bisection; put L for this line, and D for the semi-diagonal and a^2 for the sum of the squares of the lines drawn to the extremities of the diagonal; then by geometry, $a^2 = 2D^2 + 2L^2$; in the same way, if b^2 = the sum of the squares of the lines drawn from the point to the extremities of the other diagonal, we shall have $b^2 = 2D^2 + 2L^2$ $\therefore a^2 = b^2$ as required.

SECOND SOLUTION, *by Alfred.*

Let ABCD be any rectangle, having $AB = a$, $AC = b$. Then if we make AB the axis of x, AC the axis of y, and the axis of z vertical, any point P may be determined by its co-ordinates x, y, z. We shall also have for the co-ordinates of the point A, 0, 0, 0; for those of B, a, 0, 0; for those of C, 0, b, 0; and for those of D, a, b, 0.

$$\text{Hence } AP^2 = x^2 + y^2 + z^2,\ BP^2 = (a-x)^2 + y^2 + z^2,$$
$$CP^2 = x^2 + (b-y)^2 + z^2,\ DP^2 = (a-x)^2 + (b-y)^2 + z^2$$
$$\therefore AP^2 + DP^2 = x^2 + (a-x)^2 + y^2 + (b-y)^2 + 2z^2 = BP^2 + CP^2$$

which was to be proved.

——Almost in precisely the same manner was the question solved by Messrs. Benedict, Barton, Catlin, Ketchum, Montgomery, and Perkins.

QUESTION IV., BY MR. P. BARTON, JUN., ORANGE, FRANKLIN COUNTY, MASS.

The sum of the diameters of the bases of a conical frustum is 4, the excess of the altitude above the difference of the diameters is 24, and the distance of the centre of gravity from the less end is 17; what are its altitude and diameters?

FIRST SOLUTION, *by the Proposer.*

Put $b = 4$, $a = 17$, $n = 24$, $x =$ less diameter, $y =$ greater, $z =$ altitude, then the general formula for the distance of the centre of gravity from the less end, is $\frac{z}{4} \times \frac{(x+y)^2 + 2y^2}{(x+y)^2 - xy}$; hence,

$$\frac{z}{4} \times \frac{(x+y)^2 + 2y^2}{(x+y)^2 - xy} = a \quad . \quad . \quad . \quad . \quad (1.)$$

$$x + y = b \quad . \quad . \quad . \quad . \quad (2.)$$

$$z + x - y = n \quad . \quad . \quad . \quad . \quad (3.)$$

Eliminating x, I obtain

$$z(b^2 + 2y^2) = 4a(b^2 - by + y^2) \quad . \quad . \quad (4.)$$

$$z = 2y + n - b \quad . \quad . \quad . \quad . \quad (5.);$$

and the elimination of z gives

$$4y^3 + 2(n - b - 2a)y^2 + 2b(b + 2a)y + b^2(n - b - 4a) = 0 \quad (6.)$$

By substituting for a, b, n, their proper values, (6) becomes

$$y^3 - 7y^2 + 76y - 192 = 0 \quad . \quad . \quad . \quad . \quad (7.)$$

Its three roots are $3, 2 + 2\sqrt{-15}, 2 - 2\sqrt{-15}$, $\therefore y = 3$, and from equations (2) and (5), $x = 1$, $z = 26$.

SECOND SOLUTION, *by Mr. P. Ketchum, Hamilton College.*

Let a be the diameter of the larger base, $b = 4 - a$ that of the less one, and $h = a - (4 - a) + 24 = 2a + 20$ the altitude of the frustum. Then since, in any conical frustum, the distance of its centre of gravity from the less base is $\frac{h}{4} \times \frac{a^2 + 2ab + 3b^2}{a^2 + ab + b^2}$, this quantity must, in the present case, be equal to $h - 17 = 2a + 3$; and substituting the values of h and b,

$$\frac{a + 10}{2} \times \frac{2a^2 - 16a + 48}{a^2 - 4a + 16} = 2a + 3,$$

$$\therefore a^3 - 7a^2 + 76a - 192 = (a - 3)(a^2 - 4a + 64) = 0;$$

hence $a = 3$, $b = 4 - a = 1$, and $h = 2a + 20 = 26$.

QUESTION V. BY ——.

Given the roots of the equation

$$x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} + \cdot \quad . \quad . \quad . \quad . \quad + u = 0;$$

to solve the two inequalities

$$x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} + \quad . \quad . \quad . \quad . \quad + u > 0.$$

$$x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} + \quad . \quad . \quad . \quad . \quad + u < 0.$$

FIRST SOLUTION, *by Dr. T. Strong.*

Inequations or inequalities become equations by supposing any of their factors to vanish or to $= 0$, hence the origin of the method of resolving a quantity into its factors, viz. put it $= 0$, then find the roots of the equation thus obtained by the ordinary rules for solving equations; this method is much used in finding the integrals of differentials where the denominator is a rational quantity supposing the differential to be under a fractional form; its denominator is put $= 0$, and then its roots are found as stated above. Let then φx denote either of the given inequalities, (which are both the same,) then put $\varphi x = 0$ and find its roots, let them be

denoted by x_1, x_2, x_3, x_4, and so on; also suppose that $x_1 > x_2 > x_3 > x_4$, &c., then the general form of either of the given inequalities is

$$\varphi x = (x - x_1).(x - x_2).(x - x_4). \text{ \&c.} \quad . \quad . \quad . \quad . \quad (1.)$$

the number of factors being equal to the number of units in the positive integer n, which is the greatest exponent of x in the inequalities, or which comes to the same thing, the number of factors is equal to the number of roots x_1, x_2, &c., of the equation $\varphi x = 0$. We shall first suppose that x_1, x_2, x_3, &c., are real quantities, but they may be positive or negative; and in the case of x negative we shall consider any numerical value as being greater than any other negative value which is numerically greater; also when φx is positive, we shall represent it by $\varphi x > 0$, and when φx is negative by $\varphi x < 0$.

It is hence evident that when x is less than each of any odd number of the roots, x_1, x_2, x_3 &c., but greater than each of the remaining roots, we shall have $\varphi x < 0$, since it contains an odd number of negative factors; but if x is less than each of an even number of the roots x_1, x_2, x_3, &c., but greater than each of the remaining roots we shall have $\varphi x > 0$, since it contains an even number of negative roots; and it is easy to see that what has been said applies where there are equal roots in $\varphi x = 0$, or when two or more of the roots x_1, x_2, &c., are equal to each other. Imaginary factors arise from $\varphi x = 0$, by supposing that it involves factors of the forms $x^2 + a^2$, $(x + c)^2 + b^2$, then putting the first of these equal to 0, we have $x^2 + a^2 = 0$, or $x = \pm a\sqrt{-1}$ $\therefore x^2 + a^2 = (x + a\sqrt{-1}).(x - a\sqrt{-1})$, and by putting $(x + c)^2 + b^2 = 0$, we have $(x + c)^2 + b^2 = (x + c + b\sqrt{-1})$ $(x + c - b\sqrt{-1})$, it is hence evident that imaginary roots will enter (1) in pairs of the above forms, so that any corresponding pair when multiplied together will give quantities of the form $x^2 + a^2$ or $(x + c)^2 + b^2$; but the signs of quantities of these forms will remain the same whether x is positive or negative, provided x be real. It is hence evident that when x is real, and all the roots x_1, x_2, x_3, &c., are imaginary, we shall always have $\varphi x > 0$, and that whether x is positive or negative; also by omitting the imaginary roots what has been said above will be applicable to the real roots, supposing (1) to have some of its roots imaginary; that is to say the demonstration above given applies when there are imaginary roots provided x is real.

Cor. 1. It is hence evident that when x is greater than each of the roots we shall have $\varphi x > 0$, also when there are an even number of real roots, or when n is an even number we shall have $\varphi x > 0$, when x is less than each of the real roots; also when n is an odd number (since the imaginary roots enter (1) in pairs) there will be an odd number of real roots, $\therefore$ when x is less than each of them, we shall have $\varphi x < 0$.

Cor. 2. Since when there are an odd number of roots greater than x we have $\varphi x < 0$, and when there are an even number of them greater than x we have $\varphi x > 0$, it is evident that an odd number of roots will lie between $\varphi x < 0$ and $\varphi x > 0$; $\therefore$ if by substituting any number for x in (1) we have $\varphi x < 0$; and if by substituting a greater number for x we have $\varphi x > 0$, we shall be certain that an odd number of roots will lie between the two numbers which were substituted for x; but if two num-

bers are substituted for x, and φx has the same sign in both of the results, then there will be either an even number of roots between the two values of x or there will be no root between those values. We hence see the origin of the usual method of obtaining an approximate value of any of the real roots of an equation.

Cor. 3. If n is an odd number, then if we take x positive we may suppose it so great that x^n shall be greater than all the other terms in φx, $\therefore$ we shall have $\varphi x > 0$, and if x is negative we may take x such that x^n shall be numerically greater than all the other terms of φx, $\therefore \varphi x < 0$, hence when n is odd, there will be an odd number of roots between x very great positive and negative. Also if the absolute term, or that which does not involve x is negative, then by supposing $x = 0$, we shall have $\varphi x < 0$, and by taking x very great, we shall have $\varphi x > 0$, $\therefore$ there will be an odd number of roots between $x = 0$ and x very great, which roots will of course be positive since x is supposed positive; but if n is an even number then as before there will be an odd number of positive roots, but by taking x negative and very great we shall have $\varphi x > 0$, and when $x = 0$, $\varphi x < 0$, $\therefore$ there will an odd number of roots lie between $x = 0$ and x very great and negative, $\therefore$ there will be an odd number of negative roots; hence when n is even and the absolute number negative there will be an odd number of positive roots, and an odd number of negative roots. Again, if n is an odd number, and the absolute term positive, then by putting $x = 0$, we have $\varphi x > 0$, and by taking x negative and very great we have $\varphi x < 0$, $\therefore$ there will in this case be an odd number of negative roots.

Cor. 4. If φx has but one change of sign, then the absolute term will have a contrary sign from the first term x^n, therefore by the last Cor. there will be an odd number of positive roots, and there will be but one such root; for supposing x to be positive and to increase by indefinitely small increments, there will be two values of x, viz. x and $x + dx$ between which φx will change its sign, that is for x we shall have $\varphi x < 0$, and when x becomes $x + dx$ we shall have $\varphi x > 0$, $\therefore$ there will be an odd number of roots between x and $x + dx$ which are all positive, $\therefore$ by supposing dx to be infinitely small relative to x, these roots will be equal, $\therefore$ if we denote the value of x (or the root,) by a, we shall have $(x - a)^m$ for a factor of φx in this case; now the factors which give imaginary roots are all positive and those which give negative roots are also positive when x is positive, $\therefore$ all the terms by which $(x - a)^m$ is multiplied in φx are positive, let $\varphi' x$ denote them, now since m is a positive integer we have $(x - a)^m = x^m - mx^{m-1}a + \dfrac{m \cdot (m-1)}{2} x^{m-2}a^2$ &c. to $m + 1$ terms, but a quantity of this kind when multiplied by $\varphi' x$ whose terms are all positive will evidently give $(x - a)^m \times \varphi' x = \varphi x$, such that there will be more than one change of sign when m is greater than 1, but φx has but one change of sign by supposition; $\therefore m = 1$; hence φx has but one positive root in the case of this Cor., for since when x becomes $x + dx$ we have $\varphi x > 0$, it is evident that x continuing to increase we shall always have $\varphi x > 0$. In conclusion we would remark, that La Grange's proof of this Cor., given at pp. 5, 6, of his Traite De La Resolution Des Equations Numeriques appears to us to be defective.

SECOND SOLUTION, *by Professor B. Peirce, Harvard University, Cambridge.*

Let the first member of the inequality be reduced to its factors, the imaginary roots being contained in pairs in the quadratic factors as follows

$$((x+a)^2+b^2)((x+a')^2+b'^2)\ldots(x-c)^m(x-c')^{m'}.$$

Now the quadratic factors may be omitted because they are necessarily positive being the sums of two squares; also the other factors of which m m' are even, because they are squares themselves; and where m m' are odd numbers of the form $2n+1$, the $2n$ may be omitted for the same reason; and the sign of the given inequality will be the same as that of

$$(x-c)(x-c')(x-c'')$$

$(x-c)$, &c. being the factors whose exponents in the given ineqality are odd numbers.

If now x is taken greater than either of the roots c, c', &c., each of the factors is positive, and consequently the inequality is so.

But if the number of roots greater than x is even, the number of negative factors is even and the inequality is positive; but if it is odd, the inequality is negative.

Consequently if x is less than either of the roots, the inequality is negative when the whole number of roots is odd, but positive when this number is even.

Corollary 1. As n exceeds the number of the roots c, c', &c., by an even number, the number of these roots is even when n is even, and odd when n is odd.

Corollary 2. The number of changes of sign in the inequality as x passes from positive to negative infinity is the same with that of the roots c, c'; &c., which are all real.

Corollary 3. If the inequality is ever negative, the given equation must have at least one real root.

Corollary 4. If the inequality is never negative, it must be an exact square unless it has imaginary roots.

Corollary 5. If the number of changes of sign is equal to n, all the roots of the equation must be real and unequal.

Corollary 6. If n is even, all the negative values of the inequality correspond to values of x contained between the extreme roots of the equation.

QUESTION VI. BY ALFRED, ATHENS, OHIO.

To find the n unknown quantities x, y, z, &c., there are given n equations, the first members of which are the sums of the squares and the products, two by two, of every $(n-1)$ of the numbers, and the second members are the known numbers, a, b, c, &c., thus:

$$y^2+z^2+w^2+\&c.\ldots\ldots+yz+yw+zw+\&c.\ldots=a,$$
$$x^2+z^2+w^2+\&c.\ldots\ldots+xz+xw+zw+\&c.\ldots=b,$$
$$x^2+y^2+w^2+\&c.\ldots\ldots+xy+xw+yw+\&c.\ldots=c,$$
$$\&c.,\qquad\qquad \&c.$$

FIRST SOLUTION, *by Mr. T. Montgomery, Institute, Flushing.*

Let $\Sigma_1 = a + b + c + \&c.,$

$$\Sigma_2 = a^2 + b^2 + c^2 + \&c.,$$
$$s_1 = x + y + z + \&c.,$$
$$s_2 = x^2 + y^2 + z^2 + \&c.,$$
$$s_3 = xy + xz + yz + \&c.,$$

then will $s_2 + 2s_3 = s_1^2$, ... (1.)
Since $y^2 + z^2 + \&c. = s_2 - x^2$, and $yz + yw + \&c., = s_3 - (s_1 - x)x = s_3 - s_1 x + x^2$, the first member of the first equation becomes $s_2 + s_3 - s_1 x$, and similarly for the other equations, hence they may be written

$$\left.\begin{array}{l} s_2 + s_3 - s_1 x = a, \\ s_2 + s_3 - s_1 y = b, \\ s_2 + s_3 - s_1 z = c, \\ \quad \&c. \end{array}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2.)$$

and the squares of these equations are

$$\left.\begin{array}{l} (s_2 + s_3)^2 - 2s_1 (s_2 + s_3)x + s_1^2 x^2 = a^2, \\ (s_2 + s_3)^2 - 2s_1 (s_2 + s_3)y + s_1^2 y^2 = b^2, \\ \quad \&c. \end{array}\right\} \quad (3.)$$

By adding equations (2) and equations (3) separately together we get

$$n(s_2 + s_3) - s_1^2 = \Sigma_1 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (4.)$$
$$n(s_2 + s_3)^2 - 2s_1^2 (s_2 + s_3) + s_1^2 s = \Sigma_2 \quad . \; . \; . \quad (5.)$$

Equations (1), (4), (5) enable us to determine s_1, s_2, s_3 and then the required quantities are had from equations (2).

Multiply equation (1) by n, and (4) by 2, and subtract,

$$ns_2 + (n-2)s_1^2, = 2\Sigma_1 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (6.);$$

Subtract the square of (4) from n times (5),

$$ns_1^2 s_2 - s_1^4 = n\Sigma_2 - \Sigma_1^2 \quad . \; . \; . \; . \; . \; . \; . \quad (7.)$$

Multiply (6) by s_1^2, and subtract (7) from it,

$$(n-1)s_1^4 - 2\Sigma_1 s_1^2 = \Sigma_1^2 - n\Sigma_2 \quad . \; . \quad (8.)$$
$$\therefore s_1^2 = \frac{\Sigma_1 \pm \sqrt{n\Sigma_1^2 - n(n-1)\Sigma_2}}{n-1} \quad . \; . \quad (9.)$$

and, from (4), $s_2 + s_3 = \dfrac{s_1^2 + \Sigma_1}{n} = \dfrac{\Sigma_1}{n-1} \pm \dfrac{\sqrt{\Sigma_1^2 - (n-1)\Sigma_2}}{(n-1)\sqrt{n}}$ (10.)

Whence x, y, z, &c., are had directly from equations (2.)
——Prof. Peirce's solution was very like this.

SECOND SOLUTION, *by Prof. Farrand N. Benedict.*

Denoting the unknown quantities by $x_1, x_2, \ldots . x_n$, and the corresponding second members by $a_1, a_2, \ldots . a_n$, we have

$$x_2^2 + x_3^2 + \&c. \ldots + x_2 x_3 + x_2 x_4 + x_3 x_4 + \&c. = a_1,$$
$$x_1^2 + x_3^2 + \&c. \ldots + x_1 x_3 + x_1 x_4 + x_3 x_4 + \&c. = a_2,$$
&c., to n equations,

put $a_1 + a_1 + \ldots a_n = d$, $a_1^2 + a_2^2 + a_n^2 = h^2$, $x_1 + x_2 + \ldots . x_n = s$, and let the sum of all the products, two by two, of x_1, x_2, &c., be denoted by p. If $p + x_1(s - x_1) + x_1^2$ be added to both members of the first equation; the first member evidently becomes $(x_1 + x_2 + \ldots x_n)^2$, and $\therefore$

$$s^2 = a_1 + p + sx_1, \text{ or } x_1 = \frac{s^2 - a_1 - p}{s},$$

for the same reason $x_2 = \dfrac{s^2 - a_2 - p}{s}$, and generally

$$x_m = \frac{s^2 - a_m - p}{s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

These n equations added together give

$$s = \frac{ns^2 - d - np}{s}, \text{ or } s^2 = \frac{np + d}{n - 1} \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

But $s^2 = x_1{}^2 + x_2{}^2 + \ldots x_n{}^2 + 2p$. Square the n equations whose general term is (1), add and reduce, and there results

$$s^2 = \frac{n(s^2 - p)^2 + h^2 - 2d\,(s^2 - p)}{s^2} + 2p \quad . \quad . \quad (3.)$$

(2) and (3) compared give

$$p = \sqrt{\frac{d^2 - (n-1)h^2}{n}}, \text{ and } s^2 = \frac{\sqrt{nd^2 - n\,(n-1)\,h^2} + d}{n - 1},$$

These values of p and s^2 substituted in (1) give

$$x_m = \frac{(n-1)\sqrt{d^2 - (n-1)\,h^2} + d\sqrt{n} - (n-1)\,a_m\sqrt{n}}{\{n\,(n-1)\sqrt{nd^2 - n\,(n-1)\,h^2} + (n-1)\,nd\}^{\frac{1}{2}}}.$$

THIRD SOLUTION, *by William Lenhart, Esq., York, Penn.*

Let $x + y + z$ &c., $= s$, $a + b + c$, &c., $= p$ and $a^2 + b^2 + c^2$, &c., $= q$. Now, if to each equation the deficient square and the deficient products two by two be added, the first members of the equations will evidently be identical, and the second members become

$$a + x^2 + xy + xz + xw, \text{ \&c.}, = a + xs \quad . \quad . \quad . \quad (1.)$$
$$b + y^2 + xy + yz + yw, \text{ \&c.}, = b + ys \quad . \quad . \quad . \quad (2.)$$
$$c + z^2 + xz + yz + zw, \text{ \&c.}, = c + zs \quad . \quad . \quad . \quad (3.)$$

which are therefore equal to each other, and consequently their sum will be equal to n times either of them: that is the sum of (1) (2) (3) &c., or which is the same thing

$$p + s^2 = n\,(a + xs) = n\,(b + ys) = n\,(c + zs) \text{ \&c.},$$

$$\text{Hence } x = \frac{s^2 + p - na}{ns} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

$$y = \frac{s^2 + p - nb}{ns} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

$$z = \frac{s^2 + p - nc}{ns} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

&c., &c.

Again $x^2 + y^2 + z^2$, &c., $+ xy + xz + yz$, &c., $= \frac{1}{2}(s^2 + x^2 + y^2 + z^2$, &c.), which being equated to (1) (2) or (3), &c., or to $\frac{p + s^2}{n}$, we shall find

$$x^2 + y^2 + z^2, \text{ \&c.}, = \frac{2p - (n-2)\,s^2}{n}.$$

But from (4) (5) (6), &c., we have

$$x^2 + y^2 + z^2 \text{ \&c.}, = \frac{s^4 - p^2 + nq}{ns^2}, \text{ consequently}$$

$$\frac{2p - (n-2)s^2}{n} = \frac{s^4 - p^2 + nq}{ns^2} \text{ from which we obtain}$$

$$s^4 - \frac{2p}{n-1}s^2 = \frac{p^2 - nq}{n-1} \text{ and thence}$$

$$s = \left(\frac{p \pm \sqrt{n(p^2 - (n-1)q)}}{n-1}\right)^{\frac{1}{2}}$$

From (4), (5), (6), &c., we get x, y, z, &c.

QUESTION VII., BY P.

To cut a given cone of revolution, by a plane passing through a given point in its surface, so that the area of the resulting elliptical section may be given or a *minimum*.

FIRST SOLUTION, *by Mr. O. Root.*

Let $2A$ = vertical angle of the cone, r = radius of the circular section through the given point, x = radius of the circular section through the other extremity of the transverse diameter of the required ellipse, and s the radius of a circular section which has the given area; then we readily see that $(r-x)^2 \cot^2 A + (r+x)^2$ = the square of the transverse diameter, and $4rx$ = the square of the conjugate diameter of the required ellipse, hence we have

$$4s^4 = rx\{(r-x)^2 \cot^2 A + (r+x)^2\}.$$

$$\text{and } x^3 - 2rx^2 \cos 2A + r^2 x - \frac{4s^4 \sin^2 A}{r} = 0 \quad . \quad . \quad (1.)$$

The roots of this equation will determine the position of the cutting plane, since one extremity of its transverse axis passes through the given point, and the other through a point at the distance x cosec. A from the vertex of the cone. For instance, if $s = r$, the three roots are

$$x = r, \text{ and } x = r\,\{\cos 2A - \tfrac{1}{2} \pm \sqrt{(\cos 2A + \tfrac{1}{2})^2 - 2}\}$$

and they are all three real when $\cos 2A > \sqrt{2} - \frac{1}{2}$, or when $A < 11^\circ\, 57'\, 10''$. The sections resulting from the two last roots when real will both be contained between the circular section and the vertex; because, since $\cos 2A < 1$, $x = r\,\{\cos 2a - \frac{1}{2} + \sqrt{(\cos 2A + \frac{1}{2})^2 - 2}\} < r\,(\frac{1}{2} + \frac{1}{2}) < r$.

If we take the differential of (1) and equate with zero, we have

$$3x^2 - 4rx \cos 2A + r^2 = 0.$$

$$\therefore\ x = \tfrac{1}{3}r\,\{\,2\cos 2A \pm \sqrt{4\cos^2 2A - 3}\} \quad . \quad . \quad . \quad . \quad (2.)$$

Now since if $u = (1)$, we have for these values of x

$$\frac{d^2 u}{dx^2} = 6x - 4r\cos 2A = \pm\, 2r\sqrt{4\cos^2 2A - 3},$$

the upper sign of (2) will give the position of the cutting plane when the ellipse is a minimum, and the lower sign when a maximum. Hence, if $4\cos^2 2A - 3 > 0$, or $2A < 30^\circ$, there will be both a maximum and minimum ellipse; when $2A = 30^\circ$ or $> 30^\circ$ there will be neither a maximum nor minimum ellipse.

——The proposer, in a similar solution, shows that when $A < 15^\circ$ there may be sections cut in three different positions, having a given area, provided the area be such that $\left(\frac{s}{r}\right)^4$ is within the limits

$$\frac{\cos 2A(9-8\cos^2 2A)-(4\cos^2 2A-3)^{\frac{3}{2}}}{54\sin^2 A} \text{ and }$$

$$\frac{\cos 2A\,(9-8\cos^2 2A)+(4\cos^2 2A-3)^{\frac{3}{2}}}{54\sin^2 A},$$

but if $\left(\frac{s}{r}\right)^4$ be less than the least or greater than thc greatest of these quantities only one such section can be cut. He also shows that although if $2A > 150°$, $4\cos^2 2A - 3 > 0$, yet the resulting values of x from (2) will necessarily be negative, and therefore maxima and minima ellipses can only exist when $A < 15°$.

SECOND SOLUTION, *by Mr. Geo. R. Perkins, Clinton Liberal Institute.*

Let the transverse diameter of the elliptic section pass through the given point, at the distance a from the vertex, making the angle φ with a circular section through that point; let ω be the angle which the side of the cone makes with the circular section, and c the distance from the vertex to where the axis of the cone pierces the section. Then will the equation of this section be (Davies' Anal. Geom. p. 314.)

$$y^2\tan^2\omega + x^2\cos^2\varphi(\tan^2\omega - \tan^2\varphi) + 2cx\sin\varphi = c^2 \quad . \quad . \quad (1.)$$

or referring it to its centre and axes,

$$y^2\tan^2\omega + x^2\cos^2\varphi\,(\tan^2\omega - \tan^2\varphi) = \frac{c^2\tan^2\omega}{\tan^2\omega - \tan^2\varphi} \quad (2).$$

Hence the semi-axes of the section are

$$\frac{c\tan\omega}{\cos\varphi\,(\tan^2\omega - \tan^2\varphi)} \text{ and } \frac{c}{(\tan^2\omega - \tan^2\varphi)^{\frac{1}{2}}},$$

$$\text{and its area} = \frac{\pi c^2\tan\omega}{\cos\varphi\tan^2\omega-\tan^2\varphi^{\frac{3}{2}}} = \frac{\pi c^2\sin\omega\cos^2\omega\cos^2\varphi}{\sin^{\frac{3}{2}}(\omega+\varphi)\sin^{\frac{3}{2}}(\omega-\varphi)} \quad (3.)$$

But since $c\cos\varphi = a\sin(\omega+\varphi)$, if A = given area, (3) becomes

$$\frac{\pi a^2\sin\omega\cos^2\omega\sin^{\frac{1}{2}}(\omega+\varphi)}{\sin^{\frac{3}{2}}(\omega-\varphi)} = A \quad . \quad . \quad (4.)$$

$$\therefore \frac{\sin(\omega+\varphi)}{\sin^3(\bar{\omega}-\varphi)} = \frac{A^2}{\pi^2 a^4\sin^2\omega\cos^4\omega} \quad . \quad . \quad . \quad (5).$$

This expression will be in a more convenient form for finding φ, if we take the logarithms of both members, then

$$\log.\sin(\omega+\varphi) - 3\log.\sin(\omega-\varphi) = 2\log.A - 2\log.(\pi a^2\sin\omega\cos^2\omega)\ldots(6).$$

from which φ may be found by a few trials.

When the area is a minimum, $dA = 0$, and by differentiating (6.)

$$\frac{\cos(\omega+\varphi)}{\sin(\omega+\varphi)} + \frac{3\cos(\omega-\varphi)}{\sin(\omega-\varphi)} = 0,$$

$$\therefore \cos(\omega+\varphi)\sin(\omega-\varphi) + 3\sin(\omega+\varphi)\cos(\omega-\varphi) = 0, \quad (7.)$$

But $\cos(\omega+\varphi)\sin(\omega-\varphi) = \frac{1}{2}\sin 2\omega - \frac{1}{2}\sin 2\varphi$,

and $3\sin(\omega+\varphi)\cos(\omega-\varphi) = \frac{3}{2}\sin 2\omega + \frac{3}{2}\sin 2\varphi$;

$$\therefore 2\sin 2\omega + \sin 2\varphi = 0,$$

$$\text{and } \sin 2\varphi = -2\sin 2\omega \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

Since $\omega < 90°$, $2\omega < 180°$; therefore $2\varphi > 180°$ and $\varphi > 90$, or the

minimum section will lie wholly above the circular section through the given point.

——It will be seen that Mr. Perkins supposes the angles φ and ω to be counted in contrary directions from the circular section. Professor Peirce, in a solution on like principles, gives the following method of solving equation (6) which may be useful in similar cases;—Put $\varphi=\omega-\varphi'$, $P=\frac{A}{\pi a^2 \sin\omega\cos^2\omega}$ then (6) becomes

$$\log.\sin(2\omega-\varphi')-3\log.\sin\varphi'=2\log.P.$$

Let an approximate value of φ', which can easily be obtained by inspection from the logarithmic tables, be represented by φ'', and let P' be such that

$$\log.\sin(2\omega-\varphi'')-3\log.\sin\varphi''=2\log.P',$$

and we shall have for a second approximation to φ'

$$\varphi'=\varphi''+\frac{2(P'-P)}{P\{\cot(2\omega-\varphi'')+3\cot\varphi''\}}.$$

Dr. Strong, by adapting his solution to Question XVIII. to this case, finds the area of *any* elliptic section passing through the given point to be $\frac{a(1+t\cos\omega\tan\varphi)^2}{\cos\varphi\,(1-t^2\tan^2\varphi)^{\frac{3}{2}}}$, where a is the area of a circular section through the given point, φ is the angle the plane of the ellipse makes with the circular section, and t and ω being as in the solution referred to, ω indicating the longitude of the perihelion of the ellipse counted from a plane through the axis of the cone and the given point. Hence there requires something more than the area to determine the position of the ellipse. If $\omega=0$, the case will become the one considered in the preceding solutions.

(28.) QUESTION VIII. *by* Δ.

If r_1, r_2, be two radius vectors of a parabola, and α the angle included between them, show that the distance from the focus to the vertex of the parabola is

$$=\frac{r_1 r\ \ \sin^2\frac{1}{2}\alpha}{r_1+r_2\pm2\sqrt{r_1 r_2}.\cos\frac{1}{2}\alpha},$$

and tell the meaning of the ambiguous sign.

FIRST SOLUTION, *by Professor M. Catlin, Hamilton College, Clinton, N. Y.*

Let z = the angle which r makes with the axis of the parabola and $\alpha-z$ = the angle which r' makes with it. Then by a well known property of the parabola we shall have

$$\frac{2c}{1+\cos z}=r\ldots.(1);\ \text{and}\ \frac{2c}{1+\cos(\alpha-z)}=r'\ldots.(2).$$

$$\therefore\ c=r\cos^2\tfrac{1}{2}z\ldots.(3);\ \text{and}\ c=r'\cos^2\tfrac{1}{2}(\alpha-z)\quad.\quad(4.)$$

Equation (3) is easily reduced to $c=\frac{rr'\sin^2\frac{1}{2}\alpha}{r'\sin^2\frac{1}{2}\alpha\sec^2\frac{1}{2}z}$. (5.)

Dividing (4) by (3), $\pm\sqrt{\frac{r}{r'}}=\frac{\cos\frac{1}{2}(\alpha-z)}{\cos\frac{1}{2}z}=\cos\frac{1}{2}\alpha+\sin\frac{1}{2}\alpha\tan\frac{1}{2}z$ (6.)

$$\therefore\ \sqrt{r'}.\sin\tfrac{1}{2}\alpha\tan\tfrac{1}{2}z=\pm\sqrt{r}-\sqrt{r'}.\cos\tfrac{1}{2}\alpha$$

$$r \sin^2 \tfrac{1}{2}\alpha \tan^2 \tfrac{1}{2}z = r \pm 2\sqrt{rr'}.\cos \tfrac{1}{2}\alpha + r' \cos^2 \tfrac{1}{2}\alpha,$$

and adding $r' \sin^2 \frac{1}{2}\alpha$ to each member,

$$r' \sin^2 \tfrac{1}{2}\alpha \sec^2 \tfrac{1}{2}z = r + r' \pm 2\sqrt{rr'}.\cos \tfrac{1}{2}\alpha \quad . \quad . \quad (7.)$$

$$\therefore (5) \text{ becomes } c = \frac{rr' \sin^2 \frac{1}{2}\alpha}{r + r' \pm 2\sqrt{rr'}.\cos \frac{1}{2}\alpha} \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

From (3) and (4) we get $\pm\sqrt{rr'} = \dfrac{c}{\cos \frac{1}{2}z \cos \frac{1}{2}(\alpha - z}$. Hence, since c is always positive, $\sqrt{rr'}$ will be positive, or negative according as $\cos \frac{1}{2}z$ and $\cos \frac{1}{2}(\alpha-z)$, have the same or different signs; or when z and $\alpha - z$ are both greater or both less than 180°, $\sqrt{rr'}$ is positive; but when one is greater and the other less than 180, $\sqrt{rr'}$ is negative. That is $\sqrt{rr'}$ is positive when r and r' are on opposite sides of the axis, and negative when they are on the same side. Hence, in the former case, the lower sign in the given equation is to be taken and the upper in the latter case.

Cor. When $\alpha = 180°$, $c = \dfrac{rr'}{r + r'}$. If $r = r'$, then $c = \frac{1}{2}r\,(1 \mp \cos \frac{1}{2}\alpha.)$

SECOND SOLUTION, *by* Δ.

Let v be the distance from the focus to the vertex, and φ the angle which that line makes with r'_1; the angles φ and α are both counted from the fixed line r_1 and in the same direction.

$$\text{Then } \cos^2 \tfrac{1}{2}\varphi = \frac{v}{r_1} \text{ and } \cos^2 \tfrac{1}{2}(\alpha - \varphi) = \frac{v}{r_2};$$

$$\therefore \frac{\cos^2 \frac{1}{2}(\alpha - \varphi)}{\cos^2 \frac{1}{2}\varphi} = \frac{r_1}{r_2},$$

$$\text{and } \frac{\cos \frac{1}{2}(\alpha-\varphi)}{\cos \frac{1}{2}\varphi} = \cos \tfrac{1}{2}\alpha + \sin \tfrac{1}{2}\alpha \tan \tfrac{1}{2}\varphi = \pm \sqrt{\frac{r_1}{r_2}},$$

$$\therefore \tan \tfrac{1}{2}\varphi = \left(\pm \sqrt{\frac{r_1}{r_2}} - \cos \tfrac{1}{2}\alpha\right) \operatorname{cosec} \tfrac{1}{2}\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

$$\text{Hence } v = r_1 \cos^2 \tfrac{1}{2}\varphi = \frac{r_1}{1 + \tan^2 \frac{1}{2}\varphi} = \frac{r_1\, r_2 \sin^2 \frac{1}{2}\alpha}{r_1 + r_2 \mp 2\sqrt{r_1\, r_2}.\cos \frac{1}{2}\alpha} \quad (2.)$$

These equations show that two parabolas may be described through the extremities of r_1 and r_2 having their common focus at the intersection of these lines, the upper signs belonging to one, and the lower to the other. When $\cos \frac{1}{2}\alpha = \sqrt{\dfrac{r_1}{r_2}}$, the vertex of one of the parabolas is on the extremity of r_1; in all other cases the axis of one of the two parabolas passes between the two distances, its vertex being in the angle α, when $\cos \frac{1}{2}\alpha$ is between the magnitudes of $\sqrt{\dfrac{r_1}{r_2}}$ and $\sqrt{\dfrac{r_2}{r_1}}$, and in the angle opposite to α when $\cos \frac{1}{2}\alpha$ is less than either of them.

Hence in determining the form of a comet's orbit, by two distances from the sun and the angle included between them, it is necessary to know in what parts of the orbit the distances have been taken.

(27.) QUESTION IX. (COMMUNICATED BY J. F. MACULLY.)

Find four affirmative numbers, such that the sum of the first and second, the sum of the second and third, the difference of the squares of the second and third, and their difference, may be four square numbers in continued proportion; the sum of the rectangles of every two of the last three together with the square of the first, a square; and the sum of the first, third, fourth, and twice the second a square.

⁂ This was published in the Belfast Almanac, but an erroneous solution was given to it.

FIRST SOLUTION, *by Mr. Geo. R. Perkins.*

Let a, b, c, d, be the numbers, then
$a + b = \square$ (1), $b + c = \square$ (2), $b^2 - c^2 = \square$ (3), $b - c = \square$ (4.),
$bc + bd + cd + a^2 = \square$ (5), $a + c + d + 2b = \square$ . . . (6.)
And since the expressions (1), (2), (3), (4), are in continued proportion we must also have
$(a + b)(b^2 - c^2) = (b + c)^2$, or $(a + b)(b - c) = b + c$ (7.)
$(b + c)(b - c) = (b^2 - c^2)^2$, or $1 = b^2 - c^2$ . . . . (8.)

Make $b + c = m^2$, and $b - c = \dfrac{1}{m^2}$, and these with (7) give

$$b = \frac{m^4 + 1}{2m^2},\ c = \frac{m^4 - 1}{2m^2},\ a = \frac{2m^6 - m^4 - 1}{2m^2}; \quad . \ . \ . \quad (9.)$$

and since $a + b = m^4$, all the conditions except (5) and (6) are satisfied, and these by substituting (9), become

$$d + 2Q + m^4 + m^2 + 1 = \square\ (10),\ d + m^4 + m^2 = \square\ (11.);$$

where Q is put for $\frac{1}{2}m^6 - m^4 - \frac{1}{4}m^2 - 1 + \dfrac{1}{4m^2}$ for the sake of brevity, make $d + m^4 + m^2 = P^2$, and then $d = P^2 - m^4 - m^2$, and (10) becomes $P^2 + 2Q + 1 = \square = (P + 1)^2 = P^2 + 2P + 1$; hence $P = Q$, and

$$d = Q^2 - m^4 - m^2 \quad . \ . \ . \ . \quad (12.)$$

If, for example, we take $m = 2$, then $Q = 14\frac{1}{16}$, and we find $a = 13\frac{7}{8}$, $b = 2\frac{1}{8}$, $c = 1\frac{7}{8}$, $d = 177\frac{1}{2}\frac{3}{5}\frac{3}{6}$.

SECOND SOLUTION, *by Mr. N. Vernon.*

Let $w + x = P^4$, $x + y = P^2$, $x^2 - y^2 = 1$, $x - y = \dfrac{1}{P^2}$.

These four equations evidently satisfy the first four conditions of the question; and we get $w = P^4 - \frac{1}{2}P^2 - \dfrac{1}{2P^2}$, $x = \frac{1}{2}P^2 + \dfrac{1}{2P^2}$, $y = \frac{1}{2}P^2 - \dfrac{1}{2P^2}$. Again, let

$$xy + xz + yz + w^2 = r^2,\ w + y + z + 2x = s^2;$$

$$\text{then } z = \frac{r^2 - xy - w^2}{x + y} = s^2 - w - y - 2x,$$

$$\text{and } s^2 = \frac{r^2 + 2x^2 - w^2 + y^2 + wx + wy + 2xy}{x + y}.$$

Let $m = 2x^2 - w^2 + y^2 + wx + wy + 2xy$, $n = w - y - 2x$; then

$$s^2 = \frac{r^2 + m}{P^4} = \left(\frac{r - v}{P}\right)^2 = \frac{r^2 - 2rv + v^2}{P^2},$$

and we get $r = \dfrac{v^2 - m}{2v}$, $s = \dfrac{v^2 + m}{2vP}$, and $z = \dfrac{(v^2 + m)^2}{4v^2P^2} - n$.

By taking $P = 2$ and $v = 1$ we get $w = \frac{111}{8}$, $x = \frac{17}{8}$, $y = \frac{15}{8}$, $r = \frac{235}{4}$, $s = \frac{231}{8}$, and $z = \frac{52081}{64}$.

(30.) QUESTION X., by Wm. Lenhart, Esq. York, Pa.

It is required to find four integers such that the sum of every two of them may be a cube.

SOLUTION, *by the Proposer.*

Three of the conditions will evidently be answered by assuming for the numbers required $m^3 - x$, $n^3 - x$, $r^3 - x$ and x; and the three remaining ones will be expressed by the formulas $m^3 + n^3 - 2x = \text{cube} \ldots (1.)$ $m^3 + r^3 - 2x = \text{cube} \ldots (2.)$ and $n^3 + r^3 - 2x = \text{cube} \ldots (3.)$ Equate (1.) to s^3, then $x = \dfrac{m^3 + n^3 - s^3}{2}$, and by substitution, (2) and (3) become $r^3 + s^3 - n^3 = \text{cube} = a^3$, $r^3 + s^3 - m^3 = \text{cube} = b^3$, and thence $r^3 + s^3 = a^3 + n^3 = b^3 + m^3$. Now this condition, were it not that the three lesser cubes, in order to obtain positive integers to answer, must be such that the sum of every two shall be greater than the third, could easily be fulfilled by our Table of numbers composed of two cubes, because there are many numbers in it composed of three and more pairs of cubes, but with the above restriction there are but few to be found to answer. We have however, several pairs of cubes to suit, one set of which we shall here note down, viz., $46969 = \left(\frac{95}{7}\right)^3 + \left(\frac{248}{7}\right)^3 = \left(\frac{149}{12}\right)^3 + \left(\frac{427}{12}\right)^{3*} = \left(\frac{341899}{30291}\right)^3 + \left(\frac{1081640}{30291}\right)^3$. The three lesser cubes, it will be seen, approach to an equality, and being greater in proportion than $(6)^3$, $(7)^3$, and $(8)^3$, necessarily possess the properties to render the numbers positive; we may therefore assume $n = \frac{95}{7}$, $m = \frac{149}{12}$, $s = \frac{341899}{30291}$, $a = \frac{248}{7}$, $b = \frac{427}{12}$, and $r = \frac{1081640}{30291}$; or reducing to the same denominator and rejecting it,

$$n = 11510580,\ m = 10531171,\ s = 9573172,$$
$$a = 30048672,\ b = 30179933,\ r = 30285920.$$

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* These roots were obtained by substituting the roots (41) and (28) that are in the equation $46969 = (41)^3 - (28)^3$ in the formulas $\dfrac{a(a^3 - 2b^3)}{a^3 + b^3}$, $\dfrac{b(2a^3 - b^3)}{a^3 + b^3}$, which are well known to be the roots of two cubes whose sum is equal to $a^3 - b^3$. We mention this and insert the formulas here, so that the contributors to the Miscellany may have them to refer to on any future occasion.

by means of which we obtain the following integers to answer, viz.:

I. 2080913082956455142636. II. 4937801347510680732948.

III. 7262810476410016163052. IV. 214972108693241589340948.

Proof.

$$\text{I.} + \text{II.} = (19146344)^2, \quad \text{I.} + \text{III.} = (21062342)^3,$$
$$\text{I.} + \text{IV.} = (60097344)^3, \quad \text{II.} + \text{III.} = (23021160)^3,$$
$$\text{II.} + \text{IV.} = (60359866)^3, \quad \text{III.} + \text{IV.} = (60571840)^2,$$

We have in our Table four pairs of cubes that are equal to each other, and the lesser cubes in any three pairs of the four are such that the sum of every two of them is greater than the third, which are the proper requisites; consequently, as four things can be combined four different ways three at a time, we shall be able from these pairs of cubes to find four different sets of integers to answer the question; but neither set will be of a denomination as low as the set we have given above.

(31.) QUESTION XI., BY RICHARD TINTO, ESQ., GREENVILLE, OHIO

Find the locus of the centre of a given sphere, so that its shadow on a given plane, made by a light fixed in a given position, may have a given magnitude.

FIRST SOLUTION, *by Dr. Strong.*

We shall here (for brevity) refer to the second solution of Question XVIII. in the last Miscellany; then by (8), we have the given area $s = \dfrac{pc^2t^2(1 + A^2 + B^2)^{\frac{1}{2}}}{\{1-(A^2 + B^2)t^2\}^{\frac{3}{2}}}$. Imagine the plane on which the shadow falls to be horizontal, $h =$ the perpendicular upon it from the light, R the radius of the sphere, x, y, z, the rectangular co-ordinates of its centre, their origin being at the light, and h the axis of z. Let φ denote the angle made by a circular section of the shadow with the horizontal plane, and θ half the vertical angle of the visual cone. Then we shall have $\sec^2\varphi = 1 + A^2 + B^2 = \dfrac{x^2 + y^2 + z^2}{z^2}$, $A^2 + B^2 = \dfrac{x^2 + y^2}{z^2}$,

$$t^2 = \tan^2\theta = \frac{R^2}{x^2 + y^2 + z^2 - R^2},$$

$$\therefore 1 - (A^2 + B^2)t^2 = \frac{(z^2 - R^2)(x^2 + y^2 + z^2)}{z^2(x^2 + y^2 + z^2 - R^2)};$$

now, by (2) of the solution referred to, $c =$ the axis of the cone produced from the light to intersect the horizontal plane $= h\sec\varphi$, and

$$c^2\,(1 + A^2 + B^2)^{\frac{1}{2}} = h^2\sec^3\varphi = h^2\left(\frac{x^2 + y^2 + z^2}{z^2}\right)^{\frac{3}{2}}.$$

Hence we easily find $s = ph^2R^2.\sqrt{\dfrac{x^2 + y^2 + z^2 - R^2}{(z^2 - R^2)^3}}$ or if $\dfrac{s}{pR^4} = m$, we have

$$m^2(z^2 - R^2)^3 - h^4(z^2 - R^2) = h^4(x^2 + y^2) \quad . \quad . \quad . \quad . \quad (a),$$

which is the equation of the sought surface, and it is evidently formed by the revolution of a curve of the sixth order around the axis of z; the

sections of the surface by planes parallel to the horizon being circles, and those by planes perpendicular to the horizon being lines of the sixth order.

SECOND SOLUTION, *by Professor C. Avery, Hamilton College, Clinton, N. Y.*

Let a be the perpendicular distance of the light from the shadow plane, r the distance of the centre of the sphere from the light, taken as the origin, r' the radius of the sphere, θ the angle which r makes with a, and a', b' the semi-axes of the shadow. Then $a' = \dfrac{a \tan e \sec^2\theta}{1-\tan^2 e \tan^2\theta}$, and $b'^2 = \dfrac{a^2 \tan^2 e \sec^2\theta}{1-\tan^2 e \tan^2\theta}$, where $\tan e = \dfrac{r'}{\sqrt{r^2 - r'^2}}$; therefore

$$\text{area} = c = \pi a'b' = \frac{\pi a^2 \tan^2 e \sec^3\theta}{(1-\tan^2 e \tan^2\theta)^{\frac{3}{2}}} \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

If we square (1), reduce, and put $c = \pi a^2 r'^2 d$, we get

$$r^2 - r'^2 = d^2\,(r^2 \cos^2\theta - r'^2)^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

By substituting $r \cos\theta = x$, and $r^2 = x^2 + y^2 + z^2$ it becomes

$$x^2 + y^2 + z^2 - r'^2 = d^2(x^2 - r'^2)^3 \quad . \quad . \quad . \quad . \quad (3.)$$

which is the rectangular equation of the surface, the plane of $y\,z$ passing through the light, parallel to the given plane. If $z = 0$, we have

$$y^2 = d^2(x^2 - r'^2)^3 - (x^2 - r'^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

which is the equation of the generating curve. To determine the limits of the curve in the direction of the abscissa, let $y = 0$, then $x = \pm r'$, $x = \pm\sqrt{r'^2 - \frac{1}{d}}$ $x = \pm\sqrt{r'^2 + \frac{1}{d}}$; the latter are the only values that apply to the question, because when $x = \pm r'$ the shadow $= \infty$, and when $x = \pm\sqrt{r'^2 - \frac{1}{d}}$ the light is within the sphere and no shadow exists; therefore the curve crosses x at a distance from the origin $= \pm\sqrt{r'^2 + \frac{1}{d}}$, and never approaches nearer to the light. The double sign shows that there is another similar curve on the opposite side of the light, and the double sign of (4) when solved for y shows that the axis of x is an axis of the curve. By differentiating (4) we have

$$\frac{dy}{dx} = \frac{x}{y}\,\{3d^2\,(x^2 - r'^2)^2 - 1\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

When $y = 0$, $\dfrac{dy}{dx} = \infty$, or the curve cuts the axis of x perpendicularly, and when $\dfrac{dy}{dx} = 0$, $x^2 - r'^2 = \dfrac{1}{d\sqrt{3}}$, and $y^2 = \dfrac{-2}{3d\sqrt{3}}$, which is imaginary, and therefore y is unlimited, in the direction of both x and y, x being $= \infty$, when $y = \infty$; hence the curve, after passing the plane of projection at a point $y^2 = d^2\,(a^2 - r'^2)^3 - a^2 + r'^2$, passes on to infinity. Since

$$y^3.\frac{d^2y}{dx^2} = \{3d^2(x^2-r'^2)^2 + 12d^2(x^2-r'^2)x^2 - 1\}y^2 - x^2\{3d^2(x^2-r'^2)^2 - 1\}^2,$$

if there be a point of contrary flexure in the curve, it will be when the second member of this equation $= 0$.

THIRD SOLUTION, *by Prof. F. N. Benedict, University of Vermont.*

Let P (fig 2.) be the position of the light, C the centre of the given sphere, CM its radius, PFD that axial section of the cone formed by rays from P tangent to the sphere which is perpendicular to the plane of the shadow BRD. Draw PA perpendicular to ABD the common section of the triangular and shadow planes. Let V be the intersection of the diameters BD, IH of the shadow and circular section IRH, and VR their common section. Draw BK parallel to FD, and put PB $= e$, PD $= f$, BD $= a$, BK $= i$, BV $= x'$, VR $= y'$. Comparing the similar triangles DBK and DVH, PBK and PFD, FDB and IVB, we have VH $= \frac{i}{a}(a - x')$, FD $= \frac{if}{e}$, IV $= \frac{ifx'}{a}$, and consequently, since VR is evidently perpendicular to IH and BD, VI.VH $=$ VR^2, or $\frac{i^2 f(ax' - x'^2)}{ea^2} = y'^2$ .... (1.) This is the equation of the elliptic shadow, and therefore when $x' = \frac{1}{2}a$, $2y' = b = i\sqrt{\frac{f}{e}}$, its conjugate, and its area $= A^2 = \pi ai\sqrt{\frac{f}{e}}$ .... (2.), where $\pi = 0{,}7854$. Draw CM perpendicular to PF, PQ to BK, and CL to PA, and put CP $= z$, CM $= r$, AP $=$ R, $<$ APC $= \psi$, $<$MPC $= <$ CPK $= \delta$. AP being radius, we have AD $= \frac{R^2(\tan\psi + \tan\delta)}{R^2 - \tan\psi\tan\delta}$, and AB $= \frac{R^2(\tan\psi - \tan\delta)}{R^2 + \tan\psi\tan\delta}$,

$$\therefore \frac{R^2(\tan\psi + \tan\delta)}{R^2 - \tan\psi\tan\delta} - \frac{R^2(\tan\psi - \tan\delta)}{R^2 + \tan\psi\tan\delta} = \frac{2R^2\sec^2\psi\tan\delta}{R^4 - \tan^2\psi\tan^2\delta} = BD = a.$$

Substituting the above values of AD and AB in the equations PD^2 $=$ AP^2 $+$ AD^2, and PB^2 $=$ AP^2 $+$ AB^2, there results after obvious reductions PD $= f = \frac{R\sec\psi\sec\delta}{R^2 - \tan\psi\tan\delta}$ and PB $= e = \frac{R\sec\psi\sec\delta}{R^2 + \tan\psi\tan\delta}$; also from the similar triangles PCM, PQB, we have BK $=$ 2BQ $= i = \frac{2r.\text{PB}}{z} = \frac{2rR\sec\psi\sec\delta}{z(R^2 + \tan\psi\tan\delta)}$. These values of a, i, f, e, being substituted in (2) we shall have after squaring both members and reducing $A^4 = \frac{16\pi^2 r^2 R^6\sec^6\psi\tan^2\delta\sec^2\delta}{z^2(R^4 - \tan^2\psi\tan^2\delta)^3}$. Eliminate $\tan^2\delta$ and $\sec^2\delta$ by the equations R$z = \sec\delta\sqrt{z^2 - r^2}$, R$r = \tan\delta\sqrt{z^2 - r^2}$, derived from the triangle PCM, and put $\frac{16\pi^2 r^4 R^4}{A^4} = c^4$, then

$$\{R^2(z^2 - r^2) - r^2\tan^2\psi\}^3 = c^4(z^2 - r^2)\sec^6\psi \quad . \quad . \quad (3).$$

Which is the polar equation of the locus of the centre of the sphere in any plane passing through the luminous point perpendicular to the plane of the shadow. The centre of the sphere therefore will be confined in space to the surface of revolution of (3). A more simple and convenient expression is derived from the transformation of (3) to the rectangular co-ordinates LP, LC, by eliminating $z\sec\psi$, $\tan\psi$ by means of the equations $z^2 = x^2 + y^2$, $x\sec\psi = R\sqrt{x^2 + y^2}$, and $x\tan\psi = Ry$ resulting from the triangle PCL. This furnishes, after obvious reductions,

$$(x^2-r^2)^3 = c^4\,(x^2+y^2-r^2), \text{ or } y^2 = \frac{(x^2-r^2)}{c^4} - x^2 + r^2 \quad (4.)$$

If a determination of the locus of the sphere's centre on any given plane be required, it remains to investigate the sections of any plane with the surface of revolution of (4). To accomplish this we will resolve the general problem, to determine the sections of a plane given in position with any surface of revolution, the equation of whose generating curve is represented by $y^2 = \text{F}.x$. Let MI (fig. 3.) be the section of the plane MIV with the surface of revolution of the curve A$\text{M}m$, A′P the axis and A′ the origin of x. Imagine A$\text{M}m$, $m\text{I}g$ to be two sections of the surface by planes of which the first contains the axis of x and the second perpendicular to it, the first being likewise perpendicular to the plane MIV; and let TV, mp, VI be the common sections respectively of the planes A$\text{M}m$ and MVI, A$\text{M}m$ and pmI, MVI and pmI. Put $\angle$MTA $=\omega$, A′P $= g$, MP $= g'$, MV $=$ x, VI $=$ y. The triangles MNV, TPM, TpV give MN $=$ Pp $=$ x $\cos\omega$, PT $= g' \cot\omega$, pT $=$ x $\cos\omega + g'\cot\omega$, V$p$ $= \tan\omega\,(\text{x}\cos\omega + g'\cot\omega)$, A′$p$ $= x = g + \text{x}\cos\omega$, and therefore $\text{F}.x = pm^2 = \text{F}\,(g + \text{x}\cos\omega)$. Substitute these values of VI, pm, pV in the equation $\text{VI}^2 = mp^2 - \text{V}p^2$ derived from the circular section mpI and we have

$$\text{y}^2 = \text{F}\,(g+\text{x}\cos\omega) - \tan^2\omega\,(\text{x}\cos\omega + g'\cot\omega)^2 \quad . \quad (5.)$$

which is the section of a plane with any surface of revolution.

To apply this to the sections of the surface of revolution of (4); we have $\text{F}.x = \dfrac{(x^2-r^2)^3}{c^4} - x^2 + r^2$, and consequently $\text{F}\,(g+\text{x}\cos\omega) = \dfrac{\{(g+\text{x}\cos\omega)^2 - r^2\}^3}{c^4} - (g+\text{x}\cos\omega)^2 + r^2$. Substitute this in (5) and

$$\text{y}^2 = \frac{\{(g+\text{x}\cos\omega)^2-r^2\}^3}{c^4} - (g+\text{x}\cos\omega)^2 + r^2 - (g'+\text{x}\sin\omega)^2 \quad (6.)$$

If the plane in which the centre of the sphere moves is perpendicular to the plane of the shadow, then $\omega = 0$ and

$$\text{y}^2 = \frac{\{g+\text{x})^2 - r^2\}^3}{c^4} - (g+\text{x})^2 + r^2 - g'^2 \quad . \; . \; . \; . \quad (7.)$$

which becomes (4) when the plane passes through the luminous point. If the plane is parallel to the plane of the shadow, $\omega = 90°$

$$\therefore \text{y}^2 = \frac{(g^2-r^2)^3}{c^4} - g^2 + r^2 - (g'+\text{x})^2,$$

The equation of a circle. If the plane passes through the luminous point, then A′P = TP, or $g' = g\tan\omega$, and (6) becomes

$$\text{y}^2 = \frac{\{(g+\text{x}\cos\omega)^2 - r^2\}^3}{c^4} - (g\sec\omega + \text{x})^2 + r^2 \quad . \quad (8.)$$

(32.) QUESTION XII., BY P.

The surface of a polyedron is composed of a triangular, b quadrangular, c pentagonal, &c., planes; to find the number of diagonals that can be drawn in the polyedron.

FIRST SOLUTION, *by Professor Peirce.*

Let s = the number of solid angles,
H = the number of faces,
A = the number of edges,
B = the number of diagonals which can be drawn on the different faces,
N = the number of diagonals sought.

Also let Σ denote the sum of all expressions of a similar kind,
a_n denote the number of faces of n sides;

We have the number of lines which can be drawn by joining each vertex with every other $= \frac{1}{2}\, s(s-1)$; whence

$$N = \tfrac{1}{2}\, s\,(s-1) - (A+B),$$
$$B = \Sigma.\, \tfrac{1}{2}\, n\,(n-3)a_n$$
$$A = \Sigma.\, \tfrac{1}{2} n a_n,$$
$$H = \Sigma.\, a_n,$$

But, from Legendre's Geometry, $s + H = A + 2$, or

$$s = A - H + 2 = \Sigma.\, \tfrac{1}{2}(n-2)\, a_n + 2,$$
$$N = \tfrac{1}{2}\{\Sigma.\tfrac{1}{2}(n-2)\, a_n\}^2 + \tfrac{3}{2}\, \Sigma.\, \tfrac{1}{2}(n-2)a_n + 1 - \Sigma.\tfrac{1}{4}n(n-2)a_n$$
$$= \tfrac{1}{8}\{\Sigma.(n-2)a_n\}^2 + \tfrac{1}{4}\Sigma.(3-2n)\,(n-2)a_n + 1.$$

EXAMPLE 1. If all the faces have the same number of sides, $H = a_n$,
and $N = \frac{1}{8}(n-2)\, H[(n-2)H - 4n + 6] + 1.$

Hence, when the faces are triangles, $N = \frac{1}{8}H(H-6) + 1$;
when they are quadrilateral, $N = \frac{1}{2}H(H-5) + 1$;
when they are pentagonal, $N = \frac{3}{8}H(3H - 14) + 1.$

EXAMPLE 2. In the case of a prism or the frustum of a pyramid.

$$N = \tfrac{1}{8}\{2(a_4 - 2) + 2a_4\}^2 + \tfrac{1}{4}\{2(3 - 2a_4)\,(a_4 - 2) - 10a_4\} + 1$$
$$= 2(a_4 - 1)^2 - \tfrac{1}{2}(2a_4{}^2 - 2a_4 + 6) + 1$$
$$= a_4\,(a_4 - 3).$$
= twice the number on either base.

SECOND SOLUTION, *by Professor Catlin.*

Let s = the number of solid angles; E = the number of edges; F = the number of faces, and N = the number of diagonals required.

Then $2E = 3a + 4b + 5c + 6d + \&c.$ . . . . (1.)
and $s = E - F + 2$ . . . . (2)

(See *Livre* VII., *Prop.* 25, Leg. Geom., 2nd Ed. Paris.) The number of diagonals equals the number of combinations, taken two and two, of the s solid angles, minus E, and the number of diagonals in all the plane faces. The number of diagonals in the quadrilaterals $= 2b$; in the pentagons $= 5c$; in the hexagons $= 9d$, &c.

$\therefore N = \frac{1}{2}\, s\,(s-1) - \frac{1}{2}(3a+4b+5c+\&c.) - (2b+5c+9d+\&c.)$
$= \frac{1}{2}\, s\,(s-1) - \frac{1}{2}(3a+8b+15c+24d+\&c.)$ . . . (3.)

Equations (2) and (3) completely determine the problem.

(33.) QUESTION XIII. (FROM THE LADIES' DAIRY FOR 1836.)

At two given points within a spherical shell (incapable of reflection) are placed two given unequal lights. It is required to assign the points in the interior surface which are respectively most and least enlightened, and the locus of the points where the light is of any specified intensity.

Let the radius of the shell $= 1$,

a and a' = the distances of the lights from the centre,

m and m' = the quantity of light given by each light at the unit of distance,

φ and φ' = the angles made by a and a' with the radius drawn to the illuminated point;

and we shall have the distance from the light m to the illuminated point $= (1 - 2a\cos\varphi + a^2)^{\frac{1}{2}}$, the quantity of light at this distance $= \frac{m}{1-2a\cos\varphi+a^2}$, and this is to be multiplied by the cosine of the angle made by the radius drawn from the illuminated point with the line drawn from the illuminated point to the light m; and its cosine $= \frac{1 - a\cos\varphi}{(1-2a\cos\varphi+a^2)^{\frac{1}{2}}}$; therefore the quantity of light received from m

$$= m\,(1 - a\cos\varphi)(1-2a\cos\varphi + a^2)^{-\frac{3}{2}},$$

and that from $m' = m'(1 - a'\cos\varphi')(1-2a'\cos\varphi' + a'^2)^{\frac{3}{2}}$

Calling the sum of these lights A, we have

$$\text{A} = \frac{m(1-a\cos\varphi)}{(1-2a\cos\varphi+a^2)^{\frac{3}{2}}} + \frac{m'1-{}^2a'\cos\varphi')}{(1-a'\cos\varphi'+a'^2)^{\frac{3}{2}}} \quad . \quad . \quad (1).$$

Let 2β = the angle between the lines a and a'.
Draw a radius to bisect this angle.

Let ψ = the angle made by this radius with the radius drawn to the illuminated point, ψ being counted positively from 0 to 180°;

Let I = the angle made by the plane of these two radii with the plane of a and a', I being counted positively, beginning from the side of a from 0 to 360°.

$$\left.\begin{aligned}\text{Then } \cos\varphi &= \cos\beta\cos\psi + \sin\beta\sin\psi\cos\text{I} = x + y\\ \cos\varphi' &= \cos\beta\cos\psi - \sin\beta\sin\psi\cos\text{I} = x - y\end{aligned}\right\} \quad . \quad . \quad (2.)$$

making $x = \cos\beta\cos\psi$, and $y = \sin\beta\sin\psi\cos\text{I}$,

and these values of $\cos\varphi$, $\cos\varphi'$ substituted in the equation (1), give the equation of the locus.

1. *Corollary.* When $\beta = 0$, we have

$$\cos\varphi = \cos\psi = \cos\varphi',$$
$$\varphi = \psi = \varphi',$$

and (1) becomes

$$\text{A} = \frac{m\,(1-a\cos\psi)}{(1-2a\cos\psi+a^2)^{\frac{3}{2}}} + \frac{m'(1-a'\cos\psi)}{(1-2a'\cos\psi+a'^2)^{\frac{3}{2}}} \quad . \quad . \quad (3),$$

whence the locus is the circumference of a small circle, perpendicular to the radius passing through the lights. And if, moreover, $a=-a'=1$,

$$\text{A} = m(2-2\cos\psi)^{\frac{1}{2}} + m'(2+2\cos\psi)^{-\frac{1}{2}}$$
$$= \frac{m}{2\sin\frac{1}{2}\psi} + \frac{m'}{2\cos\frac{1}{2}\psi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

2. *Cor.* The differential of (1), being equated to zero, gives

$$0 = \text{v}d.\cos\varphi + \text{v}'d.\cos\varphi' \quad . \quad . \quad . \quad . \quad . \quad (5.),$$

$$\left.\begin{array}{l}\text{making } \text{v} = ma(2-a\cos\varphi-a^2)(1-2a\cos\varphi+a^2)^{-\frac{5}{2}} \\ \text{v}' = m'a'(2-a'\cos\varphi'-a'^2)(1-2a'\cos\varphi'+a'^2)^{-\frac{5}{2}}\end{array}\right\} (6.)$$

Hence by (2), $\text{v}dx + \text{v}dy + \text{v}'dx - \text{v}'dy = 0$,

or $(\text{v} + \text{v}')dx + (\text{v} - \text{v}')\,dy = 0$,

and $-(\text{v} + \text{v}')\cos\beta\sin\psi d\psi + (\text{v} - \text{v}')\sin\beta\cos\psi\cos\iota d\psi$

$- (\text{v} - \text{v}')\sin\beta\sin\psi\sin\iota d\iota = 0$

or, since ψ and ι are *altogether* independent of each other,

$$\left.\begin{array}{l}-(\text{v} + \text{v}')\cos\beta\sin\psi + (\text{v} - \text{v}')\sin\beta\cos\psi\cos\iota = 0. \\ (\text{v} - \text{v}')\sin\beta\sin\psi\sin\iota = 0\end{array}\right\} \quad . \quad (7.)$$

It follows from this second equation that, in general, when β is not $= 0$,

$$\text{v} - \text{v}' \gtrless 0, \text{ or } \psi = 0, \text{ or } \iota = 0.$$

If $\text{v} - \text{v}' = 0$, while $\psi > 0$ and $\iota > 0$, we have from the first of (7)

$-(\text{v} + \text{v}')\cos\beta\sin\psi = 0$; whence $\text{v} + \text{v}' = 0$, or $\text{v} = 0$, $\text{v}' = 0$.

That is from (6)

$$a(2 - a\cos\varphi - a^2) = 0,\ a'(2 - a'\cos\varphi' - a'^2) = 0$$

whence $a = 0$ or $a'\cos\varphi = 2 - a^2$,

and $a' = 0$ or $a'\cos\varphi' = 2 - a'^2$.

But we cannot have $a\cos\varphi = 2 - a^2$ or $a'\cos\varphi' = 2 - a'^2$, unless $a = 1$, $a' = 1$, which gives $\varphi = 0$, or $\varphi = 180°$ and $\varphi' = 0$, or $\varphi' = 180°$.

Hence either both of the lights are at the centre, or one is at the centre, which comes under the case of $\beta = 0$.

Hence, *in general*, $\iota = 0$, for this includes the case $\psi = 0$; that is the points of greatest and least illumination are in the plane of a and a'. And the first of (7) becomes

$$-(\text{v} + \text{v}')\cos\beta\sin\psi + (\text{v} - \text{v}')\sin\beta\cos\psi = 0,$$

$$\text{or } \text{v}\sin(\psi - \beta) + \text{v}'\sin(\psi + \beta) = 0 \quad . \quad . \quad . \quad . \quad . \quad (8.),$$

and (2) becomes $\cos\varphi = \cos\beta\cos\psi + \sin\beta\sin\psi = \cos(\psi - \beta)$,

$\cos\varphi' = \cos\beta\cos\psi - \sin\beta\sin\psi = \cos(\psi + \beta)$;

whence $\varphi = \psi - \beta$, $\varphi' = \beta + \psi$,

$$\varphi' - \varphi = 2\beta, \text{ and } \text{v}\sin\varphi + \text{v}'\sin\varphi' = 0 \quad . \quad . \quad . \quad (9.)$$

3 *Cor.* When $\beta = 0$, we have $\varphi' = \varphi$, whence

$(\text{v} + \text{v}')\sin\varphi = 0$; $\varphi = \varphi' = 0$, or $\text{v} + \text{v} = 0$.

When, moreover, $a = -a' = 1$, we have $\text{v} = \dfrac{m}{16\sin^3\frac{1}{2}\varphi}$, $\text{v} = \dfrac{-m'}{16\cos^3\frac{1}{2}\varphi}$;

$$\frac{m}{m'} = \tan^3\tfrac{1}{2}\varphi, \text{ and } \tan\tfrac{1}{2}\varphi = \sqrt[3]{\frac{m}{m'}};$$

and if $m = m'$, $\frac{1}{2}\varphi = 45°$, $\varphi = 90°$.

4. *Corollary.* When $a = a' = 1$, (9) becomes

$$m\cos\tfrac{1}{2}\varphi\operatorname{cosec}^2\tfrac{1}{2}\varphi + m'\cos\tfrac{1}{2}\varphi'\operatorname{cosec}^2\tfrac{1}{2}\varphi' = 0,$$

whence

$$m\cos\tfrac{1}{2}\varphi + m'\cos\tfrac{1}{2}\varphi' - \cos\tfrac{1}{2}\varphi\cos\tfrac{1}{2}\varphi'\ (m\cos\tfrac{1}{2}\varphi' + m'\cos\tfrac{1}{2}\varphi) = 0,$$

if, moreover, $m = m'$, we have

$$(\cos\tfrac{1}{2}\varphi + \cos\tfrac{1}{2}\varphi')\ (1 - \cos\tfrac{1}{2}\varphi\cos\tfrac{1}{2}\varphi') = 0.$$

But $1 - \cos\frac{1}{2}\varphi\cos\frac{1}{2}\varphi' = 0$ is impossible,

therefore $\cos\frac{1}{2}\varphi + \cos\frac{1}{2}\varphi' = 0$, and $\psi = 0$.

that is, the points of greatest and least illumination are in the line from which ψ is counted.

——Professor Benedict investigates the equation of the lines of equal intensity on any surface of revolution, when two lights are placed within it. Dr. Strong gives two sets of results; one like those of Professor Peirce, on the hypothesis "that the intensity is expressed by the light received on a very minute portion of the surface," and the other on the hypothesis that the "intensity is expresseed by the force of each particle multiplied by the number of particles, which," he remarks, "appears to us the more rational hypothesis." In this case the intensity on surfaces at the same distance would vary as the square of the sine of the incident angle and equation (1) of the preceding solution would be

$$A = m.\frac{(1 - a\cos\varphi)^2}{(1 - 2a\cos\varphi + a^2)^2} + m'.\frac{(1 - a'\cos\varphi')^2}{(1 - 2a'\cos\varphi' + a'^2)^2},$$

also (6) becomes $V = ma(1 - a^2)(1 - a\cos\varphi)(1 - 2a\cos\varphi + a^2)^{-3}$,

$$V' = m'a'(1 - a'^2)(1 - a'\cos\varphi')(1 - 2a'\cos\varphi' + a'^2)^{-3}$$

from which similar results are easily obtained. It is worthy of remark that, on this hypothesis, when the two lights are placed at the surface, or $a = \pm a' = 1$, the intensity of the light at every point of the surface will be the same.

When, in the preceding solution, (2) is substituted in (1), it will be the polar equation of the lines of equal intensity on the surface of a sphere; the prime meridian is the circumference of a great circle through the two lights, and the pole is the extremity of a radius bisecting the angle of the radii through the lights; ψ is the spherical radius rector, and χ the polar angle.

(34.) QUESTION XIV., by *Investigator*.

A given cone of revolution is attached, by its vertex and a point in the circumference of its base, to two fixed points in the same horizontal line, and then placed in the position of unstable equilibrium. If the equilibrium be suddenly disturbed, find when the pressures, in different directions, on the points of suspension of the system will be least, or when they will be entirely destroyed.

FIRST SOLUTION, by *Dr. Strong*.

We shall denote the point at the vertex of the cone by A, and the other by B; and we shall take A for the origin of the rectangular co-ordinates x, y, z, the axis of revolution for that of x, and we shall suppose the axis of y to be horizontal, that of z vertical to the horizontal plane and directed upwards. Put $g = 32.2 =$ the force of gravity, $t =$ the time from the origin of the motion, $\varphi =$ the angle described around the axis of x in the time t, reckoned from the axis of z; put dm for any element of the solid, r for the radius of the circle described by dm around the axis of x; then $dt\ Sgdmy =$ the sum of all the rotary forces which tend to turn the system in the instant dt, about the axis of x; but $\frac{rd^2\varphi}{dt} \times dm =$ the momentum received by dm in the instant dt, which acts

at the extremity of the lever r, $\therefore$ the rotary pressure communicated to dm (in the instant dt) $= \frac{d^2\varphi}{dt} \times r^2\, dm$, whose integral with respect to dm equals $\frac{d^2\varphi}{dt} Sr^2 dm$ = the sum of all the rotary pressures received, which must equal those communicated;

$$\therefore \frac{d^2\varphi}{dt} Sr^2 dm = dt\, Sgdmy, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Let $Y = R \sin\varphi$ = the distance of the centre of gravity from the plane x, z at the time t, then by the nature of that point we have $Sdmy = MY = MR \sin\varphi$, where M denotes the mass of the system, and R the radius of the circle described by the centre of gravity; put $Sr^2 dm = Mk^2$, then (1) is easily changed to

$$\frac{d^2\varphi}{dt^2} = \frac{R \sin\varphi g}{k^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

put $\frac{k^2}{R} = R'$ = the distance of the centre of oscillation from the axis of x,

$$\text{and (2) becomes } \frac{d^2\varphi}{dt^2} = \frac{g}{R'} \sin\varphi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

which will enable us to find the angular velocity of the system at any time; (3) is a well known formula, and we might have obtained it by the formula of Dynamics, but the method which we have used appears to us to be preferable, because it is in our opinion much more simple.

Put $\frac{d\varphi}{dt} = v$, then imagine any section of the solid at right angles to the axis of x, and we shall have $rv^2 dm$ for the centrifugal force of the element dm, of the section, r being the distance of dm from the axis of x, then $v^2 zdm$, $v^2 ydm$ are the centrifugal forces of dm, when resolved in the directions of z and y, then $v^2 Szdm$, $v^2 Sydm$ are the sum of the centrifugal forces of all the elements of the section in those directions; put $Szdm = mz'$, $Sydm = my'$, m = the mass of the section, and z', y' will be the distances of the centre of gravity of the section from the axis of x when estimated on the axes of z and y respectively; these forces may be supposed to be immediately applied to the axis of x. Let x denote the distance of the point of application of the forces $v^2 mz'$, $v^2 my'$ from A, then by the nature of the lever $v^2 mxz'$, $v^2 mxy'$ will express their efforts to turn the system about the point A, when estimated in the planes $x, z \,..\, x, y$; $\therefore$ $v^2 Smxz' \,..\, v^2 Smxy'$ express the efforts of the centrifugal forces of all the particles of the system to turn the body about the point A in the aforesaid planes; let r' denote the distance of the centre of gravity of the section from the axis of x, then $z' = r' \cos\varphi$, $y' = r' \sin\varphi$, put $Smxy' = X\, Smy'$, $Smxz' = X' Smz'$, then $X = \frac{Smxr'}{MR}$, $X' = \frac{Smxr'}{MR}$, $\therefore X' = X$ = the distance of the point of application of the resultants of the aforesaid forces from A, when estimated on the axis of x. Let $AB = l$, l' = the distance of the centre of gravity of the solid from A when estimated on the axis of x; also let $W = Mg$ = the weight of the solid, then

$v^2\,\frac{x\,Smz' - wl'}{l} = \frac{MR}{l}\left(v^2\, x \cos\varphi - \frac{gl'}{R}\right)$, and $\frac{v^2 Sxmy'}{l} = \frac{MR}{l}\, x \sin\varphi$ are the pressures at B in the planes $x, z, . y, z$, and they are perpendicular to the axis of x, put $R_,$ for their resultant or the whole pressure at B, then

$$R_, = \frac{MR}{l}\left[\left(v^2\, x\cos\varphi - \frac{gl'}{R}\right)^2 + v^4\, x^2 \sin^2\varphi\right]^{\frac{1}{2}}, \ . \ . \ (a),$$

in a similar way if $R_{,,}$ denote the pressure at A,

$$R_{,,} = \left[\left(v^2\,(l - x)\cos\varphi - \frac{g\,(l - l')}{R}\right)^2 + v^4\,(l - x)^2 \sin^2\varphi\right]^{\frac{1}{2}}, \ . \ (b);$$

if $R_, = 0$, then $\sin\varphi = 0$, $v^2\, x\cos\varphi - \frac{gl'}{R} = 0$, $\therefore \varphi = 0$, $v^2 = \frac{gl'}{Rx}$, also $R_{,,} = w\, .\left(\frac{x - l'}{x}\right)$. Put $v =$ the value of v, at the origin of the motion (when $\varphi = 0$,) then by (3)

$$\left(\frac{d\varphi}{dt}\right)^2 = v^2 = v^2 + \frac{2g}{R'}(1 - \cos\varphi) = v^2 + \frac{4g}{R'}\sin^2\tfrac{1}{2}\varphi \ . \ (4),$$

then by (4), (a) is easily changed to

$$R_, = \frac{MR}{l}\left[\frac{x}{R}\left((Rx + R'l')\,v^4 - (R'v^2 + 2g)\,l'v^2\right) + \frac{g^2 l'^2}{R^2}\right]^{\frac{1}{2}}, \ . \ (a'),$$

hence when R_1 is a max. or min. we have

$$vdv\,[2(Rx + R'l')\,v^2 - (R'v^2 + 2g)l'] = 0, \ . \ . \ . \ (c),$$

$\therefore v^2 = \frac{(Rv^2 + 2g)l'}{2(Rx + R'l')}$, then φ is found by (4) which gives the position of the body when the pressure is a min. (c) is also satisfied by putting $dv = 0$, which gives $\varphi = 0$, or $\varphi = 180°$, which gives the maxima pressures $\varphi = 180°$ giving the absolute maximum. If we would find when $R_{,,} = 0$, or when it is a max. or min. we have only to change in the above results x into $l - x$, l' into $l - l'$. It is evident that the above formulæ are applicable to the cone, cylinder, and various other solids, revolving about two fixed points in the same horizontal right line, and to adapt them to the solid, the constants R, l, l', x, R', must be determined from the nature of the body. In the cone, if $h =$ its attitude, $\theta =$ half its vertical angle, $a =$ the radius of its base, we have $a = h\tan\theta$, $R = \frac{3}{4}h\sin\theta$, $l' = \frac{3}{4}h\cos\theta$, $l = \sqrt{a^2 + h^2}$, $x = \frac{Smxr'}{MR}$ which is easily found by the ordinary methods of the integral calculus, $R' = \frac{k^2}{R}$, where $Mk^2 = Sr^2 dm$; put $m' = \frac{3}{10}Ma^2 =$ the moment of inertia of the cone revolving about its axis, $m'' = \frac{3}{4}Mh\left(\frac{4}{5}h + \frac{a^2}{5h}\right) =$ the moment of inertia of the cone revolving about a line which passes through its vertex at right angles to its axis, then $Mk^2 = m'\cos^2\theta + m''\sin^2\theta$, $\therefore R'$ is easily determined, and every thing required becomes known.

SECOND SOLUTION, *by Professor Catlin.*

It may be well to remark, before proceeding to the investigation of this problem, that the pressures in any given direction upon the two given points are equivalent to a single force applied at some point in the axis of rotation, and parallel to the direction of the two pressures. Whenever the point of application falls between the two fixed points, the pressures on those points will have the same sign, or will be in the same direction; when it falls on one of the fixed points, the pressure on the other will be nothing in that direction; when it falls beyond the fixed points on one side or the other, the pressures on the two points will be in opposite directions — that is, one will be positive, and the other negative.

Let h = the distance of the centre of gravity of the cone from the axis of rotation; θ = the angle which h makes with the vertical in the varying position of the cone. Let each particle dm be referred to the rectangular co-ordinates x, y, and z, the origin being at the intersection of h with the axis of rotation, z being in the direction of that axis; the axis of x passing through the centre of gravity, and that of y perpendicular to the plane (xz). Let P and P' represent the pressures on the axis of rotation at the distances p and p' from the origin in directions parallel to the axes of x and y. Then we shall have

$$\left.\begin{aligned} \text{P} &= \int\left(\text{X} - \frac{d^2x}{dt^2}\right) dm \\ \text{P}' &= \int\left(\text{Y} - \frac{d^2y}{dt^2}\right) dm \\ \text{P}p &= \int\left(\text{X} - \frac{d^2x}{dt^2}\right) z\,dm \\ \text{P}'p' &= \int\left(\text{Y} - \frac{d^2y}{dt}\right) z\,dm \end{aligned}\right\} \quad \ldots\ldots \quad (1).$$

Put v = the angular velocity at the end of the time t, $\text{M}k^2$ = the moment of inertia for the axis of rotation. Then we shall obviously have

$$\left.\begin{aligned} \frac{dv}{dt} &= \frac{\int(x\text{Y} - y\text{X})\,dm}{\text{M}k^2} \\ \frac{d^2x}{dt^2} &= -y\frac{dv}{dt} - xv^2 \\ \frac{d^2y}{dt^2} &= x\frac{dv}{dt} - yv^2 \end{aligned}\right\} \quad \ldots\ldots \quad (2).$$

Substituting (2) in (1)

$$\left.\begin{aligned} \text{P} &= v^2\!\int xdm + \int \text{X}dm + \frac{\int ydm\int(x\text{Y} - y\text{X})\,dm}{\text{M}k^2} \\ \text{P}' &= v^2\!\int ydm + \int \text{Y}dm + \frac{\int xdm\int(x\text{Y} - y\text{X})\,dm}{\text{M}k^4} \\ \text{P}p &= v^2\!\int xzdm + \int z\text{X}dm + \frac{\int yzdm\int(x\text{Y} - y\text{X})\,dm}{\text{M}k^2} \\ \text{P}'p' &= v^2\!\int yzdm + \int z\text{Y}dm - \frac{\int xzdm\int(x\text{Y} - y\text{X})\,dm}{\text{M}k^2} \end{aligned}\right\} \quad \ldots \quad (3).$$

But in this case

$$\left.\begin{array}{l} X = -g\cos\theta \;\therefore\; \int X dm = -Mg\cos\theta \\ Y = \quad g\sin\theta \;\therefore\; \int Y dm = \quad Mg\sin\theta \end{array}\right\} \quad \ldots \quad (4);$$

and since the co-ordinates of the centre of gravity are $(h, 0, 0,)$ we shall have

$$v^2\int xdm = Mhv^2, \text{ and } v^2\int ydm = 0. \quad \ldots \quad (5).$$

Since the plane (xz) divides the cone into two similar parts

$$\int yzdm = 0. \quad \text{Put } B = \int xzdm \quad \ldots \quad (6).$$

By virtue of (4), (5) and (6), equations (3) become

$$\left.\begin{array}{l} P = Mghv^2 - Mg\cos\theta \\ P' = Mg\sin\theta\left(1 - \dfrac{h^2}{k^2}\right) \\ Pp = v^2 B \\ P'p' = \dfrac{Bgh\sin\theta}{k^2} \end{array}\right\} \quad \ldots \quad (7).$$

But $v = \dfrac{d\theta}{dt} \therefore \dfrac{dv}{dt} = \dfrac{d^2\theta}{dt^2}$. Also by means of the 1st of (2) and (4) we find $\dfrac{dv}{dt} = \dfrac{gh\sin\theta}{k^2} \therefore \dfrac{d^2\theta}{dt^2} = \dfrac{gh\sin\theta}{k^2}$. Multiplying by $2d\theta$, and integrating, we get

$$v^2 = \frac{2hg \text{ vers. } \theta}{k^2}, \quad \ldots \quad (8).$$

Hence

$$\left.\begin{array}{l} P = \dfrac{2Mgh^2}{k^2} - Mg\left(1 + \dfrac{2h^2}{k^2}\right)\cos\theta \\ P' = Mg\sin\theta \quad \left(1 - \dfrac{h^2}{k^2}\right) \\ Pp = \dfrac{2ghB(1 - \cos\theta)}{k^2} \\ P'p' = \dfrac{gh\sin\theta B}{k^2} \end{array}\right\} \quad \ldots \quad (9).$$

Put $a = \dfrac{2Mgh^2}{k^2}$; $b = Mg\left(1 + \dfrac{2h^2}{k^2}\right)$; $a' = \dfrac{2ghB}{k^2}$ and $c = Mg\left(1 - \dfrac{h^2}{k^2}\right)$. Then we shall have

$$p = \frac{a'(1 - \cos\theta)}{a - b\cos\theta} \text{ and } p' = \frac{a'}{c}. \quad \ldots \quad (10).$$

Suppose $P = 0$. We find from the 1st of (7)

$$\cos\theta = \frac{2h^2}{2h^2 + k^2} \quad \ldots \quad (11).$$

Equation (11) shows the position of the cone when the pressures on the two points in a direction parallel to x are both nothing at the same time. When $P' = 0$, $\sin\theta = 0$. Hence, when $\theta = 0$ or $180°$, the pressures parallel to y are both zero. Let m and n represent the distances of the

fixed points, at the vertex and base of the cone from the origin of the co-ordinates. Then when $p = m$ we shall have by substitution in (10)

$$m = \frac{a'(1-\cos\theta)}{a - b\cos\theta} \therefore \cos\theta = \frac{ma - a'}{mb - a'} \quad . \quad . \quad . \quad . \quad (12).$$

In like manner when $p = n$, $\cos\theta = \dfrac{na - a'}{nb - a'}$ . . . . (13).

Equation (12) shows when the pressure parallel to x at the base only is nothing, and (13) shows when the pressure in the same direction at the vertex only is nothing.

Since p' is independent of θ, the point of application of the force P' is fixed for a given cone; hence it is evident that if one of the fixed points be at the distance of p' from the origin of co-ordinates, the pressure on the other point in a direction parallel to y will be constantly equal to zero, $\therefore$ the pressure in the direction of x, will be equal to the whole pressure on that point.

Let us proceed to the general solution. Let R and S be the pressures on the point at the base parallel to x and y, and R$'$ and S$'$ those at the vertex. Then we shall have (putting s = slant height of the cone)

$$R = \frac{m-p}{s}.P;\ R' = \frac{n+p}{s}.P;\quad S = \frac{m-p'}{s}.P';\ S' = \frac{n+p'}{s}.P' \quad (14).$$

Let θ' = the angle which any axis x' (at right angles with the axis of rotation) makes with the vertical.

Then $\theta' - \theta$ = the angle contained by x' and x. Resolving R and S, R$'$ and S$'$ in the direction x', we have

$$R\cos(\theta' - \theta) + S\sin(\theta' - \theta) = T \quad . \quad . \quad . \quad . \quad (15),$$
$$R'\cos(\theta' - \theta) + S'\sin(\theta' - \theta) = T' \quad . \quad . \quad . \quad . \quad (16),$$

where T and T$'$ represent the pressures parallel to x' at the base and vertex. To determine when T and T$'$ are zero, we have

$$\frac{R}{S} = -\tan(\theta' - \theta)\ .\ \text{and}\ \frac{R'}{S'} = -\tan(\theta' - \theta)\ . \quad . \quad (17).$$

The first of (17) determines the position of the cone when $T = 0$; and the second when $T' = 0$. Or if θ' be considered as variable (18) determines the directions in which the pressures on the two points are nothing, for any given position of the cone. For instance, if $\theta = 0$ or $180°$, we find $\theta' = \pm 90°$ as it should. Whenever for a given value of θ', imaginary values of θ, indicate that the pressures T and T$'$ can never be nothing, their least values may be found from the differentials of (16) and (17) equated to zero.

We have considered the constants k^2 and B as known. By the ordinary methods we easily find (β = half the vertical angle)

$$k^2 = \tfrac{1}{15}h^2(5 + \sec^2\beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$
$$B = \tfrac{1}{15}Mh^2(\cot\beta - 4\tan\beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad (19).$$

THIRD SOLUTION, *by Mr. O. Root.*

Let P, P$_{/}$ = the pressures on the given points of the axis, p, $p_{/}$ = the angles those pressures make with the vertical; r = radius of the cone's base; s = its slant height; θ = the angle through which the centre of

gravity of the cone has moved in the time t; m = a particle whose distance from the axis of motion is $r_{\prime}$ and whose rectangular co-ordinates are x, y, z; x being vertical and z measured along the axis of motion, the origin being at the vertex of the cone. Now by equating the impressed and effective forces respectively parallel to x and y we shall have

$$(1) \quad \frac{d\theta^2}{dt^2} \int x dm + \frac{d^2\theta}{dt^2} \int y dm = mg + \text{P} \cos\varphi + \text{P}_{\prime} \cos\varphi_{\prime}$$

$$(2) \quad \frac{d\theta^2}{dt^2} \int y dm - \frac{d^2\theta}{dt^2} \int x dm = \qquad \text{P} \sin\varphi + \text{P}_{\prime} \sin\varphi_{\prime}$$

where g = gravity tending to diminish x.

Also by equating the impressed and effective forces which tend to turn the system around x, y, z respectively, we shall have

$$(3) \quad \frac{d\theta^2}{d\theta^2} \int yz dm - \frac{d^2\theta}{dt^2} \int xz dm = \text{SP}_{\prime} \sin\varphi_{\prime}$$

$$(4) \quad \frac{d\theta^2}{dt^2} \int xz dm + \frac{d^2\theta}{dt^2} \int yz dm = \text{SP}_{\prime} \cos\varphi_{\prime} + g \int z dm$$

$$(5) \quad \ldots\ldots\ldots \frac{d^2\theta}{dt^2} \int r_{\prime}^2 dm = \qquad g \int y dm$$

If h = the distance of the centre of gravity from the axis of motion, then we may put $\int x dm = mh \cos\theta$; $\int y dm = mh \sin\theta$; $\int z dm = \frac{3m}{4\text{s}}(\text{s}^2 - r^2)$ and $\int r_{\prime}^2 dm = mk^2$; k being the radius of gyration, and if we assume $x = x_{\prime} \cos\theta - y_{\prime} \sin\theta$; $y = x_{\prime} \sin\theta + y_{\prime} \cos\theta$; $z = z_{\prime}$ then $\int xz dm = \cos\theta \int x_{\prime} z_{\prime} dm$ and $\int yz dm = \sin\theta \int x_{\prime} z_{\prime} dm$ because $\int y_{\prime} z_{\prime} dm = 0$ hence put $\int x_{\prime} z_{\prime} dm = \text{A}$; therefore $\int xz dm = \text{A} \cos\theta$ and $\int yz dm = \text{A} \sin\theta$; by substituting for $\int r_{\prime}^2 dm$ and $\int y dm$ their values; (5) becomes $\frac{d^2\theta}{dt^2} = \frac{gh \sin\theta}{k^2}$. Multiply by $2d\theta$ and integrate we have $\left(\frac{d\theta}{dt}\right)^2 = \frac{2gh}{k^2}(1 - \cos\theta)$; making these substitutions in (1) (2) (3) (4) and we shall have the following,

$$(6) \quad \frac{mgh^2}{k^2}(1 + 2\cos\theta - 3\cos^2\theta) \quad = \text{P} \cos\varphi + \text{P}_{\prime} \cos\varphi_{\prime} + mg$$

$$(7) \quad \frac{mgh^2}{k^2}(2 - 3\cos\theta)\sin\theta \quad = \text{P} \sin\varphi + \text{P}_{\prime} \sin\varphi_{\prime}$$

$$(8) \quad \frac{gh\text{A}}{k^2}(2 - 3\cos\theta)\sin\theta \quad = \text{S.P}_{\prime} \sin\varphi_{\prime}$$

$$(9) \quad \frac{gh\text{A}}{k^2}(1 + 2\cos\theta - 3\cos^2\theta) = \text{S.P}_{\prime} \cos\varphi_{\prime} + \frac{3mg}{4\text{s}}(\text{s}^2 - r^2$$

and by obvious reductions these will give,

$$(10) \quad \text{P} \sin\varphi = \frac{gh}{\text{s}k^2}(\text{s}mh - \text{A})(2 - 3\cos\theta)\sin\theta$$

$$(11) \quad \text{P} \cos\varphi = \frac{gh}{\text{s}k^2}(\text{s}mh - \text{A})(1 + (2 - 3\cos\theta)\cos\theta) - \frac{mg}{4\text{s}}(\text{s}^2 - r^2$$

$$(12) \quad \text{P}_{\prime} \sin\varphi_{\prime} = \frac{gh\text{A}}{\text{s}k^2}(2 - 3\cos\theta)\sin\theta$$

$$(13)\quad \text{P}_{,}\cos\varphi_{,} = \frac{gh\text{A}}{sk^2}(1 + (2 - 3\cos\theta)\cos\theta) - \frac{3mg}{4s}(s^2 - r^2).$$

Now it is plain that by squaring and adding (10) and (11) we shall find P^2 or the pressure on the point at the cone's vertex in functions of θ, and from (12) and (13) in the same way $\text{P}_{,}$ the pressure on the fixed point at the base of the cone will be found; and as I understand the question we must find θ when $d\text{P} = 0$ and when $d\text{P}_{,} = 0$, which is attended with no difficulty except the length of the process. (Mr. R. then proceeds to show how k and A may be found by the usual methods.)

Cor. Equation (10) expresses the pressure on the point at the vertex resolved parallel to the horizon, hence when this is nothing $\theta = 0$ or 180° or $\cos\theta = \frac{2}{3}$; and from (12) we infer the same for the point at the cone's base.

——Professor Avery's solution was also very elegant and complete; we are sorry our limits would not allow us to insert it. Professor Peirce's was unfortunately rendered unfit for publication, by a slight numerical error which had crept into the body of the investigation.

(35.) QUESTION XV. BY ——.

Two given circles touch each other internally; it is required to find the sum of the areas of all the circles that can be inscribed between them, so that each one shall touch the two adjacent ones, and also the two given circles; the centre of one of the inscribed circles being given in position.

FIRST SOLUTION, *by the Editor.*

Let R, r be the radii of the given circles, and d the distance between their centres which, when the circles touch internally, $= \text{R} - r$. Let a line through the two centres be the axis of x, and a perpendicular to it, through the centre of the inner circle r be the axis of y; let also r_1, r_2, r_3, &c. be the radii of the inscribed circles, and y_1, x_1; y_2, x_2; y_3, x_3; &c., be the co-ordinates of their respective centres. From the contact of these circles with the circle whose radius is r we have the equations

$$\left.\begin{array}{l} 1.\ y_1^2 + x_1^2 = (r + r_1)^2 \\ 2.\ y_2^2 + x_2^2 = (r + r_2)^2 \\ 3.\ y_3^2 + x_3^2 = (r + r_3)^2 \\ \qquad \text{\&c.} \qquad\qquad \text{\&c.} \end{array}\right\} \quad (\text{A});$$

from their contact with the circle, radius R, the equations

$$\left.\begin{array}{l} 1.\ y_1^2 + (x_1 - d)^2 = (\text{R} - r_1)^2 \\ 2.\ y_2^2 + (x_2 - d)^2 = (\text{R} - r_2)^2 \\ 3.\ y_3^2 + (x_3 - d)^2 = (\text{R} - r_3)^2 \\ \qquad \text{\&c.} \qquad\qquad \text{\&c.} \end{array}\right\} \quad (\text{B});$$

and, from their contact with each other, the equations

$$\left.\begin{array}{l} 1.\ (y_1 - y_2)^2 + (x_1 - x_2)^2 = (r_1 + r_2)^2 \\ 2.\ (y_2 - y_3)^2 + (x_2 - x_3)^2 = (r_2 + r_3)^2 \\ 3.\ (x_3 - y_4)^2 + (x_3 - x_3)^2 = (r_3 + r_4)^2 \\ \qquad \text{\&c.} \qquad\qquad\qquad \text{\&c.} \end{array}\right\} \quad (\text{C}).$$

By taking 1 of (B) from 1 of (A), and reducing by the equation $d = \text{R} - r$,

$$2dx_1 = 2(\text{R} + r)r_1 - 2rd, \text{ or } x_1 = \frac{\text{R}+r}{d} r_1 - r \quad . \quad . \quad . \quad (1).$$

Writing this in 1 of (A) and reducing

$$y_1^2 = \frac{4\text{R}rr_1^2}{d^2}\left(\frac{d}{r_1} - 1\right) \quad . \quad . \quad . \quad . \quad . \quad (2).$$

In a similar manner the second of equations (A) and (B) give us

$$x_2 = \frac{\text{R}+r}{d} r_2 - r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

$$y_2^2 = \frac{4\text{R}rr_2^2}{d^2}\left(\frac{d}{r_2} - 1\right) \quad . \quad . \quad . \quad . \quad . \quad (4).$$

If we put in equations (2) and (4)

$$\frac{d}{r_1} - 1 = k_1^2 \text{ and } \frac{d}{r_2} - 1 = k_2^2 \quad . \quad . \quad . \quad (5)$$

then multiply them together and extract the root, we get

$$y_1 y_2 = \frac{4\text{R}rr_1r_2k_1k_2}{d^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Now write the values of x_1, x_2, y_1^2, y_2^2 in equation 1 of (C) and we find

$$y_1 y_2 = \frac{2\text{R}rr_1r_2}{d^2}\left(\frac{d}{r_1} + \frac{d}{r_2} - 2 + \frac{d^2}{\text{R}r}\right)$$

$$= \frac{2\text{R}rr_1r_2}{d^2}(k_1^2 + k_2^2 - h^2) \quad . \quad . \quad . \quad (7),$$

where $$h^2 = \frac{d^2}{\text{R}r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Equations (6) and (7) give us

$$k_1^2 + k_2^2 - h^2 = 2k_1 k_2,$$
$$\therefore k_2 - k_1 = \pm h,$$
$$\text{and } k_2 = k_1 \pm h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

In like manner, if we put

$$\frac{d}{r_3} - 1 = k_3^2, \frac{d}{r_4} - 1 = k_4^2, \text{ \&c.}, \quad . \quad . \quad (10).$$

the equations 2 and 3 of (A) and (B) with 2 of (C) give us

$$k_3 = k_2 \pm h = k_1 \pm 2h \quad . \quad . \quad . \quad . \quad . \quad (11),$$

the equations 3 and 4 of (A) and (B) with 3 of (C) give us

$$k_4 = k_3 \pm h = k_1 \pm 3h \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

&c.

Now if we take the circle whose radius is r_1 for the one the position of whose centre is given in the question, x_1 will be given, and since $\frac{d}{r_1} = 1 + k_1^2$ if we write this in equation (1) and reduce, we find

$$k_1^2 = \frac{\text{R} - x_1}{r + x_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

which determines the quantity k_1 and consequently the dependent ones k_2, k_3, k_4, &c.

With regard to the ambiguous signs, it is evident that the two values of k_2 in (9) belong to the two circles which touch the circle r_1, the two values of k_3 in (11) belong to the two circles which touch the last two,

and so on. Hence if r_2, r_3, r_4, &c., be the radii of the circles, in their order, which are situated on one side of the circle r_1, and r_2', r_3', r_4', &c., those of the circles which have corresponding positions on the other side of r_1, we shall have from (5) and (10)

$$\left.\begin{array}{ll} & r_1 = \dfrac{d}{k_1^2+1}, \\ r_2 = \dfrac{d}{(h+k_1)^2+1}, & r_2' = \dfrac{d}{(h-k_1)^2+1}, \\ r_3 = \dfrac{d}{(2h+k_1)^2+1}, \text{ \&c. } & r_3' = \dfrac{d}{(2h-k_1)^2+1}, \text{ \&c.} \\ r_x = \dfrac{d}{(\overline{x-1}.h+k_1)^2+1}, & r_x' = \dfrac{d}{(\overline{x-1}h-k_1)^2+1}. \end{array}\right\} \ldots (14)$$

If we put s for the sum of the areas of all these circles, and designate by $\Sigma.u$ the infinite sum of a series whose xth term is u_x, then

$$s = \pi\{r_1^2 + \Sigma.r^2_{x+1} + \Sigma.r'^2_{x+1}\}$$
$$= d^2\pi\left[\frac{1}{(k_1^2+1)^2} + \Sigma.\frac{1}{\{(xh+k_1)^2+1\}^2} + \Sigma.\frac{1}{\{(xh-k_1)^2+1\}^2}\right] (15).$$

In order to sum these series we will take the two following well known equations of Euler's:

$$\cos\frac{z\pi}{2n} + \tan\frac{m\pi}{2n}.\sin\frac{z\pi}{2n}$$
$$= \left(1+\frac{z}{n-m}\right)\left(1-\frac{z}{n+m}\right)\left(1+\frac{z}{3n-m}\right)\left(1-\frac{z}{3n+m}\right) \text{ \&c.} \quad (16).$$

$$\cos\frac{z\pi}{2n} + \cot\frac{m\pi}{2n}.\sin\frac{z\pi}{2n}$$
$$= \left(1+\frac{z}{m}\right)\left(1-\frac{z}{2n-m}\right)\left(1+\frac{z}{2n+m}\right)\left(1-\frac{z}{4n-m}\right)\left(1+\frac{z}{4n+m}\right) \text{\&c.} (17)$$

Now put

$$s_1 = \frac{1}{n-m} - \frac{1}{n+m} + \frac{1}{3n-m} - \frac{1}{3n+m} + \text{\&c.},$$
$$s_2 = \frac{1}{(n-m)^2} + \frac{1}{(n+m)^2} + \frac{1}{(3n-m)^2} + \frac{1}{(3n+m)^2} + \text{\&c.},$$
$$t_1 = \frac{1}{m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{4n-m} + \frac{1}{4n+m} + \text{\&c.},$$
$$t_2 = \frac{1}{m^2} + \frac{1}{(2n-m)^2} + \frac{1}{(2n+m)^2} + \frac{1}{(4n-m)^2} + \frac{1}{(4n+m)^2} + \text{\&c.},$$

so that

$$t_1 - s_1 = \frac{1}{m} + \frac{1}{n+m} + \frac{1}{2n+m} + \frac{1}{3n+m} + \frac{1}{4n+m} + \text{\&c.},$$
$$- \frac{1}{n-m} - \frac{1}{2n-m} - \frac{1}{3n-m} - \frac{1}{4n-m} + \text{\&c.},$$
$$= \frac{1}{m} + \Sigma.\frac{1}{xn+m} - \Sigma.\frac{1}{xn-m} \quad \ldots\ldots\ldots \quad (18).$$

$$\text{and } t_2 + s_2 = \frac{1}{m^2} + \frac{1}{(n+m)^2} + \frac{1}{(2n+m)^2} + \frac{1}{(3n+m)^2} + \text{\&c.,}$$

$$+ \frac{1}{(n-m)^2} + \frac{1}{(2n-m)^2} + \frac{1}{(3n-m)^2} + \text{\&c.,}$$

$$= \frac{1}{m^2} + \Sigma . \frac{1}{(xn+m)^2} + \Sigma . \frac{1}{(xn-m)^2} \quad . \quad . \quad . \quad (19).$$

In the formulas (16) and (17), instead of $\cos \dfrac{z\pi}{2n}$ and $\sin \dfrac{z\pi}{2n}$ write their developments according to the powers of z, and actually multiply their second members, then they become

$$1 + \frac{z\pi}{2n} . \tan \frac{m\pi}{2n} - \frac{\pi^2 z^2}{2.4n^2} + \text{\&c.} = 1 + s_1 z + \tfrac{1}{2}(s_1{}^2 - s_2)z^2 + \text{\&c.,} \quad (20).$$

$$1 + \frac{\pi z}{2n} . \cot \frac{m\pi}{2n} - \frac{\pi^2 z^2}{2.4n^2} + \text{\&c.} = 1 + t_1 z + \tfrac{1}{2}(t_1{}^2 - t_2)z^2 + \text{\&c.,} \quad (21).$$

and equating the co-efficients of the like powers of z in the two members of each equation, we shall find

$$s_1 = \frac{\pi}{2n} \tan \frac{m\pi}{2n}, t_1 = \frac{\pi}{2n} \cot \frac{m\pi}{2n},$$

$$s_1{}^2 - s_2 = -\frac{\pi^2}{4n^2}, t_1{}^2 - t_2 = -\frac{\pi^2}{4n^2};$$

$$\text{therefore } s_2 = s_1{}^2 + \frac{\pi^2}{4n^2} = \frac{\pi^2}{4n^2}\left(\tan^2 \frac{m\pi}{2n} + 1\right) = \frac{\pi^2}{4n^2} \sec^2 \frac{m\pi}{2n},$$

$$\text{and } t_2 = t_1{}^2 + \frac{\pi^2}{4n^2} = \frac{\pi^2}{4n^2}\left(\cot^2 \frac{m\pi}{2n} + 1\right) = \frac{\pi^2}{4n^2} \operatorname{cosec}^2 \frac{m\pi}{2n};$$

$$\text{hence } t_1 - s_1 = \frac{\pi}{2n}\left(\cot \frac{m\pi}{2n} - \tan \frac{m\pi}{2n}\right) = \frac{\pi}{n} \cot \frac{m\pi}{n},$$

$$\text{and } t_2 + s_2 = \frac{\pi^2}{4n^2}\left(\operatorname{cosec}^2 \frac{m\pi}{2n} + \sec^2 \frac{m\pi}{2n}\right) = \frac{\pi^2}{n^2} \operatorname{cosec}^2 \frac{m\pi}{n}.$$

Substitute these last results in (18) and (19), we have

$$\frac{1}{m} + \Sigma . \frac{1}{xn+m} - \Sigma . \frac{1}{xn-m} = \frac{\pi}{n} \cot \frac{m\pi}{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad (22).$$

$$\frac{1}{m^2} + \Sigma . \frac{1}{(xn+m)^2} + \Sigma . \frac{1}{(xn-m)^2} = \frac{\pi^2}{n^2} \operatorname{cosec}^2 \frac{m\pi}{n} \quad . \quad . \quad (23).$$

Multiply equation (23) by a and add it to (22), then

$$\frac{m+a}{m^2} + \Sigma . \frac{xn+m+a}{(xn+m)^2} - \Sigma . \frac{xn-m-a}{(xn-m)^2}$$

$$= \frac{\pi}{n} \cot \frac{m\pi}{n} + \frac{a\pi^2}{n^2} \operatorname{cosec}^2 \frac{m\pi}{n} \quad . \quad . \quad . \quad (24).$$

write in this equation, first $a + b$, and then $a - b$ for m, and we get

$$\frac{2a+b}{(a+b)^2} + \Sigma . \frac{xn+2a+b}{(xn+a+b)^2} - \Sigma . \frac{xn-2a-b}{(xn-a-b)^2}$$

$$= \frac{\pi}{n} \cot \frac{(a+b)\pi}{n} + \frac{a\pi^2}{n^2} \operatorname{cosec}^2 \frac{(a+b)\pi}{n} \quad (25).$$

$$\frac{2a-b}{(a-b)^2}+\Sigma.\frac{xn+2a-b}{(xn+a-b)^2}-\Sigma.\frac{xn-2a+b}{(xn-a+b)^2}$$
$$=\frac{\pi}{n}\cot\frac{\overline{a-b}\,.\,\pi}{n}+\frac{a\pi^2}{n^2}\operatorname{cosec}^2\frac{\overline{a-b}.\pi}{n} \quad (26).$$

Now since $\Sigma.\dfrac{xn+2a+b}{(xn+a+b)^2}-\Sigma.\dfrac{xn-2a+b}{(xn-a+b)^2}=\Sigma.\dfrac{4a^3}{\{(xn+b)^2-a^2\}^2}$,

and $\Sigma.\dfrac{xn+2a-b}{(xn+a-b)^2}-\Sigma.\dfrac{xn-2a-b}{(xn-a-b)^2}=\Sigma.\dfrac{4a^3}{\{(xn-b)^2-a^2\}^2}$,

if we add equations (25) and (26) together, and divide by $4a^3$, we shall have

$$\frac{1}{(b^2-a^2)^2}+\Sigma.\frac{1}{\{(xn+b)^2-a^2\}^2}+\Sigma.\frac{1}{\{(xn-b)^2-a^2\}^2}$$
$$=\frac{\pi}{4na^3}\left(\cot\frac{(a+b)\pi}{n}+\cot\frac{(a-b)\pi}{n}\right)+\frac{\pi^2}{4n^2a^2}\left(\operatorname{cosec}^2\frac{(a+b)\pi}{n}+\operatorname{cosec}^2\frac{(a-b)\pi}{n}\right)$$
$$=\frac{\alpha}{4a^4}.\frac{\sin\alpha}{\cos\beta-\cos\alpha}+\frac{\alpha^2}{4a^4}.\frac{1-\cos\alpha\cos\beta}{(\cos\beta-\cos\alpha)^2}\quad .\ .\ .\ .\ \ (27).$$

where, for brevity, we have put

$$\frac{2a\pi}{n}=\alpha \text{ and } \frac{2b\pi}{n}=\beta \quad .\ .\ .\ .\ .\ .\ (28).$$

Now put $\quad a=c\sqrt{-1}$, and $\alpha=\gamma\sqrt{-1}$, so that

$$\frac{2c\pi}{n}=\gamma \quad .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ (29).$$

$\therefore \sin\alpha=\sin\gamma\sqrt{-1}=\dfrac{\varepsilon^{-\gamma}-\varepsilon^{\gamma}}{2\sqrt{-1}}$ and $\cos\alpha=\cos\gamma\sqrt{-1}=\dfrac{\varepsilon^{-\gamma}+\varepsilon^{\gamma}}{2}$;

and equation (27) becomes, after some little reduction,

$$\frac{1}{(b^2+c^2)^2}+\Sigma.\frac{1}{\{(xn+b)^2+c^2\}^2}+\Sigma.\frac{1}{\{(xn-b)^2+c^2\}^2}$$
$$=\frac{\gamma}{4c^4}.\frac{\varepsilon^{2\gamma}-1}{\varepsilon^{2\gamma}-2\cos\beta\varepsilon^{\gamma}+1}+\frac{\gamma^2\varepsilon^{\gamma}}{2c^4}.\frac{\cos\beta(\varepsilon^{2\gamma}+1)-2\varepsilon^{\gamma}}{\{\varepsilon^{2\gamma}-2\cos\beta\varepsilon^{\gamma}+1\}^2}\ (30).$$

Multiply this equation by c^4, and put $\dfrac{b}{c}=k_1$ and $\dfrac{n}{c}=h$, so that from (28) and (29), (8) and (13),

$$\beta=\frac{2k_1\pi}{h}=2\pi\sqrt{\frac{\text{R}r(\text{R}-x_1)}{d^2(r+x_1)}},\text{ and }\gamma=\frac{2\pi}{h}=\frac{2\pi\sqrt{\text{R}r}}{d}\quad (31).$$

and it becomes, finally,

$$\frac{1}{(k_1^2+1)^2}+\Sigma.\frac{1}{\{(xh+k_1)^2+1\}^2}+\Sigma.\frac{1}{\{(xh-k_1)^2+1\}^2}$$
$$=\tfrac{1}{4}\gamma.\frac{\varepsilon^{2\gamma}-1}{\varepsilon^{2\gamma}-2\cos\beta\varepsilon^{\gamma}+1}+\tfrac{1}{2}\gamma^2\varepsilon^{\gamma}.\frac{\cos\beta(\varepsilon^{2\gamma}+1)-2\varepsilon^{\gamma}}{\{\varepsilon^{2\gamma}-2\cos\beta\varepsilon^{\gamma}+1\}^2}\ .\ \ (32);$$

hence, by (15), the sum of all the areas is

$$s=\tfrac{1}{2}d\pi^2\sqrt{Rr}.\frac{\varepsilon^2\gamma-1}{\varepsilon^2\gamma-2\cos\beta\varepsilon\gamma+1}+2Rr\pi^2\varepsilon\gamma.\frac{\cos\beta(\varepsilon^2\gamma+1)-2\varepsilon^\gamma}{\{\varepsilon^2\gamma-2\cos\beta\varepsilon\gamma+1\}^2}\ (33),$$

the quantities β and γ being given by (31) and ε being the number whose hyperbolic logarithm is unity.

Cor. 1. If the centre of r_1 be on the axis of x, so that $y_1=0$; then, by (2), $\frac{d}{r_1}-1=k_1{}^2=0$, therefore $\beta=0$, and (33) reduces to

$$s=\tfrac{1}{2}d\pi^2\sqrt{Rr}+\frac{d\pi^2\sqrt{Rr}}{\varepsilon\gamma-1}+\frac{2Rr\pi^3\varepsilon\gamma}{(\varepsilon\gamma-1)^2}\quad .\ .\ .\ (34).$$

Cor. 2. If the centre of r_1 touches the axis of x, so that $y_1=r_1$; then, by (2) $\frac{d^2}{Rr}=4\left(\frac{d}{r_1}-1\right)$, or $h^2=4k_1{}^2$ and $h=2k_1$, therefore $\beta=\pi$, and

$$s=\tfrac{1}{2}d\pi^2\sqrt{Rr}-\frac{d\pi^2\sqrt{Rr}}{\varepsilon\gamma+1}-\frac{2Rr\pi^3\varepsilon\gamma}{(\varepsilon\gamma+1)^2}\quad .\ .\ .\ .\ (35).$$

Cor. 3. If the circle r_1 be so situated that $h=4k_1$, which is the case when $x_1=r.\frac{16R^2-d^2}{16Rr+d^2}$ and $y_1=\frac{8Rrd}{16Rr+d^2}$; then $\beta=\frac{1}{2}\pi$, and

$$s=\tfrac{1}{2}d\pi^2\sqrt{Rr}-\frac{d\pi^2\sqrt{Rr}}{\varepsilon^2\gamma+1}-\frac{4Rr\pi^3\varepsilon^2\gamma}{(\varepsilon^2\gamma+1)^2}\quad .\ .\ .\ .\ .\ (36).$$

SECOND SOLUTION, *by Dr. Strong.*

Put R = the radius of the smaller circle, R' that of the larger, $D=R'-R$ = the distance of their centres, r = the radius of one of the sought circles, d = the distance of its centre from that of the smaller of the given circles, d' = the distance of the same point from that of the other circle. Then since the circle (rad. r.) touches the smaller circle externally, and the other internally, $d=R+r$, $d'=R'-r$, or $d+d'=R+R'$, ∴ the locus of the centres of the sought circles is an ellipse whose foci are at the centres of the given circles, semitransverse axis $=\frac{1}{2}(R+R')=A$, semiconjugate $=\sqrt{R'R}=B$, ratio of the eccentricity to the semitransverse $=\frac{R'-R}{R'+R}=e$. Let x, y be the abscissa and ordinate of the centre of the circle (rad. r,) their origin being at the point of contact of the given circles, x being reckoned on the transverse axis, then the equation of the ellipse which is the locus of the centres of the tangent circles is

$$y^2=\frac{B^2}{A^2}(2Ax-x^2)\quad .\ .\ .\ .\ .\ .\ .\ (1).$$

let r' be the radius of the circle which touches the circle (rad. r,) r'' that of the next circle, and so on successively, x', x'', &c., their abscissæ, y', y'', &c., their ordinates and suppose $x'>x$, then because the circles, radii r, r', touch each other, we have

$$(r'-r)^2=(x'-x)^2+(y'-y)^2,\quad .\ .\ .\ (2).$$

Put $x=A(1-\cos u)$, $x'=A(1-\cos u')$, &c., then $2Ax-x^2=A^2\sin^2u$, ∴ by (1) $y=B\sin u=A\sqrt{1-e^2}\sin u$, in the same way y'

$= A\sqrt{1-e^2}.\sin u'$, & so on; also $d^2 = (R+r)^2 = (x-R)^2 + y^2$, but $R = A(1-e)$ hence

$(R+r)^2 = A^2(e-\cos u)^2 + A^2(1-e^2)\sin^2 u = A^2(1-e\cos u)^2$,

or $\qquad r = Ae(1-\cos u) = ex$;

in the same way $r' = ex'$, and so on; $\therefore x'-x = A(\cos u - \cos u')$, $y'-y = A\sqrt{1-e^2}(\sin u' - \sin u)$, $r'+r = Ae(2-\cos u-\cos u')$, and (2) is easily changed to

$(\cos u - \cos u')^2 + (1-e^2)(\sin u' - \sin u)^2 = e^2(2-\cos u - \cos u')^2$
$= 4e^2\{1-(\cos u+\cos u') + \cos u\cos u'\} + e^2(\cos u - \cos u')^2$,

or $(1-e^2)\{(\cos u - \cos u')^2 + (\sin u - \sin u')^2\} = 4e^2(1-\cos u)(1-\cos u')$,

or by an easy reduction we have

$(1-e^2)\{1-\cos(u'-u)\} = 2e^2(1-\cos u)(1-\cos u')$,

or $\qquad \sin^2\frac{1}{2}(u'-u) = b\sin^2\frac{1}{2}u\sin^2\frac{1}{2}u'$, where $b = \dfrac{4e^2}{1-e^2}$

$\qquad \therefore \sin\frac{1}{2}(u'-u) = \sqrt{b}.\sin\frac{1}{2}u\sin\frac{1}{2}u'$,

or $\qquad \cot\frac{1}{2}u - \cot\frac{1}{2}u' = \sqrt{b}$, and $\cot\frac{1}{2}u' = \cot\frac{1}{2}u - \sqrt{b}$.

In the same way $\quad \cot\frac{1}{2}u'' = \cot\frac{1}{2}u' - \sqrt{b}$, $\cot\frac{1}{2}u''' = \cot\frac{1}{2}u'' - \sqrt{b}$, &c.

$\therefore$ by adding these equations and reducing we have

$$\cot\tfrac{1}{2}u_n = \cot\tfrac{1}{2}u - n\sqrt{b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

where u_n denotes the n^{th} angle in the series of increasing angles u', u'', u''', &c. corresponding to the n^{th} radius r_n on that side of the circle (radius r) to which u corresponds, and which we shall take for the circle whose centre is given, it is evident that when the angles decrease we shall, instead of (3), have

$$\cot\tfrac{1}{2}{}_nu = \cot\tfrac{1}{2}u + n\sqrt{b},$$

corresponding to the n^{th} radius ${}_nr$ taken on the side of the decreasing angles. Now $r^n = Ae(1-\cos u_n) = 2Ae\sin^2\frac{1}{2}u_m$, hence, and by (3),

$$r_n = \frac{2Ae}{1+(\cot\frac{1}{2}u - n\sqrt{b})^2}, \text{ and } {}_nr = \frac{2Ae}{1+(\cot\frac{1}{2}u + n\sqrt{b})^2}.$$

Put $\operatorname{cosec}^2\frac{1}{2}u = a$, $2\cot\frac{1}{2}u = \sqrt{c}$, $2Ae = T$, then

$$r_n = \frac{T}{a+bn^2-n\sqrt{bc}}, \text{ and } {}_nr = \frac{T}{a+bn^2+n\sqrt{bc}} \quad . \quad . \quad (4).$$

Put $r_n + {}_nr = 2TQ$, then, by (4)

$$Q = \frac{a+bn^2}{(a+bn^2)^2-bcn^2} = \frac{1}{a+bn^2} + \frac{bcn^2}{(a+bn^2)^3} + \frac{b^2c^2n^4}{(a+bn^2)^5} + \dots \frac{b^mc^mn^{2m}}{(a+bn^2)^{2m+1}} (5),$$

where the last term is put for the general term of the series which involves c; if we multiply (5) by $2T$, then suppose m infinite, and put $n = 1, 2, 3, 4$, and so on, to infinity, then add the series, thus formed, and to their sum add the radius of the circle whose centre is given, we shall have the sum of the radii of all the sought circles.

Now it is evident that $\dfrac{bcn^2}{(a+bn^2)^3} = \dfrac{bc}{1.2} \times \dfrac{d^2(a+bn^2)^{-1}}{db.da}$, $\dfrac{b^2c^2n^4}{(a+bn^2)^5} = \dfrac{b^2c^2}{1.2.3.4} \times \dfrac{d^4.(a+bn^2)^{-1}}{db^2.da^2}$, and so on to $\dfrac{b^mc^mn^{2m}}{(a+bn^2)^{2m+1}} = \dfrac{b^mc^m}{1.2\dots 2m} \times \dfrac{d^{2m}.(a+bn^2)^{-1}}{db^m.da^m}$; hence, using the general term, (5) is changed to

$$Q = \frac{1}{a+bn^2} + \frac{b^m c^m \times d^{2m}.\,(a+bn^2)^{-1}}{1\,.\,2\,.\,.\,.\,.\,2mdb^m\,.\,da^m} \quad . \quad . \quad . \quad (6),$$

where if we put $m = 1, 2, 3$, &c., successively, then take the differentials indicated by the formula, we shall have (5), then putting $n = 1, 2, 3$, and so on successively, then add the series thus obtained, we shall find the sum of the radii as above; or using the summatory sign S, we shall obtain

$$SQ = S\left(\frac{1}{a+bn^2}\right) + \frac{b^m c^m \times d^{2m}.\,\{S\,(a+bn^2)^{-1}\}}{1\,.\,2\,.\,3\,.\,.\,.\,.\,2mdb^m\,.\,da^m} \quad . \quad (7).$$

But, by Euler's Analysis of Infinities, p. 143, or by La Croix's Calcul. Integral, vol. 3, p 449, we have

$$S\,(a+bn^2)^{-1} \text{ or } S\left(\frac{1}{a+bn^2}\right) = -\frac{1}{2a} + \frac{p}{2t}\left(\frac{\varepsilon^v+1}{\varepsilon^v-1}\right) \quad . \quad (8),$$

which is the sum of the series $\frac{1}{a+b} + \frac{1}{a+4b} + \frac{1}{a+9b}$ + &c. to infinity, and is obtained from $\frac{1}{a+bn^2}$ by putting $n = 1, 2, 3$, and so on to infinity; where, for brevity, we have put $2p\sqrt{\frac{a}{b}} = v, \sqrt{ab} = t$; p being $= 3{,}14159$, &c., and ε = the base of the hyperbolic system of logarithms. Now the radius whose centre is given $= \frac{2Ae}{a} = \frac{T}{a}$, if we add this to (7) multiplied by $2T$, and substitute the value of $S\,(a+bn^2)^{-1}$ from (8), we have

$$\frac{T}{a} + S({}_nr + r_n) = \frac{pT}{t}\left(\frac{\varepsilon^v+1}{\varepsilon^v-1}\right) + \frac{pTb^mc^m}{1.2.3....2m} \times \frac{d^{2m}.\{(\varepsilon^v+1)+t(\varepsilon^v-1)\}}{db^m\,.\,da^m} \;. \,(9),$$

which by substituting the values of v and t, then putting $m = 1, 2, 3$, &c., successively, and taking the differentials indicated by the formula, gives the sum of the radii of all the circles. If $u = p$, $\cot \frac{1}{2}u = 0$, which give $c = 0$, and (9) becomes

$$\frac{T}{a} + S\,({}_nr + r_n) = \frac{pT}{t}\left(\frac{\varepsilon^v+1}{\varepsilon^v-1}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (10);$$

in this case the centre of the given circle is on the line joining the centres of the given circles, and if we multiply (9) or (10) by $2p$, we shall have the sum of the circumferences of the sought circles; if V be that sum, (10) gives

$$V = \frac{2p^2T}{t}\left(\frac{\varepsilon^v+1}{\varepsilon^v-1}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

In applying the formula (9), should the quantity $\frac{\cot\frac{1}{2}u}{\sqrt{b}}$ be greater than $\frac{1}{2}$, put n' = the nearest integer to the quotient $\frac{\cot\frac{1}{2}u}{\sqrt{b}}$, and assume $\cot\frac{1}{2}u - n'\sqrt{b} = \pm\cot\frac{1}{2}w$; and we may take the angle w for the angle corresponding to the circle whose centre is given, making $a = \text{cosec}^2\frac{1}{2}w$ and $\sqrt{c} = 2\cot\frac{1}{2}w$. Thus c will always be a very small quantity, and the

series in (9) will converge so rapidly that it will be necessary to take only a few of its first terms.

Again, since $r_n = \text{T}(a + bn^2 - n\sqrt{bc})^{-1}$, we have $r_n{}^2 = \text{T}^2(a + bn^2 - nbc)^{-1} = -\text{T}^2 \cdot \dfrac{d \cdot (a + bn^2 - n\sqrt{bc})^{-1}}{da} = -\text{T} \cdot \dfrac{dr_n}{da}$, and ${}_nr^2 = -\text{T} \cdot \dfrac{d \cdot {}_nr}{da}$, $\therefore {}^nr^2 + r_n{}^2 = -\text{T} \cdot \dfrac{d \cdot ({}^nr + r_n)}{da}$, and

$$\frac{\text{T}^2}{a^2} + \text{S}({}^nr^2 + r_n{}^2) = -\text{T} \cdot \frac{d\left\{\frac{\text{T}}{a} + \text{S}({}_nr + r_n)\right\}}{da} \quad . \quad . \quad (12),$$

which is the sum of the squares of the radii, and if we multiply this by p, and put $\text{A}' =$ the sum of the areas of the given circles, we shall have, by (9),

$$\text{A}' = p^2\text{T}^2 \cdot \frac{d \cdot \{(\varepsilon^v + 1) + t(\varepsilon^v - 1)\}}{da}$$
$$- \frac{p^2\text{T}^2 b^m c^m}{1.2\ldots 2m} \times \frac{d^{2m+1} \cdot \{(\varepsilon^v + 1) + t(\varepsilon^v - 1)\}}{db^m \cdot da^{m+1}} \quad . \quad (13).$$

when $u = p$, and $c = 0$, this becomes

$$\text{A}' = -p^2\text{T}^2 \cdot \frac{d \cdot \{(\varepsilon^v + 1) + t(\varepsilon^v - 1\}}{da} \quad . \quad . \quad . \quad . \quad (14).$$

It is evident that, by using a similar method, we can find the sum of the cubes, fourth powers, &c., of the radii.

NEW BOOKS.

1. An Elementary Treatise on Astronomy; second edition, enlarged and improved. By John Gummere, A. M.—Kimber and Sharpless, Philadelphia.

An improved analytical investigation of the eclipses of the sun, occultations, and transits, has been introduced in this edition.

2. *Cours de Physique de l'E'cole Polytechnique*, by M. Lamè, 2 vols. 8vo. Bachelier, Paris.

3. *Théorie Mathématique de la Chaleur*, by M. Poisson, being the second part of his *Traité de Physique Mathématique;* the third part is in the press —Bachelier, Paris.

4. *Journal Mathématiques Pures et Appliquées;* designed for the reception of memoirs on the different parts of Mathematics. Published monthly, and edited by M. J. Liouville. The first twelve numbers (for 1836) have been received.—Bachelier, Paris.

List of Contributors, and of the Questions answered by each. The figures refer to the number of the Question, as marked in No. II., Art. VII.

Prof. C. AVERY, Hamilton College, Clinton, N. Y., Ans. all the questions.
ALFRED, Athens, Ohio, Ans. 1, 2, 3, 4. 6, 7. 12.
Prof. F. N. BENEDICT, University of Vt., Burlington, Ans. 1, 2, 3. 6, 7, 8, 9. 11. 13.
P. BARTON, Jun., Orange, Franklin Co., Mass., Ans. 1, 2, 3, 4. 7, 8.
C. C., Cambridge, Mass., Ans. 1.
Prof. M. CATLIN, Hamilton College, Clinton, N. Y., Ans. all the questions.
DELTA, Ans. 8.
INVESTIGATOR, Ans. 14.
P. KETCHUM, Hamilton College, Ans. 2, 3, 4.
WM. LENHART, York, Pa., Ans. 6. 9. 10.
JAMES F. MACULLY, New-York, Ans. 9.
THOS. C. MONTGOMERY, Institute, Flushing, L. I., Ans. 1. 3. 6.
GEO. R. PERKINS, Clinton Liberal Institute, N. Y., Ans. 1 to 12 inclusive.
Prof. B. PEIRCE, Harvard University, Cambridge, Ans. all the questions.
P. Ans. 7. 12.
PETRARCH, New-York, Ans. 8. 11, 12. 14.
O. ROOT, Mathem. Tutor, Hamilton College, Ans. all the questions.
Prof. T. STRONG, L.L.D., Rutgers' Col., N. Brunswick, Ans. all the questions.
RICHARD TINTO, Greenville, Ohio, Ans. 11.
N. VERNON, Frederick, Md., Ans. 1, 2, 3. 6. 9.

*** All communications for No. IV., which will be published on the first of November, 1837, must be post paid, addressed to the Editor, at the Institute, Flushing, L. I.; and must arrive before the first of August, 1837. New questions must be accompanied with their solutions.

ERRATUM.

In Question (42) VII., page 111, instead of "and terminate in the constant sides of the right angle, shall be of a length," read "and terminate in the sides of the right angle, shall be of a constant length."

ARTICLE XII.

NEW QUESTIONS TO BE ANSWERED IN NUMBER V.

Solutions to these Questions must arrive before the first of February, 1838.

(51). QUESTION I. *By* ——.

Divide $x^4 + ax^2 + b$ into two real quadratic factors.

(52). QUESTION II. *By Mr. P. Barton, Jun., Orange, Franklin Co., Mass.*

On the base of a given right angled triangle, a series of the greatest squares are constructed (*as in fig.* 4,) each having an angular point in the hypothenuse: determine the side of the n^{th} square, and the sum of the areas of n squares, or of an infinite number of them.

(53). QUESTION III. *By Prof. Jno. Chamberlain, Oakland College, Miss.*

The distance from one of the angles of a given triangle to a point within it is d, required the lengths of the two lines drawn from the same point to the other two angles of the triangle, when the given line d is equally inclined to the required lines.

(54). QUESTION IV. *By Prof. Catlin, Hamilton College.*

If, from a given point in the plane of a given parallelogram, perpendiculars be drawn to the diagonal and to the two sides which contain this diagonal; then the product of the diagonal by its perpendicular is equal to the sum of the products of the two sides into their respective perpendiculars, when the point is taken without the parallelogram, or to their difference when the point is taken within. Required a demonstration.

(55). QUESTION V. *By A.*

Convert a^x into a series, in a more simple manner than is usually done; and then deduce Rules for finding the logarithms of numbers.

(56). QUESTION VI. *By* ——.

Def. A diameter of a curve is the locus of the middle points of a system of parallel chords.

Find the equations of the diameters of the curves represented by the general equation of the second degree between two variables; show that, in general, they all pass through a fixed point; and determine the position of those diameters which bisect their systems of chords perpendicularly.

(57). QUESTION VII. *By Wm. Lenhart, Esq., York, Pa.*

Find x, y, z, such that x^2+xy+y^2, x^2+xz+z^2, y^2+yz+z^2 shall be squares.

(58). QUESTION VIII. *By Prof. T. S. Davies, Royal Military Academy, Woolwich.*

If four points on the sphere be taken at pleasure, and all the great circles joining these be drawn to mutually intersect, they will divide one another into segments, such that the sines of the segments are in harmonical proportion.

(59). QUESTION IX. *By Prof. Catlin.*

A given cone is suspended from a given point, successively by all the points in a line drawn from the vertex to the circumference of the base, while the axis remains in a given plane; required the locus of the vertex, and also the area of the locus.

(60). QUESTION X. *By Wm. Lenhart, Esq.*

Suppose five cards to be drawn promiscuously from a pack consisting of 52 cards, namely, 13 clubs, 13 spades, 13 hearts, and 13 diamonds; what is the chance that the five cards drawn will be all of the same suit, as clubs, or spades, &c.? What the chance that three and no more of the five cards will be aces? What the chance that three of the five cards will be alike, and also the remaining two; that is, three of them to be tens or nines, &c., and the remaining two to be fours, or fives, or knaves, &c.? What the chance that four of the five cards will be alike, say aces, kings, or queens, &c.? And, lastly, what is the chance that the five cards will compose one or other of the four foregoing hands

(61.) QUESTION XI. *By Richard Tinto, Esq., Greenville, Ohio.*

Two spheres are given in magnitude and position. It is required to find the locus of a point at which a light being placed, the shadows of the spheres on a given plane may be of equal magnitude.

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(62). QUESTION XII. *By* ψ.

Let M denote the mass of the sun,

m, m' the masses of any two of the planets revolving round it,

n, n' their mean angular velocities,

a, a' their mean distances from the sun.

Show that $\dfrac{M+m}{a^3} = n^2$, and that $n^2 a^3 = n'^2 a'^3$.

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(63). QUESTION XIII. *By Prof. C. Avery, Hamilton College.*

It is required to find the time in which a rigid rod of small diameter will descend from a given, to a horizontal, position; its ends sliding along a vertical and a horizontal plane without friction.

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(64). QUESTION XIV. *By Prof. F. N. Benedict, University of Vermont.*

From a vessel of water, formed by the revolution of a curve about an axis perpendicular to the horizon, three jets issue at the same point; the first horizontally, the second in the direction of a normal, and the third in the direction of a tangent, of the generating curve at the orifice. It is required to determine the form of the vessel, such that wherever the orifice may be situated, the principal vertex of the normal or of the tangent jet shall be in a given horizontal plane; and also to determine its form, such that the area of the triangle formed by connecting the foci of the three jets, shall be a given function of the depth of the orifice below the surface of the water.

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(65). QUESTION XV. *By Prof. T. S. Davies.*

A prolate ellipsoid being described on the diameter of a given sphere, and cut by any meridional plane (*fig.* 5): if another given sphere be made to *roll* upon the ellipsoid, so that a given great circle of it constantly coincides with the meridional plane, the two spheres will intersect in all their positions, and it is required to find the *envelopes* of the circles of intersection made in each sphere.

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(66). QUESTION XVI. *By Investigator.*

A given cylindrical surface is placed with one of its linear elements in contact with a horizontal plane, and then made to oscillate on the plane according to a given law. It is required to find the motion of a material point, placed on the smooth interior surface, and subjected to the action of gravity.

ARTICLE XIII.

ON SPHERICAL GEOMETRY.

(Continued from Art. IV. p. 52.)

§ IV.

The Transformation of Spherical Co-ordinates.

44. For the other two systems of co-ordinates, I shall use the notation adopted by Mr. Davies, as given in the article referred to at page 30, and this is, (*fig. on p.* 30.)

For the *longitudinal system*, OS $= v$, OR $= \chi$;

For the *latitudinal system*, MS $= \xi$, MR $= \omega$.

This settled, the two right angled triangles, ORM, OSM, which have a common hypothenuse OM $= y$, and complemental angles at O, MOR $= x$, MOS $= \frac{1}{2}\pi - x$, give the relations.

$$\left.\begin{array}{l} 1.\ \sin\omega = \sin y \sin x \\ 2.\ \tan\chi = \tan y \cos x \\ 3.\ \sin\xi = \sin y \cos x \\ 4.\ \tan v = \tan y \sin x \end{array}\right\} \quad \ldots\ldots \quad (132).$$

Hence, to change an equation from polar co-ordinates to the longitudinal ones, we should use the equations, derived from 2 and 4 of (132)

$$\left.\begin{array}{l} \tan^2 y = \tan^2 v + \tan^2 \chi \\ \tan\ x = \tan\ v \cot\ \chi \end{array}\right\} \quad \ldots\ldots \quad (133).$$

And to change an equation from polar co-ordinates to the latitudinal one, we should use the equations derived from 1 and 3 of (132),

$$\left.\begin{array}{l} \sin^2 y = \sin^2\omega + \sin^2\xi \\ \tan\ x = \sin\ \omega\ \operatorname{cosec}\ \xi \end{array}\right\} \quad \ldots\ldots \quad (134).$$

The equations for transforming a longitudinal equation into a polar one, are 2 and 4 of (132); and those for transforming a latitudinal equation into a polar one, are 1 and 3 of (132); for transforming a longitudinal equation into a latitudinal one, we have by eliminating y and x among the equations (132),

$$\left.\begin{array}{l} \sin v = \sin\omega \sec\xi \\ \sin\chi = \sin\xi \sec\omega \end{array}\right\} \quad \ldots\ldots\ldots \quad (135);$$

and for transforming a latitudinal equation into a longitudinal one

$$\left.\begin{array}{l} \tan\omega = \tan v \cos\chi \\ \tan\xi = \tan\chi \cos v \end{array}\right\} \quad \ldots\ldots\ldots \quad (136).$$

45. If the angle POQ, contained by the meridional axes was equal to some angle, β, different from a right angle, the systems of co-ordinates would be somewhat analogous to oblique co-ordinates on the plane. I shall give here the equations for transforming a polar equation to an oblique longitudinal system; those for the oblique latitudinal system are too complicated to render them of any practical use. The two quadrantal triangles OSQ, OMQ having a common angle at Q, and the two, ORP, OMP having a common angle at P, give us severally

$$\left.\begin{array}{l} \tan \mathrm{SQO} = \sin x \tan y = \sin\beta \tan v \\ \tan \mathrm{RPO} = \sin(\beta - x) \tan y = \sin\beta \tan\chi \end{array}\right\} \quad \ldots \quad (137),$$

and from these we deduce

$$\left.\begin{aligned}\tan^2 y &= \tan^2 v + 2\cos\beta \tan v \tan \chi + \tan^2 \chi \\ \cot\ x &= \cot v \tan \chi \operatorname{cosec} \beta + \cot \beta\end{aligned}\right\} \quad . \quad (138).$$

46. In order to change the angular axis of a system of polar co-ordinates to one, making an angle α with the primitive axis, the origin remaining the same, we must write

$$x = \alpha + x' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (139).$$

In order to change the origin to another point on the same angular axis at the distance δ from the primitive origin, let the new co-ordinates be $y'x'$, the axis of x' remaining the same as that of x; then in the triangle whose sides are y, y', δ and the angles opposite the two first $\pi - x'$ and x, we should have

$$\left.\begin{aligned}\cos y &= \cos\delta \cos y' - \sin\delta \sin y' \cos x' \\ \cot x &= \cos\delta \cot x' + \sin\delta \cot y' \operatorname{cosec} x'\end{aligned}\right\} \quad . \quad . \quad . \quad (140).$$

If the new origin be not in the primitive axis, we should first change the axis to one passing through the new origin by (139), and then make the second change by (140); if the new angular axis be not the great circle passing through the two origins, a third change may be effected by (139).

The other systems may be changed with equal facility, but the transformation in particular cases can generally be effected more readily by the relations of the immediate case, than by the substitution of a general formula.

§ V.

The intersections of a Sphere by a Cylinder.

47. Let the point M (*fig. on p.* 30) with its co-ordinate arcs be orthographically projected on the plane of the circle PQ P'Q', whose pole is the origin of co-ordinates. The circles OR, OM, OS will be projected into straight lines, which will be severally equal to $\sin\chi$, $\sin y$, $\sin v$; but the circles RM, SM would be projected into ellipses, intersecting in the projection (M') of M, their transverse axes being the diameter of the sphere, and their semiconjugates the projections of OR and OS, or $\sin\chi$ and $\sin v$ respectively. Hence, neither the longitudinal nor latitudinal systems of spherical co-ordinates are projected into the rectilinear co-ordinates of the projection of M; but since the point O is projected into the centre (O') of the sphere, and the spherical angle MOQ is projected into an equal rectilinear angle M'O'Q, if we make O' the origin, and O'Q the angular axis of the polar co-ordinates (v,φ) of M' on the plane of projection, we shall have, R being the radius of the sphere,

$$\sin y = \frac{v}{R} \text{ and } x = \varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (141)$$

Therefore, if any line on the surface of the sphere whose polar equation is represented by $f(y, x) = 0$, be projected on the plane of a great circle whose pole is the origin of co-ordinates, the polar equation of the projected line will be

$$f\left(\sin^{-1}\frac{v}{R},\ \varphi\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (142),$$

the centre of the sphere being the origin of co-ordinates.

To find the projection of a given spherical curve on any given plane, we have therefore only to draw a great circle parallel to the plane, and find the polar equation $f(y, x)$ of the curve having the pole of that circle for the origin of co-ordinates; then the equation of the required projected line will be given by (142). For example, let the curve be the less-circle, whose pole is ω, φ' and distance r; then its equation, by (41), will be

$$\cos\omega\cos y + \sin\omega\sin y\cos(\varphi' - x) - \cos r = 0,$$

and projecting it on the plane of the great circle whose pole is the origin, the polar equation of the projected line is by (142)

$$\cos\omega\sqrt{R^2 - v^2} + v\sin\omega\cos(\varphi' - \varphi) - R\cos r = 0.$$

By transforming this to a system of rectangular co-ordinates y, x, originating at the centre, the axis of x making an angle φ' with the angular axis, we have

$$v^2 = y^2 + x^2, \quad v\cos(\varphi' - \varphi) = x,$$

and therefore

$$\cos\omega\sqrt{R^2 - y^2 - x^2} + x\sin\omega - R\cos r = 0,$$

or $\quad y^2\cos^2\omega + x^2 - 2rx\sin\omega\cos r + R^2(\cos^2 r - \cos^2\omega) = 0.$

This is an ellipse, the co-ordinates of its centre being 0 and $R\cos r\sin\omega$, and its semi-axes being $R\sin r$ and $R\sin r\cos\omega$. When $\omega = 0$, the projection becomes a circle whose centre is the centre of the sphere, and radius $R\sin r$.

48. From what has been said it will be evident that every curve on the surface of the sphere may be regarded as the line of intersection of the sphere, by a right cylinder, the directrix of which is the curve whose equation is (142) and its generatrix is parallel to the radius through the origin of co-ordinates of the spherical curve. As the position of the origin of co-ordinates on the sphere's surface is absolutely indeterminate, the same curve may be produced by the intersection of an infinite number of cylinders with the sphere.

49. Conversely, to find the intersections of a given sphere with a given cylinder, either right or oblique, we have only to draw a plane through the centre of the sphere, perpendicular to the generatrix of the cylinder, and find the polar equation, $f(v, \varphi) = 0$, of the resulting section of the cylinder, the origin being at the centre of the sphere; then the spherical polar equation of the intersections will be

$$f(\sin y, x) = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (143).$$

the origin being at the extremity of a radius perpendicular to the section. There will, in general, be two opposite curves on the sphere, similar and equal to each other, which may be named *conjugate curves.*

§ VI.

On the Spherical Ellipse.

50. *Definition.* If a curve be described on the surface of the sphere, such that every point in that curve has the sum of its distances from two fixed points, (called the *foci*) on the surface, counted on great circle arcs, equal to a given arc, we shall call that curve a *spherical ellipse;* from its construction being analogous to that of the ellipse on the plane, and from its possessing, as we shall see, many kindred properties.

51. *Lemma.* If F and f be two points taken at pleasure on the surface of the sphere (*fig.* 6), then if the arcs FC, fC be drawn through the point C, they will respectively pass through the points F′, f' diametrically opposite to F, f; and we shall have $fC = \pi - f'C$, and
$fC + FC = \pi - f'C + FC = \pi - (f'C - FC) = \pi + (FC - f'C)$ (144);
hence, if the point C has the sum of its distances from two given points given, the difference of its distances from one of these points and a point diametrically opposite to the other, is also given.

Moreover, if we describe a circle with the pole F and the given distance $fC + FC$ cutting FC in D, we shall have $DC = fC$, or the point C will always be at the same distance from the point f and the circle ED; and if, at the same time, $fC + FC = \frac{1}{2}\pi$, the circle ED will be a great circle.

It thus appears that the spherical ellipse is analogous in its construction to both the ellipse and the hyperbola on the plane; and when the constant sum of the distances is equal to a quadrantal arc, it is also analogous to the parabola.

Again, since $F'C + f'C = 2\pi - (FC + fC)$, the point C has also the sum of its distances from the two points F′ and f' constant, and therefore if two curves were described, the one having the sum of the distances of its points from F and f, $FC + fC = 2a$, and the other having $FC + fC = 2\pi - 2a$, these two curves would be precisely similar and equal, but placed on opposite sides of the sphere, the points F and f, or F′ and f' being indifferently the foci of both. We shall name these curves *conjugate ellipses.*

When $2a > \pi$, $2\pi - 2a < \pi$, and therefore, if we take the nearest distance $Ff = 2c$, we may consider the magnitudes c and $\frac{1}{2}\pi$ as the minor and major limits of a; for $FC + fC = 2a > Ff = 2c$, and if $FC - fC$ were $= 2a$, we have seen that the resulting curve would be an ellipse having F and f' for its foci, $FC + f'C = 2a' = \pi - 2a$, by (144); $Ff' = 2c' = \pi - 2c$, and therefore $a' > c'$. Hence, in investigating the properties of the curves, we may always have

$$a > c < \tfrac{1}{2}\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (145).$$

52. *To find the equations of the spherical ellipse.* Let C, (*fig.* 7,) the middle point of Ff, be the origin of co-ordinates, and CF the angular axis. Let M be a point in the curve, and put $FM = d_1$, $fM = d_2$, so that $d_1 + d_2 = 2a$; then we have

$$\cos d_1 = \cos c \cos y + \sin c \sin y \cos x,$$
$$\cos d_2 = \cos c \cos y - \sin c \sin y \cos x;$$

and adding and subtracting these two equations,

$$\cos a \cos \tfrac{1}{2}(d_2 - d_1) = \cos c \cos y,$$
$$\sin a \sin \tfrac{1}{2}(d_2 - d_1) = \sin c \sin y \cos x;$$

or, multiplying the first by $\sin a$, the second by $\cos a$, squaring the two equations and adding them,

$$\sin^2 a \cos^2 a = \sin^2 a \cos^2 c \cos^2 y + \cos^2 a \sin^2 c \sin^2 y \cos^2 x \quad . \quad (146),$$

which is the equation of the ellipse. When $x = 0$ or $180°$, we find

$$\sin^2 a \cos^2 a = \sin^2 a \cos^2 c \cos^2 y + \cos^2 a \sin^2 c \sin^2 y$$
$$= \sin^2 a \cos^2 c - (\cos^2 c - \cos^2 a) \sin^2 y,$$

or $$\sin^2 y = \sin^2 a, \text{ and } y = \pm a;$$

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that is, the curve intersects the axis at two points A, B, each at the distance a from the origin; we shall call AB $= 2a$, the *transverse axis* of the ellipse. If γ be any angle, the same values of y will result from supposing $x = \gamma$, $x = -\gamma$, $x = \pi - \gamma$, or $x = \pi + \gamma$, therefore the curve is symmetrical with respect to the axis AB, and all chords through the origin are bisected at that point. Also the same values of y would result from supposing $x = \frac{1}{2}\pi - \gamma$, $x = \frac{1}{2}\pi + \gamma$, $x = \frac{3}{2}\pi - \gamma$, or $x = \frac{3}{2}\pi + \gamma$, therefore the curve is also symmetrical with respect to a circle, DE, through C, perpendicular to AB; let $x = \frac{1}{2}\pi$, then this circle intersects the curve at the points D and E, when

$$\cos^2 y = \frac{\cos^2 a}{\cos^2 c},$$

or if we make CD or CE $= b$, we shall have

$$\cos b \cos c = \cos a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (147),$$

b is necessarily $< a$, because by (145), $a < \frac{1}{2}\pi$ and b and c are the sides of a spherical right angled triangle, whose hypothenuse DF $= a$. DE $= 2b$ is called the *conjugate axis* of the ellipse, and their intersection C, the centre, or spherical centre of the curve; and we shall also call all chords through the centre *diameters*.

53. Put the angles of the triangle, CFD, (*fig.* 7,) and which may be called the *eccentric angles* of the ellipse, $= \varepsilon_1$ and ε_2; that is, the angle D, opposite c, $= \varepsilon_1$, and F, opposite b, $= \varepsilon_2$, then we shall have among these quantities the following relations:

$$\left.\begin{array}{l} \sin \varepsilon_1 = \dfrac{\sin c}{\sin a}, \ \cos \varepsilon_1 = \dfrac{\tan b}{\tan a}, \ \tan \varepsilon_1 = \dfrac{\tan c}{\sin b} \\ \sin \varepsilon_2 = \dfrac{\sin b}{\sin a}, \ \cos \varepsilon_2 = \dfrac{\tan c}{\tan a}, \ \tan \varepsilon_2 = \dfrac{\tan b}{\sin c} \end{array}\right\} \quad . \quad (148).$$

Then if we solve equation (146) for $\sin^2 y$, we find

$$\begin{aligned} \sin^2 y &= \frac{\sin^2 a(\cos^2 c - \cos^2 a)}{\sin^2 a \cos^2 c - \cos^2 a \sin^2 c \cos^2 x} \\ &= \frac{\sin^2 a \cos^2 c \sin^2 b}{\sin^2 a \cos^2 c - \cos^2 a \sin^2 c \cos^2 x} \\ &= \frac{\sin^2 b}{1 - \cot^2 a \tan^2 c \cos^2 x} \\ &= \frac{\sin^2 b}{1 - \cos^2 \varepsilon \cos^2 x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (149). \end{aligned}$$

Also dividing equation (146) by $\sin^2 a \cos^2 c \sin^2 y$ and solving for $\cot^2 y$,

$$\begin{aligned} \cot^2 y &= \cos^2 b \operatorname{cosec}^2 y - \cot^2 a \tan^2 c \cos^2 x \\ &= \cot^2 b(1 - \cos^2 \varepsilon_2 \cos^2 x) - \cos^2 \varepsilon_2 \cos^2 x \\ &= \cot^2 b(1 - \sec^2 b \cos^2 \varepsilon_2 \cos x^2) \\ &= \cot^2 b(1 - \sin^2 \varepsilon_1 \cos^2 x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (150). \end{aligned}$$

Equation (150) is the one we shall principally use. By (149) we see that the spherical ellipse may be produced by the intersection of the

sphere with a right cylinder, whose directrix is on the plane PQ pq of the great circle whose pole is C, its equation being, by (142),

$$v^2 = \frac{R^2 \sin^2 b}{1 - \cos^2 \varepsilon_2 \cos^2 \varphi},$$

that is, an ellipse whose centre is at the centre of the sphere, its semiconjugate axis equal R sin b, its eccentricity $= \cos \varepsilon_2$, and its semitransverse $=$ R sin a.

54. To transform the equation (150) to the longitudinal co-ordinates CR $= \chi$ and CS $= v$, our equations (133) may be written

$$\cot^2 y = \frac{1}{\tan^2 v + \tan^2 \chi}, \cos^2 x = \frac{\tan^2 \chi}{\tan^2 v + \tan^2 \chi};$$

and (150) becomes, after multiplying by $\tan^2 v + \tan^2 \chi$, and restoring the value of ε_1,

$$\begin{aligned} 1 &= \cot^2 b(\tan^2 v + \tan^2 \chi - \sin^2 \varepsilon_1 \tan^2 \chi) \\ &= \cot^2 b(\tan^2 v + \cos^2 \varepsilon_1 \tan^2 \chi) \\ &= \cot^2 b \tan^2 v + \cot^2 a \tan^2 \chi, \end{aligned}$$

or, as it may be written,

$$\frac{\tan^2 v}{\tan^2 b} + \frac{\tan^2 \chi}{\tan^2 a} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (151),$$

which is the longitudinal equation of the spherical ellipse, referred to its centre and axes.

And to transform it to the latitudinal co-ordinates RM $= \omega$, SM $= \xi$, substituting equations (134) in (149), it becomes

$$\sin^2 \omega + \sin^2 \varepsilon_2 \sin^2 \xi = \sin^2 b$$

or, by (148), $$\frac{\sin^2 \omega}{\sin^2 b} + \frac{\sin^2 \xi}{\sin^2 a} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (152),$$

which is the latitudinal equation of the spherical ellipse, referred to its centre and axes. The analogy between either of the equations (151) or (152) and the rectangular equation of the plane ellipse is sufficiently striking. By taking either of them, as (151), and solving it successively for the two variables, we shall see that the curve is wholly included between two secondaries to Qq, at the distances $+ a$ and $- a$ from C, and two secondaries to Pp, at the distances $+ b$ and $- b$ from C, the secondaries being all drawn round the sphere, circumscribing the two conjugate ellipses, which are shown under another point of view in (fig. 11.)

55. To find the equation of the spherical ellipse when the focus F is the origin, and FQ the axis, we shall find it easier to use the generating property than to substitute the equations (140.) Let FM $= y$, MFQ $= x$, (fig. 7), then fM$=2a-y$, and $\cos(2a-y) = \cos 2c \cos y - \sin 2c \sin y \cos x$

$$\therefore \cot y = \frac{\sin 2a + \sin 2c \cos x}{\cos 2c - \cos 2a} \quad . \quad . \quad . \quad . \quad (153).$$

When $x = \frac{1}{2}\pi$, y will be half the arc of a secondary to the transverse axis through the focus, and if we call this chordal arc the *parameter* of the ellipse and put it $= 2\tilde{\omega}$, we shall have

$$\tan \tilde{\omega} = \frac{\cos 2c - \cos 2a}{\sin 2a} = \frac{\tan^2 b}{\tan a} = \cos \varepsilon_1 \tan b \quad . \quad (154),$$

and (153) may then be written in the different ways

$$\left.\begin{aligned}\cot y &= \cot \tilde{\omega}\,(1 + \frac{\sin 2c}{\sin 2a}.\cos x)\\ &= \cot \tilde{\omega}\,(1 + \frac{\sin^2 \varepsilon_1}{\cos \varepsilon_2}\cos x)\\ &= \frac{\tan a}{\tan^2 b} + \frac{\tan c}{\sin^2 b}\cos x\\ &= \frac{\tan a}{\sin^2 b}(\cos^2 b + \cos\varepsilon_2 \cos x)\end{aligned}\right\} \quad \ldots\ldots \quad (155).$$

If $\sin 2c = \sin 2a = \sin(\pi - 2a)$, or $2c = \pi - 2a$, and $c = \frac{1}{2}\pi - a$, which may be the case when $\frac{1}{2}\pi - a < a$, or $a > \frac{1}{4}\pi$ and $c < \frac{1}{4}\pi$, then $\cos b = \cot a = \sin \varepsilon_1$, or $\varepsilon_1 = \frac{1}{2}\pi - b$, and $\tan \tilde{\omega} = -2\cot 2a$, and the equation (155) becomes

$$\cot y + 4\cot 2a \cos^2 \tfrac{1}{2}x = 0 \quad \ldots\ldots \quad (156).$$

In this case, since $\cot^2 b = -\cos^2 a \sec 2a$, equation (150) becomes

$$\cot^2 y \cos 2a + \cos^2 a\,(1 - \cot^2 a \cos^2 x) = 0 \quad \ldots \quad (157),$$

and if this equation were transferred to the point P as origin, and PC as angular axis, it would be the same as Professor Peirce's equation to the locus in Question XIV., Math. Miscel., showing that the curve in that case is a spherical ellipse, whose semiaxes are A and $\cos^{-1}(\cot A)$, A being $> \frac{1}{4}\pi$.

Besides this, we shall only mention the particular cases, $a = \frac{1}{4}\pi$, and $b = c$. When $a = \frac{1}{4}\pi$, $2\cos^2 b \cos^2 c = 1$, $\tan^2 b = \cos 2c$, $\tan \tilde{\omega} = \cos 2c$, $\cos^2 \varepsilon_1 = \cos 2c$, and equations (150) and (153) become

$$\left.\begin{aligned}\cot^2 y \cos 2c &= 1 - 2\sin^2 c \cos^2 x\\ \cot\ y \cos 2c &= 1 + \ \sin 2c \cos\ x\end{aligned}\right\} \quad \ldots\ldots \quad (158),$$

And when $b = c$, $\cos^2 b = \cos^2 c = \cos a$, $\sin^2 b = \sin^2 c = 2\sin^2 \frac{1}{2}a$, $\sin^2 \varepsilon_1 = \sin^2 \varepsilon_2 = \frac{1}{2}\sec^2 \frac{1}{2}a$, $\tilde{\omega} = \frac{1}{2}a$, and (150) and (153) become

$$\left.\begin{aligned}\sin^2 a \cot^2 y &= \cos^2 a + \cos a \sin^2 x\\ \sin \tfrac{1}{2}a \cot\ y &= \cos \tfrac{1}{2}a + \frac{\cos x}{\sqrt{2}\cos a}\end{aligned}\right\} \quad \ldots \quad (159).$$

56. To draw a tangent circle (fig. 8,) through a given point N $(y_1\, x_1)$ of the spherical ellipse, we have, by (115), page 49,

$$\cot y - \cot y_1 \cos(x_1 - x) + \frac{d.\cot y_1}{dx_1}.\sin(x_1 - x) = 0;$$

but, by (150), $\quad \cot y_1 = \cot b\sqrt{1 - \sin^2 \varepsilon_1 \cos^2 x_1}$,

and $\quad \dfrac{d.\cot y_1}{dx_1} = \dfrac{\cot b \sin^2 \varepsilon_1 \sin 2x_1}{2\sqrt{1 - \sin^2 \varepsilon_1 \cos^2 x_1}}$;

therefore, substituting and reducing, the equation of the tangent is

$$\cot y \tan^2 b = \tan y_1 \{\cos^2 \varepsilon_1 \cos x_1 \cos x + \sin x_1 \sin x\} \quad . \ (160).$$

Then the equation of a tangent through a point H $(y_1, \pi + x_1)$ of the ellipse, diametrically opposite to this, is

$$\cot y \tan^2 b = -\tan y_1 \{\cos^2 \varepsilon_1 \cos x_1 \cos x + \sin x_1 \sin x\} \quad (161);$$

and if $y'x_2$ be the point of intersection of these tangents,

$$\cot y' = 0, \text{ or } y' = \tfrac{1}{2}\pi,$$

and $\quad \cos^2 \varepsilon_1 \cos x_1 \cos x_2 + \sin x_1 \sin x_2 = 0$,

or $\quad \tan x_1 \tan x_2 + \cos^2 \varepsilon_1 = 0 \quad \ldots\ldots \quad (162);$

that is, tangent circles, at the extremities of a diameter of the spherical ellipse, always intersect each other in two points T, t in the circle PQ of which the centre of the ellipse is the pole. Moreover, since the equation (162) is symmetrical with respect to the two angles x_1 and x_2, these angles interchange properties, and tangents through the vertices L, K of the diameter through T, intersect PQ in the points T′ t', where HN also intersects it. We shall call diameters which make angles with the transverse axis having the relation in (162), *conjugate diameters*, and we shall designate any system of conjugate semidiameters by a', b'.

57. Then, by (150)

$$\left.\begin{array}{l}\cot^2 a' = \cot^2 b(1-\sin^2\varepsilon_1\cos^2 x_1)\\ \cot^2 b' = \cot^2 b(1-\sin^2\varepsilon_1\cos^2 x_2)\end{array}\right\} \quad . \quad . \quad . \quad (163),$$

and multiplying these equations together

$$\cot^2 a'\cot^2 b' = \cot^4 b\{1-\sin^2\varepsilon_1(\cos^2 x_1+\cos^2 x_2)+\sin^4\varepsilon_1\cos^2 x_1\cos^2 x_2\}.$$

But, by (162), $\sin^2\varepsilon_1\cos x_1\cos x_2 = -\cos(x_2-x_1)$,

and $\tan^2 x_1\tan^2 x_2 = \cos^4\varepsilon_1 = (\sec^2 x_1 - 1)(\sec^2 x_2 - 1)$,

$$\therefore 1-\cos^2 x_1-\cos^2 x_2+\cos^2 x_1\cos^2 x_2 = \cos^4\varepsilon_1\cos^2 x_1\cos^2 x_2,$$

$$\begin{aligned}\therefore \sin^2\varepsilon_1(\cos^2 x_1+\cos^2 x_2) &= \sin^2\varepsilon_1+(1+\cos^2\varepsilon_1)\sin^4\varepsilon_1\cos^2 x_1\cos^2 x_2\\ &= \sin^2\varepsilon_1+(1+\cos^2\varepsilon_1)\cos^2(x_2-x_1)\\ &= 2-(1+\cos^2\varepsilon_1)\sin^2(x_2-x_1).\end{aligned}$$

Therefore, by substitution, and writing for $\cos\varepsilon_1$ its value in (148),

$$\cot^2 a'\cot^2 b' = \cot^2 a\cot^2 b\sin^2(x_2-x_1)$$

or

$$\tan a'\tan b'\sin(x_2-x_1) = \tan a\tan b \quad . \quad . \quad (164).$$

Moreover, by adding together equations (163), multiplying by $\tan^2 a'$ $\tan^2 b'$, and using the reductions already obtained,

$$\begin{aligned}\tan a^{2\prime}+\tan^2 b' &= \cot^2 b\tan^2 a'\tan^2 b'\{2-\sin^2\varepsilon_1(\cos^2 x_1+\cos^2 x_2)\}\\ &= \tan^2 a(1+\cos^2\varepsilon_1)\\ &= \tan^2 a+\tan^2 b \quad . \quad . \quad . \quad . \quad . \quad . \quad (165).\end{aligned}$$

These relations are analogous to those of the conjugate diameters and their angles, of the ellipse on the plane.

From (162) we also see that the angles x_1 and x_2 are always the one greater and the other less than $\frac{1}{2}\pi$, and therefore, for a system of equal conjugates, we must have, by (163), $x_2 = \pi - x_1$, therefore, by (162)

$$\tan x_1 = \cos\varepsilon_1 = \frac{\tan b}{\tan a} \quad . \quad . \quad . \quad (166),$$

and for the length of the equal conjugates

$$\tan a' = \sqrt{\tfrac{1}{2}(\tan^2 a+\tan^2 b)} \quad . \quad . \quad . \quad . \quad . \quad (167).$$

58. If we make any diameter, CN, the angular axis, the origin remaining the same, the equation will be, by (139),

$$\cot^2 y = \cot^2 b\{1-\sin^2\varepsilon_1\cos^2(x_1+x)\} \quad . \quad . \quad (168).$$

Then if we take CN and its conjugate CL as the axis of an oblique system of longitudinal co-ordinates, making the co-ordinates of the point M, CR $= \chi$, CS $= v$, the semidiameters CN $= a'$, CL $= b'$, and the angle of ordination LCN $= x_2 - x_1$, the equations of transformation (138), give

$$\begin{aligned}&\cos x = \cot y\{\tan\chi+\tan v\cos(x_2-x_1)\},\\ &\sin x = \cot y\,.\,\tan v\sin(x_2-x_1),\\ &\therefore \cos(x_1+x) = \cot y\{\tan\chi\cos x_1+\tan v\cos x_2\},\\ &\tan y^2 = \tan^2 v+\tan^2\chi+2\tan v\tan\chi\cos(x_2-x_1).\end{aligned}$$

and multiplying (168) by $\tan^2 y$, and substituting these,

$$1 = \cot^2 b(1 - \sin^2 s_1 \cos^2 x_2)\tan^2 v + \cot^2 b(1 - \sin^2 s_1 \cos^2 x_1)\tan^2\chi$$
$$+ 2\cot^2 b\{\cos^2 s_1 \cos x_1 \cos x_2 + \sin x_1 \sin x_2\}\tan v \tan\chi,$$

or, reducing by (162) and (163),

$$\frac{\tan^2 v}{\tan^2 b'} + \frac{\tan^2\chi}{\tan^2 a'} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (169),$$

which is precisely similar to the equation (151) referred to the principal diameters as axes. In case of the equal conjugates it becomes

$$\tan^2 v + \tan^2\chi = \tan^2 a' = \tfrac{1}{2}(\tan^2 a + \tan^2 b) \quad . \quad . \quad . \quad (170).$$

It would be expected, in pursuance of the analogy hitherto so perceptible, that the arcs intercepted by the ellipse, such as MM′, of the great circles through T, t, would be bisected by the diameter CN, as at R. Such however is not generally the case, as is easily proved. For, let MR $= \omega$, M′R $= \omega'$, $\angle$ ST′C $= \zeta$, $\angle$ MRT′ $= \eta$; then, for any value of χ, we have from (169,) CS $=$ CS′ $= v$, and therefore, by the quadrantal triangles ST′C, S′T′C,

$$\tan CT'S = \tan CT'S' = \tan\zeta = \tan v \sin(x_2 - x_1);$$

and, by the quadrantal triangle CRt, having RC$t = \pi - (x_2 - x_1)$,

$$\cot\eta = \cos\chi \cot(x_2 - x_1),$$

$$\therefore \cos\eta = \frac{\cos(x_2 - x_1)}{\sqrt{1 + \tan^2\chi \sin^2(x_2 - x_1)}}, \quad \sin\eta = \frac{\sec\chi \sin(x_2 - x_1)}{\sqrt{1 + \tan^2\chi \sin^2(x_2 - x_1)}}.$$

Hence, the triangles MRT′, M′RT′ which have the common side RT′ $= \frac{1}{2}\pi - \chi$, and supplemental angles at R, give

$$\cot\omega = \sec\chi \cot\zeta \sin\eta + \tan\chi \cos\eta$$
$$= \frac{\sec^2\chi \cot v + \tan\chi \cos(x_2 - x_1)}{\sqrt{1 + \tan^2\chi \sin^2(x_2 - x_1)}},$$
$$\cot\omega' = \sec\chi \cot\zeta \sin\eta - \tan\chi \cos\eta$$
$$= \frac{\sec^2\chi \cot v - \tan\chi \cos(x_2 - x_1)}{\sqrt{1 + \tan^2\chi \sin^2(x_2 - x_1)}}.$$

Hence the two arcs ω and ω' are only equal when $v = 0$, or $\chi = 0$, which indicate the tangent and diametral circles TNt, TCt; or when $x_2 - x_1 = \frac{1}{2}\pi$, that is in the case of the principle diameters, when they are equal for all values of χ.

59. If through one extremity A$(a, 0)$ of the transverse axis (fig. 9), and the point T$(\frac{1}{2}\pi, x_2)$ where a diameter intersects the circle PQ, a great circle be drawn, its equation, by (12), will be

$$\cot y \sin x_2 + \cot a \sin(x - x_2) = 0,$$

or, $$\tan x_2(\cot y - \cot a \cos x) = -\cot a \sin x.$$

And if through the other extremity B(a, π) of the transverse axis, and the point T′$(\frac{1}{2}\pi, x_1)$ where a diameter conjugate to the former intersects PQ, a great circle be drawn, its equation, by (12), will be

$$\cot y \sin x_1 - \cot a \sin(x - x_1) = 0,$$

or, $$\tan x_1(\cot y + \cos a \cot x) = \cot a \sin x.$$

Then at the point where these two circles intersect each other, we shall have, by multiplying the two equations,

$$\tan x_1 \tan x_2(\cot^2 y - \cot^2 a \cos^2 x) = -\cot^2 a \sin^2 x,$$

or, by (162), $\cos^2 \varepsilon_1 (\cot^2 y - \cot^2 a \cos^2 x) = \cot^2 a \sin^2 x,$

or, by (148), $\cot^2 y = \cot^2 b \sin^2 x + \cot^2 a \cos^2 x,$

$$= \cot^2 b\{1-(1-\frac{\tan^2 b}{\tan^2 a}) \cos^2 x\}$$

$$= \cot^2 b(1 - \sin^2 \varepsilon_1 \cos^2 x);$$

which shows, by (150), that the intersection of the two circles is a point in the ellipse. This property is analogous to that of supplemental chords in the plane ellipse, and it enables us to draw a diameter conjugate to a given one, as HN, thus:—Produce the given diameter until it intersects the equatorial circle PQ in T′, through T′ and either extremity, B, of the transverse axis, draw the great circle T′B cutting the ellipse at M, through M and the other extremity, A, of the transverse axis, draw the great circle MA, intersecting PQ in T and t, draw TCt through the centre, and it will be a diameter conjugate to HN.

The same property enables us to draw a tangent circle at a given point N of the ellipse; since, a like construction being made, a tangent circle through N or H will pass through the determined points T and t.

60. By comparing the equation (160) of a tangent circle at the point $y_1 x_1$, with equation (2), page 31, we find

$$a_1 = -\cot^2 b \cos^2 \varepsilon_1 \tan y_1 \cos x_1,\ b_1 = -\cot^2 b \tan y_1 \sin x_1;$$

then if the pole of that circle be denoted by (ω, φ), from (4),

$$\tan \varphi = \frac{b_1}{a_1} = \frac{\tan x_1}{\cos^2 \varepsilon_1} = -\cot x_2, \text{ by (162)},$$

using $y_2 x_2$ to denote the extremity of a diameter conjugate to that through $y_1 x_1$; therefore

$$\varphi = x_2 - \tfrac{1}{2}\pi, \quad \dots\dots\dots \quad (171),$$

or the pole is in a circle through C, perpendicular to y_2; also

$$\tan \omega = \sqrt{a_1^2 + b_1^2} = -\cot^2 b \tan y_1 \sqrt{\sin^2 x_1 + \cos^4 \varepsilon_1 \cos^2 x_1}$$

$$= -\cot^2 b \tan y_1 \sqrt{1 - (1 + \cos^2 \varepsilon_1) \sin^2 \varepsilon_1 \cos^2 x_1}$$

$$= -\cot^2 b \tan y_1 \sqrt{(1 + \cos^2 \varepsilon_1) \tan^2 b \cot^2 y_1 - \cos^2 \varepsilon_1}$$

$$= -\cot b \sqrt{1 + \cos^2 \varepsilon_1 - \cos^2 \varepsilon_1 \cot^2 b \tan^2 y_1}$$

$$= -\cot a \cot b \sqrt{\tan^2 a + \tan^2 b - \tan^2 y_1}, \text{ by (148)},$$

$$= -\cot a \cot b \tan y_2, \text{ by (165)}. \quad \dots\dots\dots \quad (172).$$

If we substitute the values $\cot y_2 = -\cot a \cot b \cot \omega$, and $x_2 = \varphi + \frac{1}{2}\pi$ in equation (150), we shall find

$$\cot^2 \omega = \tan^2 a (1 - \sin^2 \varepsilon_1 \sin^2 \varphi)$$

$$= \cot^2 (\tfrac{1}{2}\pi - a) \{1 - \sin^2 \varepsilon_1 \cos^2 (\tfrac{1}{2}\pi - \varphi)\} \quad . \quad (173).$$

This is the equation of the locus of the poles of the circles tangent to the ellipse. It is evidently another ellipse A′D′B′E′ (fig. 10) concentric with the original one, but its transverse axis is on Pp and its conjugate on Qq; if a'', b'' be the semiaxes we have

$$\cot b'' = \tan a, \quad \text{or } b'' = \tfrac{1}{2}\pi - a,$$

and $\cot a'' = \tan a \cos \varepsilon_1 = \tan b'$, or $a'' = \frac{1}{2}\pi - b$;

that is the axes are the supplements of the given ones.

61. Let I (fig. 10) be the point where the tangent at M intersects the transverse axis, and put CI $=\psi$, MI $= t$, then in (160) when $x=0$,

$$\cot y = \cot \psi = \cot^2 b \cos^2 \varepsilon_1 \tan y_1 \cos x_1 = \cot^2 a \tan y_1 \cos x_1,$$
$$\cos t = \cos y_1 \cos \psi + \sin y_1 \sin \psi \cos x_1$$
$$= \sin y_1 \sin \psi (\cot y_1 \cot \psi + \cos x_1)$$
$$= \sin y_1 \sin \psi \cos x_1 \operatorname{cosec}^2 a.$$

Let FM $= d_1$, fM $= d_2$; FMI $= \beta_1$, fMI $= \beta_2$; then as we have before shown in Art. 52,

$$\cos a \cos \tfrac{1}{2}(d_2 - d_1) = \cos c \cos y_1,$$
$$\sin a \sin \tfrac{1}{2}(d_2 - d_1) = \sin c \sin y_1 \cos x_1;$$

and the triangles FMI, fMI give

$$\sin d_1 \sin t \cos \beta_1 = \cos(\psi - c) - \cos d_1 \cos t,$$
$$\sin d_2 \sin t \cos \beta_2 = \cos(\psi + c) - \cos d_2 \cos t;$$

multiply the first of these equations by $\tan a \sin d_2 \operatorname{cosec} \psi$, the second by $\tan a \sin d_1 \operatorname{cosec} \psi$, and add them together, putting for brevity $\tan a \sin d_1 \sin d_2 \sin t \operatorname{cosec} \psi = 2A$, recollecting that $d_1 + d_2 = 2a$, and writing for $\cot \psi$ and $\cos t$ their values, we have

$$A(\cos\beta_1 + \cos\beta_2) = \tan y_1 \cos x_1 \cos c \cos a \cos \tfrac{1}{2}(d_2 - d_1)$$
$$+ \sin c \sin a \sin \tfrac{1}{2}(d_2 - d_1) - \sin y_1 \cos x_1$$
$$= \cos^2 c \sin y_1 \cos x_1 + \sin^2 c \sin y_1 \cos x_1 - \sin y_1 \cos x_1$$
$$= 0,$$
$$\therefore \cos\beta_1 + \cos\beta_2 = 0, \text{ and } \beta_1 = \pi - \beta_2 \quad . \quad . \quad . \quad (174),$$

or the tangent makes equal angles with the radius vectors at the point of tangency. Therefore the normal circle MP′ at the point M bisects the angle fMF, and intersects the ellipse A′B′D′E′ at the point P′ the pole of the tangent circle TMI. Thus we have another method of drawing a tangent or normal circle at a given point of the spherical ellipse.

We shall easily find the equation of the normal circle P′M by equation (12), for it passes through the points $y_1 x_1$ and $\omega\varphi$, and therefore by using the values of ω and φ in (171) and (172), and reducing by (164), we shall find

$$\cot y \sin y_1 \cos y_1 = \cos^2 y_1 \cos(x_1 - x) - \tan(x_2 - x_1)\sin(x_1 - x) \quad (175).$$

62. *To find the Area of the Spherical Ellipse.* By multiplying together equations (149) and (150) we find

$$\cos^2 y = \cos^2 b \,.\, \frac{1 - \sin^2 \varepsilon_1 \cos^2 x}{1 - \cos^2 \varepsilon_2 \cos^2 x}$$
$$= \cos^2 b \,.\, \frac{\tan^2 x + \cos^2 \varepsilon_1}{\tan^2 x + \sin^2 \varepsilon_2} \quad . \quad . \quad . \quad . \quad (176).$$

Therefore, by equation (106), page 47,

$$d\Sigma = (1 - \cos y)dx = dx - \frac{\cos y d.\tan x}{1 + \tan^2 x}$$
$$= dx - \frac{\cos b d.\tan x}{1 + \tan^2 x} . \sqrt{\frac{\tan^2 x + \cos^2 \varepsilon_1}{\tan^2 x + \sin^2 \varepsilon_2}} \quad . \quad (177).$$

Put $\qquad \tan x = \cos \varepsilon_1 \tan \varphi = \cot a \tan b \tan \varphi \quad . \quad . \quad (178),$

where x and φ commence together at 0, and pass through every successive magnitude, $\frac{1}{2}\pi$, together. Then (177) becomes

$$d\Sigma = dx - \frac{\cos b\, d\varphi \cos \varepsilon_1}{\cos^2\varphi + \cos^2\varepsilon_1 \sin^2\varphi} \cdot \frac{\cos \varepsilon_1}{\sqrt{\cos^2\varepsilon_1 \sin^2\varphi + \sin^2\varepsilon_2 \cos^2\varphi}}$$

$$= dx - \frac{\cos b \cos^2\varepsilon_1}{1 - \sin^2\varepsilon_1 \sin^2\varphi} \cdot \frac{d\varphi}{\sqrt{\sin^2\varepsilon_2 - (\sin^2\varepsilon_2 - \cos^2\varepsilon_1)\sin^2\varphi}}.$$

But, by (148) and (147),

$$\sin^2\varepsilon_2 - \cos^2\varepsilon_1 = \sin^2\varepsilon_2\left(1 - \frac{\cos^2\varepsilon_1}{\sin^2\varepsilon_2}\right) = \sin^2\varepsilon_2\left(1 - \frac{\cos^2 a}{\cos^2 b}\right)$$

$$= \sin^2\varepsilon_2\,(1 - \cos^2 c) = \sin^2\varepsilon_2 \sin^2 c,$$

and $$\frac{\cos b \cos^2\varepsilon_1}{\sin\varepsilon_2} = \cos b \cos c \cos\varepsilon_1 = \cos a \cos\varepsilon_1\,;$$

$$\therefore d\Sigma = dx - \frac{\cos a \cos \varepsilon_2}{1 - \sin^2\varepsilon_1 \sin^2\varphi} \cdot \frac{d\varphi}{\sqrt{1 - \sin^2 c \sin^2\varphi}} \quad . \quad . \quad (179),$$

$$\therefore \Sigma = x - \cos a \cos\varepsilon_1\, \Pi(-\sin^2\varepsilon_1, \sin c, \varphi) \quad . \quad . \quad (180)\,;$$

the area commencing when x and $\varphi = 0$, and using $\Pi(n, c, \varphi)$ to designate Legendre's third species of elliptic functions, whose parameter is n, modulus c, and amplitude φ. Legendre's notation of c and b for the modulus and its complement will not interfere with the arcs c and b which are elements of the spherical ellipse. Then if Σ' represent the surface of a quadrant of the ellipse,

$$\Sigma' = \tfrac{1}{2}\pi - \cos a \cos\varepsilon_1\, \Pi'(-\sin^2\varepsilon_1, \sin c) \quad . \quad . \quad (181).$$

In order to express the complete function Π' in terms of functions of the first and second species (F and E), we have, as in Art. 100, *Legendre's Exercices de Calcul Integral*,

$$n = -\sin^2\varepsilon_1 = -1 + \cos^2 c \sin^2\theta,$$

$$\therefore \sin\theta = \frac{\cos\varepsilon_1}{\cos c} = \frac{\tan b}{\tan a} \cdot \frac{\cos b}{\cos a} = \frac{\sin b}{\sin a} = \sin\varepsilon_2$$

$$\therefore \theta = \varepsilon_2.$$

Moreover $$\Delta(b, \theta) = \sqrt{1 - b^2 \sin^2\theta} = \sqrt{1 - \cos^2 c \sin^2\varepsilon_2}$$

$$= \sqrt{1 - \cos^2\varepsilon_1} = \sin\varepsilon_1$$

and $$\frac{b^2 \sin\theta \cos\theta}{\Delta(b, \theta)} = \frac{\cos^2 c \sin\varepsilon_2 \cos\varepsilon_2}{\sin\varepsilon_1} = \cos a \cos\varepsilon_1,$$

Therefore, Legendre's equation (m'), art. 101, becomes

$$\cos a \cos\varepsilon_1\, \Pi'(-\sin^2\varepsilon_1, \sin c) = \tfrac{1}{2}\pi - \mathrm{E}'(\sin c)\,\mathrm{F}(\cos c, \varepsilon_2)$$
$$+ \mathrm{F}'(\sin c)\,\{\cos a \cos\varepsilon_1 + \mathrm{F}(\cos c, \varepsilon_2) - \mathrm{E}(\cos c, \varepsilon_2)\} \quad . \quad (182),$$

and equation (181) becomes

$$\Sigma' = \mathrm{F}'(\sin c)\{\mathrm{E}(\cos c, \varepsilon_2) - \mathrm{F}(\cos c, \varepsilon_2) - \cos a \cos\varepsilon_1\} + \mathrm{E}'(\sin c)\,\mathrm{F}(\cos c, \varepsilon_2) \quad (183).$$

Both the complete and incomplete functions in this expression can be had, by inspection, from Legendre's Tables. In certain cases, the following functions will be more convenient; let, as in arts. 18 and 33 (*ibid.*),

$$\sin c \tan\varepsilon_2 \tan\psi = 1, \text{ or } \tan\psi = \frac{\cot\varepsilon_2}{\sin c} = \cot b,$$

$$\therefore \psi = \tfrac{1}{2}\pi - b, \text{ and}$$

$$\cos^2 c \sin\varepsilon_2 \sin\psi = \cos a \cos\varepsilon_1\,;$$

$$\therefore \mathrm{F}(\varepsilon_2) = \mathrm{F}' - \mathrm{F}(\tfrac{1}{2}\pi - b),$$

$$\mathrm{E}(\varepsilon_2) = \mathrm{E}' - \mathrm{E}(\tfrac{1}{2}\pi - b) + \cos a \cos\varepsilon_1\,;$$

$$\therefore \mathrm{E}(\varepsilon_2) - \mathrm{F}(\varepsilon_2) - \cos a \cos\varepsilon_1 = \mathrm{E}' - \mathrm{F}' - \mathrm{E}(\tfrac{1}{2}\pi - b) + \mathrm{F}(\tfrac{1}{2}\pi - b),$$

and equation (183) becomes

$$\begin{aligned}\Sigma' = & F'(\sin c)\,E'(\cos c) - F'(\sin c)\,E(\cos c, \tfrac{1}{2}\pi - b)\\ & - F'(\sin c)\,F'(\cos c) + F'(\sin c)\,F(\cos c, \tfrac{1}{2}\pi - b)\\ & + E'(\sin c)\,F'(\cos c) - E'(\sin c)\,F(\cos c, \tfrac{1}{2}\pi - b) \quad . \quad (184).\end{aligned}$$

But, by art. 42, equation (d'), *ibid.*

$$\tfrac{1}{2}\pi = F'(\sin c)\,E'(\cos c) + E'(\sin c)\,F'(\cos c) - F'(\sin c)\,F'(\cos c),$$

$$\begin{aligned}\therefore \Sigma' = \tfrac{1}{2}\pi + F(\cos c, \tfrac{1}{2}\pi - b)\}F'(\sin c) - E'(\sin c)\}\\ - F'(\sin c)\,E(\cos c, \tfrac{1}{2}\pi - b) \quad . \quad . \quad . \quad . \quad (181),\end{aligned}$$

and four times this will be the whole area.

63. *To find the length of the Spherical Ellipse.* We have by equation (108),

$$\begin{aligned} ds^2 &= dy^2 + \sin^2 y\, dx^2 = \sin^4 y\, dx^2 \left(\frac{d\,.\cot y^2}{dx^4} + \frac{1}{\sin^2 y}\right)\\ &= \frac{\sin^2 b\, dx^2}{(1 - \cos^2\varepsilon_2\cos^2 x)^2}\left\{\frac{\sin^2\varepsilon_1\cos^2\varepsilon_2\sin^2 x\cos^2 x}{1 - \sin^2\varepsilon_1\cos^2 x} + 1 - \cos^2\varepsilon_2\cos^2 x\right\}\\ &= \frac{\sin^2 b\, dx^2}{(1 - \cos^2\varepsilon_2\cos^2 x)^2}\cdot\frac{1 - (\sin^2\varepsilon_1 + \cos^2\varepsilon_1\cos^2\varepsilon_2)\cos^2 x}{1 - \sin^2\varepsilon_1\cos^2 x}\\ &= \frac{\sin^2 b\, d\,.\tan x^2}{(\tan^2 x + \sin^2\varepsilon_2)^2}\cdot\frac{\tan^2 x + \cos^2\varepsilon_1\sin^2\varepsilon_2}{\tan^2 x + \cos^2\varepsilon}\end{aligned}$$

$$\text{and } ds = \frac{\sin b\,.\,d\,.\tan x}{\tan^2 x + \sin^2\varepsilon_2}\cdot\sqrt{\frac{\tan^2 x + \cos^2\varepsilon_1\sin^2\varepsilon_2}{\tan^2 x + \cos^2\varepsilon_1}} \quad . \quad . \quad (186).$$

Now put $\tan x = \cos\varepsilon_1 \sin\varepsilon_2 \tan\varphi$ . . . . (187),

then

$$\begin{aligned} ds &= \frac{\sin b\cos\varepsilon_1\, d\varphi}{\cos^2\varepsilon_1\sin^2\varphi + \cos^2\varphi}\cdot\frac{1}{\sqrt{\sin^2\varepsilon_2\sin^2\varphi + \cos^2\varphi}}.\\ &= \frac{\sin b\cos\varepsilon_1\, d\varphi}{1 - \sin^2\varepsilon_1\sin^2\varphi}\cdot\frac{1}{\sqrt{1 - \cos^2\varepsilon_2\sin^2\varphi}} \quad . \quad . \quad . \quad (188).\end{aligned}$$

$$\therefore s = \sin b\cos\varepsilon_1 .\, \Pi(-\sin^2\varepsilon_1, \cos\varepsilon_2, \varphi) \quad . \quad . \quad (189),$$

$$\text{and } s' = \sin b\cos\varepsilon_1 .\, \Pi'(-\sin^2\varepsilon_1, \cos\varepsilon_2) \quad . \quad . \quad . \quad (190);$$

where s' is the length of a quadrantal arc of the ellipse, φ, x and s commencing together at 0°. In order to express this in functions of the first and second species, we will first transform it into a function whose parameter is $\frac{c^2}{n} = -\frac{\cos^2\varepsilon_2}{\sin^2\varepsilon_1} = -\cos^2 b$, by Art. 46, of Legendre. The a of that Art. $= (1+n)\left(1 + \frac{c^2}{n}\right) = \cos^2\varepsilon_1 \sin^2 b$, and therefore

$$\Pi'(-\sin^2\varepsilon_1) = F' - \Pi'(-\cos^2 b) + \frac{\pi}{2\sin b\cos\varepsilon_1}$$

and (190) becomes

$$s' = \tfrac{1}{2}\pi - \sin b\cos\varepsilon_1\{\Pi'(-\cos^2 b, \cos\varepsilon_2) - F'(\cos\varepsilon_2)\} \quad . \quad (190).$$

By making now $-\cos^2 b = -1 + \sin^2\varepsilon_2 \sin^2\theta$, or $\sin^2\theta = \frac{\sin^2 b}{\sin^2\varepsilon_2} = \sin^2 a$, and $\theta = a$; $\therefore \frac{b^2\sin\theta\cos\theta}{\Delta(b,\theta)} = \frac{\sin^2\varepsilon_2\sin a\cos a}{\cos b} = \sin b\cos\varepsilon_1$, and by equation ($m'$) of Art. 101, *ibid.*, (191) is reduced to

$$s' = \}E'(\cos\varepsilon_2) - F'(\sin c)\}\,.\,F(\sin\varepsilon_2, a) + F'(\cos\varepsilon_2)E(\sin\varepsilon_2, a) (192),$$

and $4s'$ is the whole length of the Spherical Ellipse. Δ.

ERRATUM.—The Solution to Question XIII. is by Prof. Peirce, Cambridge, Mass.

METEOROLOGICAL OBSERVATIONS,

MADE AT THE INSTITUTE, FLUSHING, L. I., FOR THIRTY-SEVEN SUCCESSIVE HOURS, COMMENCING AT SIX A. M., OF THE TWENTY-FIRST OF DECEMBER, EIGHTEEN HUNDRED AND THIRTY-SIX, AND ENDING AT SIX P. M., OF THE FOLLOWING DAY.

(*Lat.* 40° 44′ 58″ N., *Long.* 73° 44′ 20″ W. *Height of Barometer above low water mark of Flushing Bay,* 54 *feet.*)

Hour.	Barometer Corrected.	Attached Therm'ter.	External Therm'ter.	Wet Bulb Therm'ter.	Winds —from—	Clouds —to—	Strength of wind.	REMARKS. Storm began at 4½ A. M. of the 21st, and ended at 12½ P. M.—depth of rain ,34 inches.
6	29.647	55	53	.	S.	N.	Very high.	Dark clouds and rain.
7	29.642	56	53	.	"	"	"	"
8	29.635	57	54	.	"	"	"	"
9	29.583	57	51	.	"	"	"	"
10	29.569	57	52½	52	"	"	"	" more.
11	29.556	51½	51	50½	SSW.	"	"	" less.
12	29.531	57	53½	51	"	"	"	" mist and small rain.
1	29.597	58	46	42½	WNW	E.	"	Dark, driving clouds.
2	29.701	55	38	34¾	NW.	S. E.	"	"
3	29.731	49	36	31½	"	N.E.	High.	Fleecy white clouds.
4	29.795	47	33	29	WNW	E.	"	Thin white clouds.
5	29.875	50	29½	27¼	"	.	"	Clear.
6	29.937	54	27¼	25½	"	.	Very high.	"
7	29.999	52	25¾	24¼	"	.	"	"
8	30.046	51½	23½	22¼	"	.	"	"
9	30.094	51	22	20¾	"	.	"	"
10	30.174	50	21	19¼	NW.	S. E.	"	" one or two white clouds.
11	30.217	50	20	18½	"	.	"	"
12	30.290	50	17½	16½	"	"	"	" "
1	30.319	49	16	14¾	"	.	"	"
2	30.353	49	15	13¾	"	.	"	"
3	30.381	49	14	13	"	.	"	"
4	30.423	51	13½	12½	"	.	Brisk.	"
5	30.434	52	13½	12½	"	.	"	"
6	30.457	51	12¾	12	"	.	"	"
7	30.471	50	12¾	12	"	.	"	"
8	30.537	47	14	12¼	"	.	"	"
9	30 564	38	14½	12½	"	.	"	"
10	30.570	38	17	14½	NNW.	.	"	"
11	30.554	35	18½	15¼	"	.	"	"
12	30.538	35	20	17	"	.	"	"
1	30.539	37	20¾	17¾	"	.	"	"
2	30.539	36	21	18¼	"	.	"	"
3	30.544	35½	20¼	17¾	"	.	Gentle.	"
4	30.549	35	18½	16½	"	.	"	"
5	30.554	37	17¾	16¼	"	.	"	"
6	30.573	39	16½	15¼	"	.	"	"
	30.150	48	27	25¼	Means.			The Barometer is the same as that used in September.

METEOROLOGICAL OBSERVATIONS,

MADE AT THE INSTITUTE, FLUSHING, L. I., FOR THIRTY-SEVEN SUCCESSIVE HOURS, COMMENCING AT SIX A. M., OF THE TWENTY-FIRST OF MARCH, EIGHTEEN HUNDRED AND THIRTY-SEVEN, AND ENDING AT SIX P. M., OF THE FOLLOWING DAY.

(*Lat.* 40° 44′ 58″ N., *Long.* 73° 44′ 20″ W. *Height of Barometer above low water mark of Flushing Bay,* 54 *feet.*)

Hour.	Barometer Corrected.	Attached Therm'ter.	External Therm'ter.	Wet Bulb Therm'ter.	Winds —from—	Clouds —to—	Strength of wind.	REMARKS.
6	30.022	50	33	32	NE	Sd.	Gentle.	Thin clouds overspread.
7	30.027	49	34½	33¾	"	.	"	Misty.
8	30.035	48	36	35	"	.	"	"
9	30.032	51	38¼	37	"	.	"	"
10	30.042	50	39	37½	"	SW	Brisk.	Misty—clouds.
11	30.040	52	41	39	"	"	"	"
12	30.037	51	41½	39	"	"	"	"
1	30.030	44	41½	39	"	"	"	"
2	30.002	50	42	39½	"	"	"	"
3	29.991	50	41¾	39¼	"	"	"	"
4	29.989	50	40	38½	"	"	"	"
5	29.986	50	39	38	"	"	Gentle.	"
6	29.984	49	37½	36½	"	W	"	Dark clouds and rain.
7	29.965	50	38	37½	"	"	"	"
8	29.964	50	38½	38	"	"	Brisk.	"
9	29.961	50	36½	36	"	"	"	Violent rain.
10	29.954	52	36	35¼	"	"	"	"
11	29.951	51	36	35½	"	"	High.	Rain began at 5½ P. M. of the 21st, and ended at 9 P. M. of the 22d; commenced again at 1 A. M. of the 23d, and ended at 1 P. M. of the same day. It rained also from 5 to 6½ P. M. of the 23d, with a brisk wind. The wind prevailing as in the table, fell at about 3 A. M. of the 23d. An accident having happened to the rain-gauge, the depth of rain could not be measured.
12	29.926	50	35½	35	"	"	"	
1	29.839	51	35½	35	"	"	"	
2	29.831	51	36½	36	"	"	"	
3	29.800	53	37	36½	"	"	"	
4	29.794	55	39½	39	"	"	"	
5	29.801	52	42	41	"	"	"	
6	29.801	52	41	40	"	"	"	
7	29.710	52	39	38½	"	"	"	
8	29.680	52	42½	42	"	"	"	
9	29.675	51	43¾	43	"	"	"	
10	29.656	51	44	43½	"	"	Very high.	
11	29.655	51	43½	43	"	"	"	
12	29.652	51	44½	44	E	"	High.	
1	29.650	52	44½	44	"	"	"	
2	29.653	51	42½	42	"	"	"	Rain less heavy.
3	29.655	51	38½	37½	NE	SW	Brisk.	"
4	29.658	50	36	35½	"	"	"	"
5	29.659	50	35½	35	"	"	High.	"
6	29.661	49	35	34½	"	"	Very high.	"
Mean.	29.859	50⅓	39	38⅕				

THE

MATHEMATICAL MISCELLANY.

NUMBER IV.

JUNIOR DEPARTMENT.

ARTICLE VI.

SOLUTIONS TO THE QUESTIONS PROPOSED IN NUMBER III.

(7.) QUESTION I. (*Communicated by Mr. Lenhart.*)

Given $\left\{\begin{array}{r} xy = x^2 - y^2 \\ x^2 + y^2 = x^3 - y^3 \end{array}\right\}$ to determine x and y by a pure quadratic.

FIRST SOLUTION. *By Mr. Geo. K. Birely, Frederick College, Md.*

By multiplying the first equation by $x + y$, and transposing,

$$x^2y + xy^2 = x^3 + x^2y - xy^2 - y^3.$$

or,

$$2xy^2 = x^3 - y^3$$
$$= x^2 + y^2, \text{ from the second equation} \quad . \quad (1).$$

Square the first equation, and add $4x^2y^2$ to both members,

$$5x^2y^2 = x^4 + 2x^2y^2 + y^4,$$

or

$$xy\sqrt{5} = x^2 + y^2$$
$$= 2xy^2, \text{ from (1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Therefore,

$$y = \tfrac{1}{2}\sqrt{5}.$$

Add the first equation to (2), then

$$xy(\sqrt{5} + 1) = 2x^2,$$

or

$$x = \tfrac{1}{2}y(\sqrt{5} + 1) = \tfrac{1}{4}(5 + \sqrt{5}).$$

SECOND SOLUTION. *By Mr. B. Birdsall, Clinton Liberal Institute.*

Multiply the first equation by $x + y$, and we get

$2xy^2 = x^3 - y^3 = x^2 + y^2$, by comparing it with the second.

Subtracting the first equation from this,

$$xy(2y - 1) = 2y^2,$$

or

$$x = \frac{2y}{2y - 1},$$

substitute this value of x in the first equation, we find

$$4y^2 = 5, \text{ or } y = \tfrac{1}{2}\sqrt{5}.$$

Therefore $$x = \frac{\sqrt{5}}{\sqrt{5} - 1} = \tfrac{1}{4}(5 + \sqrt{5}).$$

THIRD SOLUTION. *By a Lady.*

Subtract twice the first equation from the second, and divide by $x - y$,

$$x - y = x^2 + xy + y^2 - 2(x + y) \quad . \quad . \quad . \quad . \quad (1).$$

Add the first to this, and transpose

$$y = 2x^2 - 3x \quad . \quad . \quad . \quad . \quad (2).$$

Substitute this value of y in the first equation, we get

$$4x^2 - 10x + 5 = 0 \quad . \quad . \quad . \quad . \quad (3).$$

Multiply (2) by 2, and add it to (3), member by member, then

$$2y - 10x + 5 = -6x, \text{ or } x = \tfrac{1}{4}(2y + 5) \quad . \quad . \quad . \quad . \quad (4).$$

Writing this value of x in (3) we find

$$4y^2 - 5 = 0, \text{ or } y = \tfrac{1}{2}\sqrt{5} \quad . \quad . \quad . \quad . \quad (5).$$

and, by (4), $$x = \tfrac{1}{4}(\sqrt{5} + 5) \quad . \quad . \quad . \quad . \quad (6).$$

—— Neat solutions were also received from Messrs. Bacot, Biddle, Bowden, Barton, and Ketchum.

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(8). QUESTION II. *By* ——.

Find the angle x, from the equation

$$\frac{1 + a\cos(x + \theta)}{\sin(x + \varphi)} = \frac{1 + a\cos\theta}{\sin\varphi}.$$

FIRST SOLUTION. *By Mr. R. S. Howland, Flushing Institute.*

Clear the equation of fractions, and transpose,

$$a\{\sin\varphi\cos(x + \theta) - \cos\theta\sin(\varphi + x)\} = \sin(\varphi + x) - \sin\varphi \quad . \quad (1).$$

Now $$\sin\varphi\cos(x + \theta) = \tfrac{1}{2}\sin(x + \theta + \varphi) - \tfrac{1}{2}\sin(x + \theta - \varphi),$$

and $$\cos\theta\sin(\varphi + x) = \tfrac{1}{2}\sin(x + \theta + \varphi) + \tfrac{1}{2}\sin(x - \theta + \varphi)$$

$$\begin{aligned} \therefore \sin\varphi\cos(x+\theta) - \cos\theta\sin(x+\varphi) &= -\tfrac{1}{2}\sin(x+\theta-\varphi) - \tfrac{1}{2}\sin(x - \theta + \varphi) \\ &= -\sin x\cos(\theta - \varphi) \\ &= -2\sin\tfrac{1}{2}x\cos\tfrac{1}{2}x\cos(\theta - \varphi) \quad . \quad (2); \end{aligned}$$

$$\begin{aligned} \text{Also } \sin(\varphi + x) - \sin\varphi &= 2\sin\tfrac{1}{2}x\cos(\tfrac{1}{2}x + \varphi) \\ &= 2\sin\tfrac{1}{2}x(\cos\tfrac{1}{2}x\cos\varphi - \sin\tfrac{1}{2}x\sin\varphi) \quad . \quad . \quad (3). \end{aligned}$$

Write (2) and (3) in (1), and divide by $2\sin\frac{1}{2}x\cos\frac{1}{2}x$, it becomes

$$-a\cos(\theta - \varphi) = \cos\varphi - \sin\varphi\tan\tfrac{1}{2}x,$$

or $$\tan\tfrac{1}{2}x = \cot\varphi + a\,.\frac{\cos(\theta - \varphi)}{\sin\varphi} \quad . \quad . \quad . \quad . \quad (4).$$

—— Mr. Birdsall's solution is nearly like this.

If a triangle be constructed such that one of its angles, $A = \pi - \varphi$, and the ratio of the sides which include it, $\dfrac{b}{c} = a\cos(\theta - \varphi)$, then equation (4) shows that the angle $\frac{1}{2}x$ is the complement of the angle C, of that triangle, or $x = \pi - 2C$. Also we may take $\frac{1}{2}x = \frac{3}{2}\pi - C$, or $x = 3\pi - 2C$.

SECOND SOLUTION. *By a Lady.*

Remove the denominators and expand $\cos(x+\theta)$ and $\sin(x+\varphi)$; also put $\cos(\varphi-\theta)$ for its equal $\cos\varphi\cos\theta+\sin\varphi\sin\theta$, then

$$\{a\cos(\varphi-\theta)+\cos\varphi\}\sin x+\sin\varphi\cos x=\sin\varphi,$$

which is the form solved at page 138 of the Mathematical Miscellany.

—— Good solutions were also sent by Messrs. Barton, Biddle, Birely, and Bowden.

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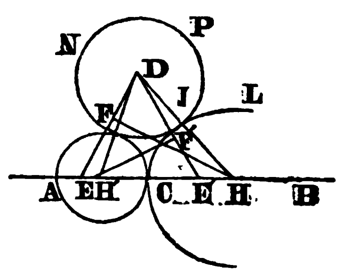
(9). QUESTION III. *By Mr. Geo. K. Birely.*

Through a given point in a right line given in position, it is required to describe a circle, having its centre on the same line, which shall touch a circle, given in position and magnitude.

FIRST SOLUTION. *By Mr. Jacob Blickensderfer, jun., Roscoe, Ohio.*

Let AB be the right line given in position, and C the given point in it; IPN the given circle, centre D. Take, on either side of C, CE or CE′ = DI the given radius; join DE or DE′ and bisect it in F or F′; erect FH or F′H′ perpendicular to DE or DE′ and cutting AB in H or H′, then either of these points is the centre of the required circle, and HC or H′C is its radius. For if we join HD, the triangle EHD is isoceles, having HD = HE, and since EC was made = DI, HC must equal HI; therefore a circle described with H as a centre and radius, HC will be tangent to the circle IPN. A similar demonstration will apply to the other centre H′.



When one of the centres, as H′, falls within the given circle, while the given point is without it, the circle whose centre is H′ will include the given circle.

—— The proposer also gave a very neat geometrcal construction.

SECOND SOLUTION. *By Mr. R. Dewar Bacot, Flushing Institute.*

Let the given line be the axis of x, and a perpendicular to it through the given point the axis of y; let the co-ordinates of the centre of the given circle be y', x', and its radius r; and let the radius of the required circle be R, the co-ordinates of its centre being 0 and $\pm$R. Then for the distance of the points y', x' and 0, $\pm$R we have

$$r+R=\sqrt{y'^2+(x'\pm R)^2},$$

or

$$r^2+2rR=y'^2+x'^2\pm 2x'R,$$

therefore

$$R=\frac{y'^2+x'^2-r^2}{2(r\mp x')}.$$

This equation may be very easily and neatly constructed. When both values of R are positive, the two required circles touch the given one externally; when either value is negative and numerically less than r, the

resulting circle will be within the given one; and when negative and numerically greater than r, the given circle will be within the required one.

—— The solutions by "A Lady," and by Messrs. Barton, Birdsall, and Bowden, were also well worthy of insertion.

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(10). QUESTION IV. (*From the Dublin Problems.*)

Express the sides and area of a plane triangle, as functions of the radius of the inscribed circle and the three angles.

FIRST SOLUTION. *By Mr. J. I. Bowden, Flushing Institute.*

Let A, B, C be the angles of the triangle, and a, b, c the sides respectively opposite them. Let O be the centre of the inscribed circle, and r its radius. Now the lines AO, BO respectively bisect the angles A and B, and therefore the two segments into which the perpendicular, r, from O, divides the side c, are $r\cot\frac{1}{2}A$ and $r\cot\frac{1}{2}B$; therefore

$$c = r(\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

and similarly for the other sides. But

$$\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B = \frac{\cos\frac{1}{2}A\sin\frac{1}{2}B + \cos\frac{1}{2}B\sin\frac{1}{2}A}{\sin\frac{1}{2}A\sin\frac{1}{2}B}$$

$$= \frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}A\sin\frac{1}{2}B} = \frac{\cos\frac{1}{2}C}{\sin\frac{1}{2}A\sin\frac{1}{2}B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

which is better adapted for logarithmic computation; then

$$c = r(\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B) = \frac{r\cos\frac{1}{2}C}{\sin\frac{1}{2}A\sin\frac{1}{2}B},$$

$$b = r(\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}C) = \frac{r\cos\frac{1}{2}B}{\sin\frac{1}{2}A\sin\frac{1}{2}C},$$

$$a = r(\cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C) = \frac{r\cos\frac{1}{2}A}{\sin\frac{1}{2}B\sin\frac{1}{2}C}.$$

Also, the area $= \frac{1}{2}r(a+b+c) = r^2(\cot\frac{1}{2}A + \cot\frac{1}{2}B + \cot\frac{1}{2}C)$

$$= \tfrac{1}{2}ab\sin C = \tfrac{1}{2}.\frac{r^2\cos\frac{1}{2}A\cos\frac{1}{2}B\sin C}{\sin\frac{1}{2}A\sin\frac{1}{2}B\sin^2\frac{1}{2}C}$$

$$= r^2\cot\tfrac{1}{2}A\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C.$$

Cor. If A, B, C be the three angles of a triangle,

$$\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C = \cot\tfrac{1}{2}A\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C.$$

SECOND SOLUTION. *By Mr. P. Barton, jun., South Orange, Mass.*

If S be the area of the triangle, r the radius of the inscribed circle, a, b, c the three sides, and A, B, C their opposite angles, we have from well known properties of the triangle

$$2S = r(a+b+c) = ab\sin C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Eliminating successively b and c, a and c, a and b from (1) by means of (2), we shall find,

$$a = r.\frac{\sin A + \sin B + \sin C}{\sin B\sin C},\ b = r.\frac{\sin A + \sin B + \sin C}{\sin A\sin C},\ c = r.\frac{\sin A + \sin B + \sin C}{\sin A\sin B},$$

which are all included in the formula,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = r.\frac{\sin A + \sin B + \sin C}{\sin A \sin B \sin C} \quad . \quad . \quad . \quad . \quad (3),$$

$$\therefore 2s = r^2.\frac{(\sin A + \sin B + \sin C)^2}{\sin A \sin B \sin C} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

—— From these two solutions we can easily obtain the property

$$\sin A + \sin B + \sin C = 4\cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C.$$

This question was also answered by "A Lady," and by Messrs. Birely, Biddle, and Birdsall.

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(11). QUESTION. *By Mr. J. Ketchum, Principal of the Acad., Gaines, Orleans Co., N. Y.*

Required the sides of a trapezoid in which the oblique sides are equal, the sum of the parallel sides is 10, two-thirds of the difference of the parallel sides is equal to their perpendicular distance, and the distance of the centre of gravity from the longer of the parallel sides is equal to $1\frac{3}{5}$.

FIRST SOLUTION. *By Mr. B. Birdsall.*

If b and b' be the parallel sides of a trapezoid whose altitude is a, then the distance of the centre of gravity from the longer parallel side is

$$\frac{ab}{b-b'} - \frac{2a}{3(b-b')}.\frac{b^3-b'^3}{b^2-b'^2} = \frac{a}{b-b'}.\left\{b - \tfrac{2}{3}.\frac{b^2+bb'+b'^2}{b+b'}\right\} = 1\tfrac{3}{5}.$$

Or, since $a = \frac{2}{3}(b-b')$, this becomes, after multiplying it by $\frac{9}{2}$,

$$3b - 2.\frac{b^2+bb'+b'^2}{b+b'} = \frac{b^2+bb'-2b'^2}{b+b'} = \frac{36}{5}.$$

But $b+b'=10$, or $b'=10-b$, and substituting this,

$$b^2 + b(10-b) - 2(10-b)^2 = 72,$$

or,

$$b^2 - 25b + 136 = 0;$$

therefore $b = 8$ or 17, of which the first value is true when the sum, and the second when the difference of the parallel sides $= 10$. Hence $b = 8$, $b' = 2$, $a = 4$.

—— Mr. Birdsall also favoured us with another good solution.

SECOND SOLUTION. *By Mr. Geo. K. Birely.*

Let $x =$ the perpendicular distance, $y =$ greater and $z =$ less of the parallel sides, and $w =$ one of the oblique sides. Then the distance of the centre of gravity from the less side will be

$$\frac{x}{3}.\frac{2y+z}{y+z} = x - \tfrac{8}{5} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Also $\quad y + z = 10$, and $y - z = \frac{3}{2}x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$

therefore $\quad y = 5 + \frac{3}{4}x$, and $z = 5 - \frac{3}{4}x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3);$

these substituted in (1) give

$$x^2 - 20x = -64 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4);$$

therefore $x = 16$ or 4, and taking $x = 4$, we get $y = 8$, $z = 2$, and $w = 5$.

—— The proposer's solution was unfortunately mislaid. "A Lady," and Messrs. Barton and Blickensderfer also answered it.

(12). QUESTION VI. *By* ——.

The semi-axes of two ellipses are 2,1 and 5,3. It is required to place them with their transverse axes on the same straight line, so that they may intersect each other at right angles.

FIRST SOLUTION. *By a Lady.*

Let x, y be the co-ordinates of the points of intersection, the origin being at the centre of the larger ellipse, d the distance between the centres of the ellipses. Then from the nature of the ellipse we have

$$25y^2 + 9x^2 = 225 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1),$$
$$4y^2 + (d - x)^2 = 4 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2).$$

The subtangent of the larger ellipse, for the point of intersection is

$$\frac{25 - x^2}{x},$$

and the subnormal of the smaller ellipse, at the same point is

$$\tfrac{1}{4}(d - x).$$

Now it is manifest that we shall satisfy the conditions of the problem by equating these expressions, that is by making

$$\frac{25 - x^2}{x} = \frac{d - x}{4}, \text{ or } d - x = \frac{100 - 4x^2}{x} \quad . \; . \; . \; . \; . \quad (3).$$

Multiply (1) by 4 and (2) by 25, and subtract; there results

$$36x^2 - 4(d - x)^2 = 800, \text{ or } 36x^2 - 4.\frac{(100 - 4x^2)^2}{x^2} = 800;$$
$$\therefore \quad 164x^4 - 19200x^2 = -250000,$$

from which we find $x = 4,8387$ nearly, and this substituted in (3) gives $d = 6,14928$, which determines the position of the ellipses.

SECOND SOLUTION. *By Mr. T. B. Biddle, Flushing Institute.*

Let the distance of their centres be z; then the equations of the two ellipses, the origin being at the centre of the larger one, are

$$25y^2 + 9x^2 = 225 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1),$$
$$4y^2 + (z - x)^2 = 4 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2).$$

The equations of the tangents of these two ellipses at a common point y, x, are

$$y' - y = -\frac{9x}{25y}(x' - x),$$
$$y' - y = \frac{z - x}{4y}(x' - x);$$

and in order that these tangents may be perpendicular to each other, we must have

$$1 - \frac{9x(z - x)}{100y^2} = 0,$$

or, $$100y^2 - 9x(z - x) = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3).$$

Multiply (1) by 4, subtract (3) from it, and divide by 9, then

$$4x^2 + x(z - x) = 100 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4).$$

Multiply (2) by 25, and subtract (3)

$$25(z - x)^2 + 9x(z - x) = 100 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (5).$$

Taking (5) from (4), $4x^2 - 8x(z-x) - 25(z-x)^2 = 0$,
and adding $29(z-x)^2$ to both members, and taking the root,

$$2x - 2(z-x) = \pm (z-x)\sqrt{29},$$

$$\therefore \frac{x}{z} = \frac{\sqrt{29} \pm 2}{\sqrt{29} \pm 4} \quad \ldots \ldots \ldots \ldots \quad (6).$$

Hence from (4) and 5,

$$100 = 3x^2 + xz = x\left(3 + \frac{z}{x}\right) = x^2 . \frac{4\sqrt{29} \pm 10}{\sqrt{29} \pm 2} \quad \ldots \quad (7),$$

$$25z^2 - 100 = 41xz - 16x^2 = x^2(41 . \frac{z}{x} - 16) = x^2 . \frac{25\sqrt{29} \pm 132}{\sqrt{29} \pm 2} \quad \ldots \quad (8).$$

Divide (8) by (7), then

$$\tfrac{1}{4}z^2 - 1 = \frac{25\sqrt{29} \pm 132}{\sqrt{29} \pm 10} = \frac{790 \pm 139\sqrt{29}}{182},$$

$$\therefore z^2 = \frac{1944 \pm 278\sqrt{29}}{91},$$

and $\qquad z = \pm 6,1493105$ or $\pm 2,2161351$.

The last two roots would not apply to the question, since the smaller ellipse would be wholly within the larger one.

—— Solutions were also received from Messrs. Bacot, Barton, and Birdsall.

ARTICLE VII.

QUESTIONS TO BE ANSWERED IN NUMBER V.

Their Solutions must arrive before February 1st, 1838.

(13). Question I. *By* ——.

To find x, y, z, there are given the three equations

$$ax + by + cz = p,$$
$$bx + cy + az = q,$$
$$cx + ay + bz = r.$$

(14). Question II. *By* θ.

Let $x =$ logarithm of N to any base,
$y =$ logarithm of N′ to the same base;
prove that $N'^x = N^y$.

(15). Question III. *By* ——.

Divide $a^4 + b^4 - 2a^2b^2 \cos 2\varphi$ by $a^2 + b^2 - 2ab \cos \varphi$.

(16). Question IV. *By Mr. Lenhart.*

Theorem. If from any point in either side of a right angled plane triangle, a straight line be drawn perpendicular to the hypothenuse; then

shall the rectangle of the segments of the hypothenuse be equal to the rectangle of the segments of the sides containing the point, together with the square of the perpendicular thus drawn.

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(17). QUESTION V. *By* ——.

Given $v = \sin^n x \{(n+2)\sin nx - n\sin(n+2)x\}$; to find $\frac{dv}{dx}$.

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(18). QUESTION VI. *By a Lady.*

Three ladies purchase a ball of exceedingly fine thread, for which they pay equally. Allowing the radius of the ball to be three inches, and the quality of the thread in each layer to vary as its distance from the centre, how much will she diminish the radius who winds off the first portion?

ARTICLE VIII.

HINTS TO YOUNG STUDENTS. (*Continued from page* 135.)

17. In commencing the study of Algebra, great care should be taken to obtain a correct and precise idea of the symbols used in the science, and the operations performed upon them. To assist you in mastering these first principles, I will translate for your use part of a Note on this subject in the "*Analyse Algébrique,*" of M. Augustus-Louis Cauchy, one of the greatest mathematicians of the age.

18. ON THE THEORY OF POSITIVE AND NEGATIVE QUANTITIES.

In the same way that the idea of *number* arises from the measure of magnitudes, we acquire the idea of *quantity* (positive or negative) when we look upon any magnitude of a given kind as used for the increase or diminution of another fixed magnitude of the same kind. In order to indicate this object, we represent a magnitude to be used as an increment by a number preceded by the sign +, and a magnitude to be used as decrement by a number preceded by the sign —. In this way the sign + or — placed before a number modifies its signification, in nearly the same manner as an adjective modifies that of its substantive. We rank the numbers which are preceded by the sign + under the name of *positive quantities*, and the numbers preceded by the sign — under the name of *negative quantities*. Lastly, it is agreed to rank absolute numbers which are preceded by no sign in the class of positive quantities; and it is for this reason that we sometimes dispense with writing the sign + before numbers which represent quantities of this kind.

In Arithmetic we always operate on numbers whose particular value is known, and which are consequently given in figures; while in Algebra, where we have to consider the general properties of numbers, we

generally represent them by letters. A quantity is thus expressed by a letter with the sign $+$ or $-$ prefixed to it.

In the case where the letter A represents a *number*, we may, from what has been before said, designate the positive quantity of which the numerical value is A, either by $+$ A, or by A only, while $-$ A designates the *opposite* quantity, that is, the negative quantity of which the numerical value is A. So also, in the case where the letter a represents a *quantity*, we regard the two expressions a and $+a$ as synonymous, and we designate by $-a$ the opposite quantity.

From this system of notation, if we represent by A either a number, or any quantity whatever, and if we make

$$a = +\mathrm{A},\ b = -\mathrm{A};$$

we shall have

$$+a = +\mathrm{A},\ +b = -\mathrm{A},$$
$$-a = -\mathrm{A},\ -b = +\mathrm{A}.$$

If in the four last equations, instead of a and b, we place within parenthesis their values contained in the two first, we obtain the formulas

$$(1)\left\{\begin{array}{l} +(+\mathrm{A}) = +\mathrm{A},\ +(-\mathrm{A}) = -\mathrm{A}, \\ -(+\mathrm{A}) = -\mathrm{A},\ -(-\mathrm{A}) = +\mathrm{A}. \end{array}\right.$$

In each of these formulas, the sign of the second member is what we call the *product* of the two signs of the first member. The *multiplication* of one sign by another, produces the product of these signs. The inspection alone of the equations (1) will be sufficient to establish *the rule of the signs*, comprised in the following theorem:

THEOREM 1. *The product of two similar signs is always* $+$, *and the product of two opposite signs is always* $-$.

It follows also from the same equations that the product of two signs, when one of the two is $+$, remains equal to the other. If then there are many signs to be multiplied together, we may suppress all the signs $+$. From this remark we can easily deduce the following propositions.

THEOREM 2. *If we multiply many signs together in any order whatever, the product will always be* $+$, *when the number of signs* $-$ *is even, and the product will be* $-$, *when the number of signs* $-$ *is odd.*

THEOREM 3. *The product of any number of signs will be the same, in whatever order they are multiplied.*

An immediate consequence of the preceding definitions, is that the multiplication of signs has no relation to the multiplication of numbers. But we ought not to be surprised that the notion of the product of two signs presents itself as the first step in analysis, since in the addition or subtraction of a monomial, we actually multiply the sign of this monomial by $+$ or by $-$.

The principles thus established will easily enable us to surmount all the difficulties that the use of the signs $+$ and $-$ can present in the operations of algebra and trigonometry. It is only necessary to distinguish carefully the operations relative to numbers from those which refer to quantities, positive or negative. We ought especially to fix in a precise manner the object of both, to define the results produced by them, and to show their principle properties. We shall now endeavour to do this, in few words, for the different operations that are ordinarily used.

19. Addition and Subtraction.

Sums and Differences of Numbers. The adding of a number A to a number B, or in other words, subjecting the number A to an increment + B, is what we call *arithmetical addition.* The result of this operation is called *sum.* It is indicated by placing after the number A its increment + B, thus:

$$A + B.$$

We cannot demonstrate, but we may admit as self-evident, that *the sum of several numbers is the same in whatever order they are added.* This is a fundamental axiom, on which arithmetic, algebra, and all the sciences of calculation rest.

Arithmetical subtraction is the inverse of an addition. It consists in taking from a first number A, or second number B, that is, in seeking a third number C which, added to the second, reproduces the first. We might also define it as the subjecting of a number A to the decrement — B. The result of this operation is called *difference.* It is indicated by placing after the number A its decrement — B, thus:

$$A - B.$$

The difference A — B is sometimes called the *excess*, or the *remainder*, or the *arithmetical ratio* between the two numbers A and B.

Sums and differences of Quantities. Two quantities being given, there can always be found a third quantity, which, taken as the increment of a fixed number, if it be positive, and for its decrement in the contrary case, will produce the same result as if the two given quantities had been employed, the one after the other, in the same manner. This third quantity which produces by itself the same effect as the two others, is called their *sum.*

Thus the two quantities — 10 and + 7 have for their sum — 3, because the diminution of a fixed number by 10 units, and an augmentation of the last result by 7 units, is simply equivalent to a diminution of the fixed number by 3 units. The *addition* of several quantities to one another forms their sum. It is easy to demonstrate, by the help of the axiom relative to the addition of numbers, the following proposition:

Theorem 4. *The sum of many quantities remains the same, in whatever order they are added.*

We indicate the sum of many quantities by the simple juxtaposition of the letters which represent either their numerical values or the quantities themselves, prefixing to each letter the sign proper to express the corresponding quantity. The several letters may always be disposed in any order we please; and we are allowed to suppress the sign + before the first letter. Let us consider, for example, the quantities

$$a, b, c, \ldots\ldots -f, -g, -h, \ldots\ldots$$

Their sum may be represented by the expression

$$a - f - g + b - h + c + \&c. \ldots\ldots$$

In such an expression, each one of the quantities

$$a, b, c \ldots\ldots -f, -g, -h, \ldots\ldots$$

is called a *monomial.* The expression itself is a *polynomial*, of which the above monomials are the different *terms.*

It is easy to prove that two polynomials, of which all the terms are equal and with contrary signs, represent two opposite quantities.

The *difference* between a first quantity and a second, is a third quantity, which added to the second, reproduces the first. From this definition we can demonstrate that, *in order to subtract from a first quantity* a *a second quantity* b, *it is sufficient to add to the first the quantity opposite to* b, *that is*, — b. We conclude that the difference of the two quantities a and b should be represented by

$$a - b.$$

Subtraction, being the inverse of addition, may always be represented in two ways. Thus, for example, to express that the quantity c is the difference of the two quantities a and b, we may write indifferently

$$a - b = c, \text{ or } a = b + c.$$

Lastly, we say that a quantity is *greater* or *less* than another, accordingly as the difference of the first and the second is positive or negative. According to this definition, positive quantities always surpass negative quantities, and these last should be considered as becoming less when their numerical values become greater.

20. Multiplication and Division.

Products and Quotients of Numbers. To *multiply* the number A by the number B, is to operate on the number A precisely as we operate on the unity to obtain the number B. The result of this operation is called the *product* of A by B. In order the better to comprehend the preceding definition of multiplication, it is necessary to distinguish the different cases, according to the species of the number B. Now this number may be either rational, that is, a whole number or a fraction, or irrational.

When B is a whole number, it is sufficient, in order to obtain B, to add unity many successive times to itself. It is necessary then, in order to form the product of A by B, to add the number A to itself a like number of times, that is, to find the sum of as many numbers equal to A as there are units in B.

When B is a fraction which has m for its numerator and n for its denominator, the operation by which we arrive at the number B, consists in dividing unity into n equal parts, and repeating the result thus formed m times. We obtain then the product of A by B, by dividing the number A into n equal parts, and repeating one of the parts m times.

When B is an irrational number, we can obtain in rational numbers, values approaching nearer and nearer to it. We easily see that on the same hypothesis the product of A by such rational numbers will necessarily approach nearer and nearer towards a certain limit. This limit will be the product of A by B. If we suppose, for example, B = 0, we shall find the limit zero, and we conclude that the product of any number by zero vanishes.

In the multiplication of A by B, the number A is called the *multiplicand*, and the number B the *multiplier*. The two numbers together are also designated by the name of *factors* of the product.

To indicate the product of A by B, we employ indifferently one of the three following notations:

$$B \times A, \quad B \cdot A, \quad BA.$$

The product of many numbers is the same in whatever order they are multiplied. This proposition, when applied to only two or three whole factors, is deduced from the axiom relative to the addition of numbers. We can then demonstrate it successively, 1°. for two or three rational factors: 2°. for two or three irrational factors: 3°. for any number of factors, rational or irrational.

To *divide* the number A by the number B, is to seek a third number of which the product by B is equal to A. The operation by which we find this is called *division*, and the result of the operation the *quotient*. Moreover, the number A takes the name of the *dividend*, and the number B that of the divisor.

To indicate the quotient of A by B, we employ at pleasure either of the two following notations:

$$\frac{A}{B}, \quad A : B.$$

Sometimes we designate the quotient $A : B$ by the name of the *geometrical ratio* or *relation* of the two numbers A and B.

The equality of two geometrical ratios $A : B$, $C : D$, or, in other words, the equation

$$A : B = C : D$$

is called a *geometrical proportion*. In general, instead of the sign $=$ we employ the sign :: which has the same value, and we write

$$A : B :: C : D.$$

Note. When B is a whole number, to divide A by B is, by the definition, to seek a number which repeated B times reproduce A. But this is to divide the number A into as many equal parts as there are units in B. We easily conclude from this remark that, if m and n designate two whole numbers, the n^{th} part of unity should be represented by

$$\frac{1}{n},$$

and the fraction which has m for its numerator, and n for its denominator, by

$$m \times \frac{1}{n}.$$

Such is, in fact, the notation by which we ought naturally to designate this fraction. But, as we can easily prove that the product $m \times \frac{1}{n}$ is equivalent to the quotient of m by n, that is, to $\frac{m}{n}$, it follows that this fraction can be represented more simply by the notation

$$\frac{m}{n}.$$

Products and quotients of quantities. The product of a first quantity bya second, is a third quantity which has for its numerical value the

product of the numerical values of the two others, and for its sign the product of their signs. The *multiplying* of two quantities by each other, forms their product. One of the two quantities is called the *multiplier*, the other the *multiplicand*, and both of them together, the factors of the product.

These definitions being admitted, we shall easily establish the following proportion.

THEOREM 5. *The product of many quantities is the same, in whatever order they are multiplied.*

To demonstrate this proposition, it is sufficient to combine the like proposition relative to numbers with the 3ᵈ theorem relative to signs.

To *divide* a first quantity by a second, is to seek a third quantity which multiplied by the second reproduces the first. The operation by which we arrive at it is called *division;* the first quantity, the *dividend*, the second, the *divisor*, and the result of the operation, the *quotient*. Sometimes we designate the quotient by the name of the *geometrical ratio* or *relative* of the two given quantities. From the preceding definition, we easily prove that *the quotient of two quantities has for its numerical value the quotient of their numerical values, and for its sign the product of their signs.*

The multiplication and division of quantities is indicated in the same way as the multiplication and division of numbers. We say that two quantities are *reciprocals* of one another, when the product of these two quantities is unity. From this definition, the quantity a has for its reciprocal $\frac{1}{a}$, and conversely.

We have before remarked that what we call a fraction in arithmetic is equal to the ratio or quotient of two whole numbers. In algebra, we designate by the name of *fraction* the ratio or quotient of any two quantities. If then a and b represent two quantities, their ratio $\frac{a}{b}$ will be an algebraic fraction.

We observe also that division, being the inverse operation of multiplication, may always be indicated in two ways. Thus, for example, to express that the quantity c is the quotient of the two quantities a and b, we may write indifferently

$$\frac{a}{b} = c, \text{ or } a = bc.$$

The products and quotients of numbers and quantities possess many general properties to which we frequently have recourse. We have already spoken of the one relative to the order of multiplication of several quantities. Other properties, not less remarkable, will be found comprised in the following formulas:

Let $a, b, c, \ldots k$; $a', b', c', \ldots$; $a'', b'', \ldots$; &c. . . .
be several series of quantities positive or negative. We shall have, for all the possible values of these quantities,

$$(2).\quad \begin{cases} k(a+b+c+\ldots.) = ka+kb+kc+\ldots\ldots, \\ \dfrac{a+b+c+\ldots.}{k} = \dfrac{a}{k}+\dfrac{b}{k}+\dfrac{c}{k}+\ldots\ldots, \\ \dfrac{a}{b}\times\dfrac{a'}{b'}\times\dfrac{a''}{b''}\times\ldots = \dfrac{a\,a'\,a''\ldots\ldots}{b\,b'\,b''\ldots\ldots}, \\ \dfrac{k}{\left(\dfrac{a}{b}\right)} = \dfrac{bk}{a} = \dfrac{b}{a}\times k. \end{cases}$$

The four preceding formulas give rise to a multitude of consequences which it would be out of place to enumerate here in detail. We may conclude, for example, from the third formula, 1°. that the fractions

$$\frac{a}{b}, \frac{ka}{kb},$$

are equal to each other, a, b, k designating any quantities whatever; 2°. that the fraction $\frac{a}{b}$ has for its reciprocal $\frac{b}{a}$; 3°. that, to divide a quantity k by another quantity a, it is sufficient to multiply k by the reciprocal of a, that is by $\frac{1}{a}$.

SENIOR DEPARTMENT.

ARTICLE XIV.

SOLUTIONS TO THE QUESTIONS PROPOSED IN ARTICLE VIII.

(36.) QUESTION I. *By Querist.*

It has been said that "in the ellipse all its circumscribing parallelograms are equal." Is this true?

FIRST SOLUTION. *By Prof. B. Peirce, Cambridge, Mass.*

Let $2A$ = transverse axis of the ellipse,
$2B$ = conjugate axis.

There is a plane upon which the projection of the ellipse is a circle whose radius is B; and the projection of the parallelogram circumscribing the ellipse, the area of which we will denote by V, is a parallelogram circumscribing the circle, the area of which we will denote by V'. Then, we have

$$V : V' = A : B.$$

But if x is either diagonal made by either diagonal of V′, with the radius drawn to either point of contact of either side of V′ with the circle, we have

$$V' = 2B^2(\tan x + \cot x) = \frac{4B^2}{\sin 2x}.$$

Whence $$V = \frac{4AB}{\sin 2x},$$

the least value of which is $V = 4AB$,
and the greatest value is $V = \infty$;
so that the parallelograms vary from $4AB$ to ∞.

Corollary. As $4AB$ = any parallelogram whose sides are parallel to two conjugate diameters, it follows that all such parallelograms are the least which can be circumscribed about the ellipse.

SECOND SOLUTION. *By Mr. P. Barton, jun., South Orange, Mass.*

Let a and b be the semiaxes of the ellipse,
a' and b' any pair of conjugate semidiameters,
β the angle they make with each other.

Then if tangents to the ellipse be drawn at the extremities of a', they will be parallel to b' and to each other, and the perpendicular distance between them will be $2a' \sin \beta$. Moreover, if a second pair of parallel tangents be drawn, the co-ordinates of their points of contact with the ellipse, referred to a' and b' as axes of co-ordinates, being y', x' and $-y'$, $-x'$, their equations will be

$$a'^2y'y + b'^2x'x = a'^2b'^2,$$
$$a'^2y'y + b'^2x'x = -a'^2b'^2,$$

and they will form with the former a parallelogram. Making $x = a'$ in these two equations we find

$$y = \frac{b'^2}{a'y'}(a' - x'),\ y = -\frac{b'^2}{a'y'}(a' + x'),$$

and the difference of these is $\frac{2b'^2}{y'}$, the intercept of one of the first tangents by the two last, or the side of the parallelogram. Hence, if s is its area, we have

$$s = \frac{4a'b'^2 \sin \beta}{y'} \quad \ldots\ldots\ldots\ldots \quad (1).$$

But, $a'b' \sin \beta = ab$, since a', b' are conjugates,

$$\therefore s = 4a \cdot \frac{b'}{y'} \quad \ldots\ldots\ldots\ldots\ldots \quad (2).$$

The factor $\frac{b'}{y'}$ varies for the same system, from 1 when $y' = b'$ its greatest value, to ∞ when $y' = 0$; therefore the circumscribed parallelograms may be of any magnitude from $4ab$, in which case the sides are parallel to any system of conjugate diameters, to infinity.

THIRD SOLUTION. *By Dr. T. Strong, Rutgers' College, New-Brunswick, N. J.*

We shall generalize this question, and for simplicity, shall consider the ellipse as the orthographic projection of a circle.

Lemma. Of all the rectilineal figures, of n sides, which circumscribe a given circle, that which is regular has the least area.

Put $P = 3{,}14159\ldots$, $2T =$ the perimeter of any irregular figure of n sides circumscribing the circle whose radius is r, $A =$ its area, then

$$A = rT \quad \ldots\ldots\ldots\ldots \quad (1);$$

also let A' denote the area of a regular figure of n side, whose perimeter is $2T$, then $\frac{2T}{n} =$ the length of one of its sides, and $\frac{T}{n}\cot\frac{P}{n} =$ the radius of the inscribed circle, therefore

$$A' = \frac{T^2}{n}\cot\frac{P}{n} \quad \ldots\ldots\ldots\ldots \quad (2).$$

But since the two figures have equal perimeters, and the same number of sides, by Le Gendre's Geometry, Art. 309, $A' > A$, or

$$\frac{T^2}{n}\cot\frac{P}{n} > rT, \text{ or } T > nr\tan\frac{P}{n}, \text{ or } rT > nr^2\tan\frac{P}{n} \quad \ldots \quad (3);$$

but if $A'' =$ the area of the regular figure of n sides which circumscribes the circle, radius r, we have $A'' = nr^2\tan\frac{P}{n}$, $\therefore$ by (1) and (3), $A > A''$ as was to be proved.

Cor. Let A''' denote the area of a regular figure of m sides, circumscribing the circle, then

$$A''' = mr^2\tan\frac{P}{m} \quad \ldots\ldots\ldots\ldots \quad (4),$$

and suppose that $n > m$, then we shall have $A''' > A''$, for imagine tangents to be drawn to the circle so as to cut off triangles towards the angular points from the figure A''', &c., until the remaining figure, whose area we shall denote by A'''', has n sides, then we shall have $A''' > A''''$, but if A'''' is regular, then $A'''' = A''$, but if it is not regular, $A'''' > A''$, by what has been shown; $\therefore$ in both cases $A''' > A''$; hence, when $n > m$ and n and m each > 2, we have

$$m\tan\frac{P}{m} > n\tan\frac{P}{n}, \text{ or } \frac{m}{n} > \tan\frac{P}{n}\cot\frac{P}{m} \quad \ldots\ldots \quad (5).$$

Hence the equilateral triangle is less than any other triangle described about the same circle, the square is less than any other quadrilateral described about the same circle, the regular pentagon is less than any other pentagon described about the same circle, and so on; also, the square is less than the equilateral triangle, the regular pentagon is less than the square, and so on; supposing them to be described about the same circle. It may further be observed, that each side of any regular figure described about a circle is bisected at the point of contact.

We will now suppose that A and B denote the semiaxes of an ellipse. Put $\frac{B}{A} = \cos\varphi$, and imagine a plane to pass through the transverse axis,

as to make the angle φ with the plane of the ellipse; then suppose a circle to be described with the radius A in the inclined plane, having its centre at the centre of the ellipse. Let then A' denote the area of any figure of n sides which circumscribes the circle thus described, then if we project the circle and the figure, orthographically, on the plane of the ellipse, the circle will be projected into the ellipse, and the figure will be projected into one of n sides, each of which will touch the ellipse, and it will therefore circumscribe the ellipse; also, if A' is a regular figure, as each of its sides is bisected at the point of contact with the circle, each side of the projected figure will be bisected at its point of contact with the projected ellipse, as is evident from the well known principles of the orthographic projection. Let $A_{,}$ denote the projection of A', and we shall have $A_{,} = A' \cos \varphi$; but all regular figures of the same number, n, of sides, described about the circle are equal to each other, therefore their projections are equal to each other, and their sides are each bisected at their points of contact with the ellipse; *hence all figures of the same number of sides described about an ellipse, such that their sides are each bisected at their points of contact are equal to each other.*

Again, since when A' is regular it is less than when it is irregular, the number of sides remaining the same; also since when A' is regular, it is less than any figure of m sides described about the circle which we shall denote by A'', supposing $n > m$; then denoting the projection of A'' by $A_{,,}$ we shall have $A_{,,} = A'' \cos \varphi$. *Hence any figure of* n *sides described about an ellipse so that each of its sides is bisected at the point of contact is less than the area of any other figure of* m *sides described about the same ellipse when* n $>$ m.

If n is an even number, then the opposite sides of A', supposing it to be a regular figure, will be parallel, therefore the opposite sides of the projected figure will be parallel; hence if $n = 4$, A' is a square, and its projection will be a parallelogram, each of whose sides is bisected at its point of contact with the ellipse, and the right lines joining the opposite points of contact will be conjugate diameters in the ellipse, and all such parallelograms are equal to each other, and each less than any other four-sided figure described about the ellipse, whether it is a parallelogram or not. Hence the proposition stated in the question should be, that *all the parallelograms found by drawing tangents at the four vertices of any two conjugate diameters of an ellipse are equal to each other.*

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(37). QUESTION II. *By* ——.

Show that if the bases of a number of different systems of logarithms are in geometrical progression, the logarithms of any given number, taken in these different systems successively, will be in harmonical progression.

FIRST SOLUTION. *Mr. B. Birdsall, Clinton Liberal Institute.*

Let a, an, an^2, an^3, &c., be the bases of the different systems of logarithms, let r be the number whose logarithm is taken, and c, c', c'', c''', &c., its logarithms in the several systems; then we shall have

$$a^c = r,\ (an)^{c'} = r,\ (an^2)^{c''} = r,\ (an^3)^{c'''} = r,\ \&c.$$

If we take the logarithms of these numbers in *any* system, we have

$$c\log a=\log r,\text{ or } c=\frac{\log r}{\log a},\text{ and }\frac{1}{c}=\frac{\log a}{\log r},$$

$$c'(\log a+\log n)=\log r,\text{ or } c'=\frac{\log r}{\log a+\log n},\text{ and }\frac{1}{c'}=\frac{\log a}{\log r}+\frac{\log n}{\log r},$$

$$c''(\log a+2\log n)=\log r,\text{ or } c''=\frac{\log r}{\log a+2\log n},\text{ and }\frac{1}{c''}=\frac{\log a}{\log r}+2.\frac{\log n}{\log r},$$

$$c'''(\log a+3\log n)=\log r,\text{ or } c'''=\frac{\log r}{\log a+3\log n},\text{ and }\frac{1}{c'''}=\frac{\log a}{\log r}+3.\frac{\log n}{\log r},$$

&c. &c.

The numbers $\frac{1}{c}, \frac{1}{c'}, \frac{1}{c''}, \frac{1}{c'''}$, &c., are in arithmetical progression, and therefore their reciprocals c, c', c'', c''', &c., or the several logarithms of the number r, are in harmonical progression.

Second Solution. *By Mr. O. Root, Hamilton College, Clinton, N. Y.*

Let a, ar, ar^2, ar^3, &c., be the bases of the different systems; n a given number, whose logarithms in the different systems are x, y, z, w, v, &c., then we have

$$n=a^x=a^y r^y=a^z r^{2z}=a^w r^{3w}=a^v r^{4v}=\text{\&c.}$$

$$\therefore\ a^{x-y}=r^y,\ a^{x-z}=r^{2z},\ a^{x-w}=r^{3w},\ a^{x-v}=r^{4v},\text{ \&c.}$$

$$\therefore\ r=a^{\frac{x-y}{y}}=a^{\frac{x-z}{2z}}=a^{\frac{x-w}{3w}}=a^{\frac{x-v}{4v}},=\text{\&c.}$$

and
$$\frac{x-y}{y}=\frac{x-z}{2z}=\frac{x-w}{3w}=\frac{x-v}{4v}=\text{\&c.},$$

therefore
$$y=\frac{2xz}{x+z},\ z=\frac{2yw}{y+w},\ w=\frac{2vz}{v+z},\text{ \&c.},$$

and x, y, z, w, v, &c., are in harmonical progression.

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(38). Question III. (*From the Cambridge Problems.*)

In what time will a given principal double itself at a given rate of compound interest, when the interest is added every instant?

First Solution. *By Professor M. Catlin, Hamilton College.*

Let r = the annual rate of interest, dt = the differential of the time; then will rdt = the increment of one dollar in the time dt; consequently, if x = variable principal, we shall have

$$rxdt=dx,\text{ or } rdt=\frac{dx}{x}\ \ldots\ldots\ldots \quad (1).$$

Integrating (1) from $x=1$, to $x=2$, we find

$$rt=\text{h. log. }2,\text{ and } t=\frac{\text{h. log.}2}{r}.$$

Cor. 1. If r = h. log. 2, t will equal one year.

Cor. 2. In the same manner we find $t=\frac{\text{h. log. }n}{r}$ for the time in which a given sum, a, will amount to na.

—— Prof. Peirce, whose solution is on the same principle, adds the following

Corollary. Let t' = the time in which a amounts to an at common compound interest, and we have

$$(1+r)^{t'} = n,$$

$$t' \times \text{h. log. } (1+r) = \text{h. log. } n = rt.$$

Whence

$$\begin{aligned} t' : t &= r : \text{h. log. } (1+r) \\ &= 1 : 1 - \tfrac{1}{2}r + \tfrac{1}{3}r^2 - \&c. \\ &= 1 : 1 - \tfrac{1}{2}r \text{ nearly, when } r \text{ is small and when} \end{aligned}$$

$r = ,06$,

$$t' : t = 1 : ,97 \text{ nearly.}$$

SECOND SOLUTION. *By Mr. O. Root.*

Let t = the required time, r = the annual rate, n = the number of intervals in a year, at the end of each of which the principal is augmented by the interest, and a = the amount of an unit at the end of t years.

Then $$\left(1+\frac{r}{n}\right)^{nt} = a \quad \dots\dots\dots\dots \quad (1).$$

Differentiate both members of (1) and we shall have

$$\frac{t\,dr}{1+\dfrac{r}{n}} = \frac{da}{a};$$

but when n is infinite $\frac{r}{n} = 0$, therefore we have

$$t\,dr = \frac{da}{a},$$

hence, by integration,

$$t = \frac{\text{h. log. } a}{r}.$$

In the present question $a = 2$, therefore

$$t = \frac{\text{h. log. } 2}{r}.$$

—— Professor Avery takes the logarithm of equation (1) of Mr. Root's solution, and finds

$$\begin{aligned} \text{h. log. } a &= nt \times \text{h. log.} \left(1 + \frac{r}{n}\right) \\ &= nt\left(\frac{r}{n} - \tfrac{1}{2}.\frac{r^2}{n^2} + \tfrac{1}{3}.\frac{r^3}{n^3} - \&c.\right) \\ &= rt\left(1 - \tfrac{1}{2}.\frac{r}{n} + \tfrac{1}{3}.\frac{r^2}{n^2} - \&c.\right) \\ &= rt, \text{ when } n \text{ is infinite.} \end{aligned}$$

$$\therefore t = \frac{\text{h. log. } a}{r}.$$

THIRD SOLUTION. *By Prof. F. N. Benedict, University of Vermont, Burlington.*

Representing the principal, rate per cent per annum, time and amount, by p, r, t, mp, we have

$$p(1+rdt)^n = mp,$$

the element dt, being contained n times in t. Developing this equation, we have

$$p(1+nr\,dt+n.\frac{n-1}{2}.r^2\,dt^2+\&c.)=mp,$$

or dividing by p and supplying ndt by its equal, t, we have

$$1+rt+\left(1-\frac{1}{n}\right).\frac{r^2t^2}{2}+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right).\frac{r^3t^3}{2.3}+\&c.=m,$$

and regarding n as infinite, it become

$$1+rt+\frac{r^2t^2}{2}+\frac{r^3t^3}{2.3}+\&c.=m,$$

or

$$e^{rt}=m,$$

where e represents the number whose hyp. log. is 1.

Therefore $$t=\frac{\log.\ m}{r\log.\ e}.$$

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(39). QUESTION IV. *By Dr. Strong.*

Prove that lines, drawn through the points of trisection of a given line, and the points of trisection of the semicircumference of a circle described upon it as a diameter, pass through the vertex of an equilateral triangle described on the opposite side of the given line.

FIRST SOLUTION. *By Mr. P. Barton, jun.*

Let the given line, $2r$, be the axis of x and a perpendicular through its middle point the axis of y. Then we shall have for the points of trisection of $2r$,

$$1.\ y=0,\ x=\tfrac{1}{3}r;\ 2.\ y=0,\ x=-\tfrac{1}{3}r;$$

and for the points of trisection of the semicircumference on $2r$,

$$3.\ y=r\sin\tfrac{1}{3}\pi=\tfrac{1}{2}r\sqrt{3},\ x=r\cos\tfrac{1}{3}\pi=\tfrac{1}{2}r;$$

$$4.\ y=r\sin\tfrac{2}{3}\pi=\tfrac{1}{2}r\sqrt{3},\ x=r\cos\tfrac{2}{3}\pi=-\tfrac{1}{2}r.$$

The equation of the right line through 1 and 3 is

$$y=\frac{\frac{1}{2}\sqrt{3}}{\frac{1}{2}-\frac{1}{3}}(x-\tfrac{1}{3}r)=3\sqrt{3}(x-\tfrac{1}{3}r \quad \cdots\cdots\cdots \quad (1),$$

and that of the right line through 2 and 4 is

$$y=\frac{\frac{1}{2}\sqrt{3}}{-\frac{1}{2}+\frac{1}{3}}(x+\tfrac{1}{3}r)=-3\sqrt{3}(x+\tfrac{1}{3}r) \quad \cdots \quad (2).$$

Hence, for the point, yx, of intersection of these lines, we have, by adding (1) and (2)

$$y=-r\sqrt{3},$$

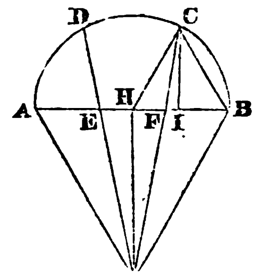
and by subtracting (1) and (2)

$$x=0,$$

which is evidently the vertex of an equilateral triangle described on the opposite side of $2r$.

SECOND SOLUTION. *By Mr. N. Vernon, Frederick, Md.*

Let AB be the given line trisected in E and F, ADCB the semicircle trisected in D and C. Join DE, CF which produce to meet in G; join AG, BG; AGB is an equilateral triangle.



Bisect AB in H, join GH, CH, and draw CI perpendicular to AB. The triangle EGF is evidently isosceles, having its base bisected in H, GH is therefore perpendicular to AB. And as CB is one-third of the semicircumference, CHB is an equilateral triangle; HF = $\frac{1}{3}$HB, and HI = $\frac{1}{2}$HB, therefore HF = 2FI.

The triangles HGF, FCI, are equiangular, each having a right angle, and CFI, HFG, vertical angles, and therefore since HF = 2FI, GH = 2CI. Also the triangles HIC, HBG are equiangular, since the sides HI, IC are the halves of HB, HG respectively, and their included angles are right angles; therefore HBG = IHC = one-third of two right angles. In the same manner it may be shown that HAG = one-third of two right angles. ABG is therefore an equilateral triangle.

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(40). QUESTION V. *By Mr. N. Vernon.*

Given the radius, to determine the arc, when the lune formed by the arc, and the semicircle described upon its chord, is the greatest possible.

FIRST SOLUTION. *By Mr. James F. Macully, New-York.*

Let 1 = radius of the circle,
$2x$ = the angle subtended by the required arc.
Then $2\sin x$ = the chord of the arc or diameter of the semicircle,
x = area of the sector comprehended between the arc and the two radii at its extremities,
$\sin x \cos x$ = the triangle formed by the chord and two radii,
$\frac{1}{2}\pi \sin^2 x$ = the area of the semicircle.
Therefore the area of the lune $= \frac{1}{2}\pi \sin^2 x - x + \sin x \cos x =$ a *max.*
Therefore $\pi \sin x \cos x - 1 + \cos^2 x - \sin^2 x = 0$,
or $\pi \sin x \cos x - 2\sin^2 x = 0$.
Therefore, first $\sin x = 0$, or $x = \pi$, and $2x = 2\pi$,
or, second $\tan x = \frac{1}{2}\pi$, or $x = 57^\circ\, 31'\, 6''$ and $2x = 115^\circ\, 2'\, 12''$.

The first root applies when the semicircle is less than the segment, and then the semicircle vanishes at the limit, and the lune is equal to the given circle; this is the absolute maximum.

SECOND SOLUTION. *By Mr. B. Birdsall.*

Since the relation between the given and required parts in this question will be the same whatever be the magnitude of the circle, we will suppose its radius to be unity. Let x be half the required arc. If the

extremities of the chord which subtends the arc $2x$ be joined with the centre of the given circle, by two radii they will form a triangle with it whose area will be $\sin x \cos x$; and the area of the sector between the same radii will be x; therefore the segment on which the lune rests $= x - \sin x \cos x$. Also the semicircle described on the chord $= \frac{1}{2}\pi \sin^2 x$.

Hence the lune $\quad = \frac{1}{2}\pi \sin^2 x + \sin x \cos x = u$,

therefore $$\frac{du}{dx} = \pi \sin x \cos x - 2\sin^2 x = 0,$$

or $\pi \cos x = 2\sin x$, $\therefore \tan x = \frac{1}{2}\pi$, and $x = 57^\circ\ 31'$ and $2x = 115^\circ\ 2'$.

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(41). QUESTION VI. *By Professor Catlin.*

Required the greatest rectangle that can be inscribed in a given circular ring.

FIRST SOLUTION. *By Mr, Lyman Abbot, jun., Niles, Cayuga Co., N. Y.*

Let $r =$ radius of the exterior circle, $r' =$ that of the interior one, and $2\varphi =$ angle subtended by the arc whose chord is one side of the rectangle, then the rectangle will be expressed by

$$2r \sin \varphi(r \cos \varphi \pm r').$$

where the ambiguous sign is $-$ when the rectangle is included between the circumferences, and $+$ when the rectangle envelopes the inner circle. This differentiated and equated to zero gives

$$2r^2 (\cos^2 \varphi - \sin^2 \varphi) \pm 2rr' \cos \varphi = 0,$$

or $$\frac{\cos 2\varphi}{\cos \varphi} = \mp \frac{r'}{r},$$

or $$\cos \varphi = \frac{\mp r' \mp \sqrt{r'^2 + 8r^2}}{4r}.$$

Were it not for want of room we might show that upper signs and lower signs all obtain together.

SECOND SOLUTION. *By Mr. J. Ketchum, Prin. of Gaines Acad., Gaines, Orleans Co., N. Y.*

Let $r =$ radius of the larger circle, $a =$ that of the smaller one, $2x =$ that side of the rectangle which is a chord of the outer circle, and $v =$ versed sine of the arc whose chord is $2x$. Then by the circle $r - v = \sqrt{r^2 - x^2}$, and the second side of the rectangle $= r - a - v = \sqrt{r^2 - x^2} - a$; therefore its area $= 2x(r - a - v) = 2x\sqrt{r^2 - x^2} - 2ax$, and differentiating and multiplying by $\dfrac{\sqrt{r^2 - x^2}}{2dx}$, we have

$$r^2 - 2x^2 - a\sqrt{r^2 - x^2} = 0,$$

$$\therefore\ 2(r - v)^2 - r^2 - a(r - v) = 0,$$

and $$r - v = \tfrac{1}{4}a \pm \tfrac{1}{4}\sqrt{a^2 + 8r^2},$$

and $r - a - v = -\frac{3}{4}a \pm \frac{1}{4}\sqrt{a^2 + 8r^2} =$ the width of the rectangle.

—— Professor Peirce and Dr. Strong make two of the angular points in one circumference, and two in the other, and perhaps this is the more appropriate way of considering it as *inscribed* within the ring. Dr. S.

shows that radii drawn at right angles to each other, will intersect the two circumferences in the angle of the greatest rectangle that can be so inscribed.

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(42). QUESTION VII. *By Mr. O. Root.*

Required the locus of all the points, so situated within a right angle, that the shortest line which can be made to pass through each of them and terminate in the sides of the right angle, shall be of a constant length.

FIRST SOLUTION. *By Mr. J. F. Macully.*

Let the sides of the right angles be axes of co-ordinates, xy the co-ordinates of a point in the required locus, θ the angle the given line, a, makes with the axes of x; then will

$$\frac{y}{\sin\theta}+\frac{x}{\cos\theta}=a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Differentiating equation (1), making θ only vary,

$$\frac{-y\cos\theta d\theta}{\sin^2\theta}+\frac{x\sin\theta\,.\,d\theta}{\cos^2\theta}=da=0,$$

or

$$\tan^3\theta=\frac{y}{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

$$\therefore \cos\theta=x^{\frac{1}{3}}(y^{\frac{2}{3}}+x^{\frac{2}{3}})^{-\frac{1}{2}},\ \sin\theta=y^{\frac{1}{3}}(y^{\frac{2}{3}}+x^{\frac{2}{3}})^{-\frac{1}{2}} \quad . \quad . \quad (3),$$

and these substituted in equation (1) gives

$$(y^{\frac{2}{3}}+x^{\frac{2}{3}})^{\frac{2}{2}\cdot\frac{3}{}}=a,$$

or

$$y^{\frac{2}{3}}+x^{\frac{2}{3}}=a^{\frac{2}{3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

the equation of the locus. It is a line of the sixth order, and resembles in form the evolute of the ellipse.

The length of the curve is $\int_0^a\sqrt{dy^2+dx^2}=a^{\frac{1}{3}}\int_0^a y^{-\frac{1}{3}}\,dxy=\frac{3}{2}a$, for each of the four equal branches; the length of the whole curve is therefore $=6a$.

If we make $x=a\sin^3\varphi$, then $y=a\cos^3\varphi$, and the area of the curve $=\int ydx=\int_0^{\frac{1}{2}\pi}3a^2\cos^4\varphi d\varphi=\frac{3}{32}a^2\int_0^{\frac{1}{2}\pi}d\varphi\,(2+\cos 2\varphi-2\cos 4\varphi-\cos 6\varphi)=\frac{3}{32}a^2\pi$ for each branch, therefore the surface bounded by the whole curve $=\frac{3}{8}a^2\pi$, or $\frac{3}{2}$ of the circle described on the given line as a diameter.

The radius of the osculating circle will be found to be $=3\sqrt[3]{axy}$ which is an elegant property of the curve, and it might easily be shown that the given line is tangent to the locus, at the point, yx, in all its positions.

—— Professor Avery's solution is very like this.

SECOND SOLUTION. *By Mr. Root, the proposer.*

The origin being taken at the given right angle,

$$y-y'=a(x-x') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

will be the equation of a right line passing through the point $y'x'$; from (1), when $x=0$, $-y=ax'-y'$, and when $y=0$, $x=\dfrac{ax'-y'}{a}$; hence,

$$(ax'-y')^2+\frac{(ax'-y')^2}{a^2}=\left(1+\frac{1}{a^2}\right)(ax'-y')^2 \quad . \quad . \quad (2),$$

will be the square of the intercepted line, which, by the question, must be a maximum; therefore differentiating and equating with zero, considering x', y' constant and a variable, we find

$$a=-\left(\frac{y'}{x'}\right)^{\frac{1}{3}};$$

this value of a substituted in (2) gives

$$y^{\frac{2}{3}}+x^{\frac{2}{3}}=c^{\frac{2}{3}},$$

the locus required, c being constant.

THIRD SOLUTION. *By Mr. Lyman Abbot, jun.*

Let x and y be the co-ordinates of the point, a = constant length of the line, z = part of it below xy, and $a-z$ = the part above; then if θ = angle included between y and z, and if θ be supposed to receive an increment while the point y, x remains fixed, *i. d.* if the line be supposed to revolve about the point y, x, we have when the line $= a =$ a maximum,

$$da=dz+d(a-z)=0.$$

But it is easily seen that,

$$dz=z\tan\theta d\theta, \text{ and } d(a-z)=-(a-z)\cot\theta d\theta;$$

substituting these values of dz and $d(a-z)$ we get

$$z\tan\theta=(a-z)\cot\theta, \text{ or } z\sin^2\theta=(a-z)\cos^2\theta;$$

therefore, $\qquad z=a\cos^2\theta$, and $a-z=a\sin^2\theta$.

But $\qquad y=z\cos\theta=a\cos^3\theta$, and $y^{\frac{2}{3}}=a^{\frac{2}{3}}\cos^2\theta$,

also $\qquad x=(a-z)\sin\theta=a\sin^3\theta$, and $x^{\frac{2}{3}}=a^{\frac{2}{3}}\sin^2\theta$;

$$\therefore y^{\frac{2}{3}}+x^{\frac{2}{3}}=a^{\frac{2}{3}}(\cos^2\theta+\sin^2)=a^{\frac{2}{3}},$$

for the equation of the locus.

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(43). QUESTION VIII. *By* ——.

Having given two series of polygonal members, of the m^{th} and n^{th} orders respectively; to find those terms, when there are such, which are common to both series. Or, to solve, when it is possible, the indeterminate equation.

$$(m-2)x^2-(m-4)x=(n-2)y^2-(n-4)y,$$

m, n, x, and y being positive integers, of which m and n are given.

SOLUTION. *By Mr. C. Gill, Institute at Flushing, L. I.*

We shall solve the more general equation

$$ax^2-a'x=by^2-b'y=\text{T}$$

or $$x(ax-a')=y(by-b')=\text{T} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where a, a', b, b' are given whole numbers, having no common divisor.

Assume $$ax - a' = \frac{p}{q}. y \quad \ldots \ldots \ldots \quad (2),$$

$$x = \frac{q}{p}. (by - b') \quad \ldots \ldots \ldots \quad (3),$$

which fulfil equation (1), and we shall find from (2) and (3),

$$x = \frac{-q(b'p + a'bq)}{p^2 - ab\,q^2},\ y = \frac{-q\,(a'p + b'aq)}{p^2 - ab\,q^2} \quad \ldots \quad (4),$$

and $$\text{T} = \frac{pq(a'p + ab'q)\,(b'p + a'bq)}{(p^2 - ab\,q^2)^2} \quad \ldots \ldots \quad (5).$$

If we take

$$p^2 - ab\,q^2 = 1 \quad \ldots \ldots \ldots \quad (6),$$

which is always possible in whole numbers, except when ab is a square number, the value of x and y in (4), taken with opposite signs, will solve, in positive integers, the equation

$$ax^2 + a'x = by^2 + b'y \quad \ldots \ldots \ldots \quad (7).$$

But to solve equation (1), it is necessary that

$$p^2 - ab\,q^2 = -1, \text{ or } -\text{N} \quad \ldots \ldots \ldots \quad (8),$$

N being same number, by which the numbers

$$b'p + ba'\,q \text{ and } a'p + ab'q$$

will both divide. If p' and q' be one particular solution of equation (8), fulfilling these conditions; that is, if

$$p'^2 - ab\,q'^2 = -\text{N} \quad \ldots \ldots \ldots \quad (9),$$

and $$\frac{a'p' + ab'\,q'}{\text{N}} = \text{A},\ \frac{b'p' + ba'q'}{\text{N}} = \text{B} \quad \ldots \ldots \quad (10),$$

A and B being integers; then we may take

$$p = p't + ab\,q'u, \quad q = q't + p'u \quad \ldots \ldots \quad (11),$$

so that $p^2 - ab\,q^2 = (p'^2 - ab\,q'^2)\,(t^2 - ab\,u^2) = -\text{N}(t^2 - ab\,u^2)$; then will

$$x = \frac{(q't + p'u)(\text{B}t + \text{A}bu)}{t^2 - abu^2},\ y = \frac{(q't + p'u)\,(\text{A}t + \text{B}au)}{t^2 - abu^2} \quad \ldots \quad (12),$$

$$\text{T} = \frac{(q't + p'u)\,(p't + abq'u)\,(\text{A}t + \text{B}au)\,(\text{B}t + \text{A}bu)}{t^2 - abu^2} \quad \ldots \quad (13).$$

The method of finding the least numbers t_1 and u_1 that make

$$t_1 - ab\,u_1{}^2 = 1, \quad \ldots \ldots \ldots \quad (14),$$

is well known;—we can then form the recurring series

$$\left.\begin{matrix} t_0,\ t_1,\ t_2,\ t_3 \ldots \ldots t_i \\ u_0,\ u_1,\ u_2,\ u_3 \ldots \ldots u_i \end{matrix}\right\} \quad \ldots \ldots \quad (15),$$

where, $t_0 = 1$, $u_0 = 0$, and

$$t_i = 2t_1 t_{i-1} - t_{i-2},\ u_i = 2t_1\,u_{i-1} - u_{i-2} \quad \ldots \ldots \quad (16);$$

then will $$t_i{}^2 - ab\,u_i{}^2 = 1;$$

and taking $t = t_i$ and $u = u_i$, we shall have from (13), a series of numbers which are of both the two forms $ax^2 - a'x$ and $by^2 - b'y$.

In particular cases, the most commodious values of p', q', N, which solve equation (9), will present themselves while finding a series of continued fractions approximating to the square root of ab; since, as is well known, the numbers N, which are less than $\sqrt{ab}$, are all found in the denominators of the complete quotients that stand in the odd places. But

in the numerous class of cases, in which

$$a - a' = b - b' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17),$$

even if there is no number N, less than $\sqrt{ab}$, fulfilling the conditions in (10), we can always take

$$p' = \sqrt{(a - a')(b - b')} = a - a' = b - b', \text{ and } q' = 1, \quad . \quad . \quad (18)$$

then will

$$N = ab' + ba' - a'b', \; A = 1, \; B = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),$$

and (12) and (13) become

$$x = \frac{(t + \overline{a - a'}.u)(t + bu)}{t^2 - abu^2}, \; y = \frac{(t + \overline{a - a'}.u)(t + au)}{t^2 - abu^2} \quad (20),$$

$$T = \frac{(t + \overline{a - a'}.u)(\overline{a - a'}.t + abu)(t + au)(t + bu)}{(t^2 - abu^2)^2} \quad (21).$$

Where we have only to take $t = t_i$ and $u = u_i$ these numbers being determined as in (16), and we shall have the general term of a series of positive whole numbers of the forms $ax^2 - a'x$ and $by^2 - b'y$; if we call this general term T_i, we shall have

$$\begin{aligned} T_i &= (t_i + \overline{a - a'}u_i)(\overline{a - a'}\,t_i + abu_i)(t_i + au_i)(t_i + bu_i) \\ &= \{(a - a')t_{2i} + \tfrac{1}{2}(\overline{a - a'}^2 + ab)u_{2i}\}\{t_{2i} + \tfrac{1}{2}(a + b)u_{2i}\} \\ &= \tfrac{1}{2}(a - a')(t_{4i} + 1) + \frac{(a + b)(c + a'b')}{16\,ab}.(t_{4i} - 1) + \tfrac{1}{4}cu_{4i} \quad (22), \end{aligned}$$

where $c = (2a - a')(2b - b')$. The first term of this series is $T_0 = a - a'$, and when the second is found from (22) the series can be continued by the property, easily deduced from (16) and (22),

$$T_{i+2} = 2t_4 T_{i+1} - T_i + \left\{\frac{(a + b)(c + a'b')}{8\,ab} - (a - a')\right\}(t_4 - 1) \quad (23),$$

so that four terms only of the series in (15) need be calculated.

The only exceptionable case is where

$$a = hk^2, \; b = hk'^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24),$$

that is when ab is a complete square, in which case equation (14) cannot be resolved, and from what has been frequently done on this class of equations, we know that the equation

$$hk^2x^2 - a'x = hk'^2y^2 - b'y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

is impossible in integers, except in a very few particular cases, one of which may be stated thus. If

$$\frac{a'^2}{k^2} = \frac{b'^2}{k'^2}, \text{ or } \frac{a'}{k} = \pm\frac{b'}{k'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (26),$$

we may multiply (25) by $4h$, and add (26), then extracting the root

$$2hkx - \frac{a'}{k} = 2hk'y - \frac{b'}{k'};$$

hat is, either

$$x = \frac{k'}{k}.y, \text{ or } = \frac{k'}{k}y + \frac{a'}{hk^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27),$$

according as the upper or under sign in (26) has place. The last of these is only possible when $\frac{a'}{hk}$ is a whole number.

In order to apply what has been said to the finding of a series of numbers $P_1, P_2, P_3 \ldots . P$, which are polygonal numbers of both the n^{th} and

m^{th} orders, we may remark that the co-efficients $n-2, n-4, m-2, m-4$, either have no common divisor, or they all divide by 2; for since $n-2$ and $n-4$ differ by 2, they can have no other common divisor. In the first case, when m and n are both odd numbers, we may take

$$a = m-2,\ a' = m-4;\ b = n-2,\ b' = n-4,$$

and since these co-efficients fulfil the condition (17), having

$$a - a' = b - b' = 2,$$

the formulas (22) and (23) fulfil the required conditions, that is we shall have $P_i = \frac{1}{2}T_i$; hence

$$P_0 = 1,\ P_1 = \tfrac{1}{2}(t_4+1) + d(t_4-1) + \tfrac{1}{8}mn\,u_4,$$

$$P_i = 2t_4 P_{i-1} - P_{i-2} + (2d-1)(t_4-1) \quad . \quad . \quad . \quad . \quad . \quad (28),$$

d being $= \dfrac{(m+n-4)(mn-2m-2n+8)}{16(m-2)(n-2)}$, and t_4, u_4 found by (16).

Secondly, when m and n are both even, we may take

$$a = \tfrac{1}{2}m-1,\ a' = \tfrac{1}{2}m-2,\ b = \tfrac{1}{2}n-1,\ b' = \tfrac{1}{2}n-2;$$

then

$$a - a' = b - b' = 1,$$

and we shall have $P_i = T_i$, in (22) and (23), therefore

$$P_0 = 1,\ P_1 = \tfrac{1}{2}(t_4+1) + d(t_4-1) + \tfrac{1}{16}mn\,u_4,$$

$$P_i = 2t_4 P_{i-1} - P_{i-2} + (2d-1)(t_4-1) \quad . \quad . \quad . \quad . \quad . \quad (29),$$

d being the same as in (28), and t_4, u_4 found from (16).

Example 1. Let $m = 3$, $n = 5$, or $a = 1$, $b = 3$; equation (14) becomes $t_1{}^2 - 3u_1{}^2 = 1$, therefore $t_1 = 2$, $u_1 = 1$, and the series in (15) are

1, 2, 7, 26, 97, 362, 1351, 5042, 18817, &c.

0, 1, 4, 15, 56, 209, 780, 2911, 10864, &c.

Then from (28) we find $P_0 = 1$, $P_1 = \frac{1}{2}(t_4+1) + \frac{7}{12}(t_4-1) + \frac{15}{8}u_4 = 210$, and

$$P_{i+2} = 2t_4 P_{i+1} - P_i + \tfrac{1}{6}(t_4-1) = 194P_{i+1} - P_i + 16,$$

whence the series of numbers which are both triangular and pentagonal, is

1, 210, 40755, 7906276, 1533776805, &c.,

the first, third, fifth, &c., terms being also hexagonal numbers. Barlow's assertion that "No triangular number, except unity, can be equal to a pentagonal number," is therefore untrue.

Example 2. Let $m = 4$, $n = 8$; then $a = \frac{1}{2}m - 1 = 1$, $b = \frac{1}{2}n - 1 = 3$; therefore t_i and u_i will be the same series as in the last example. Then from (29) we find, $P_0 = 1$, $P_1 = \frac{1}{2}(t_4+1) + \frac{3}{2}(t_4-1) + 2u_4 = 225$,

$$P_{i+2} = 194P_{i+1} - P_i + 32,$$

and the series of square numbers which are also octagonal numbers, is

1, 225, 43681, 8473921, 1643897025, &c.

These formulas show that every two series of polygonal numbers contain terms which are common to both, except those pairs of series, of the orders m and n, in which $(m-2)(n-2)$ is a complete square. The only case of this kind that falls under the condition in (26) is when $m = 3$, $n = 6$, or $a = 1$, $a' = -1$, $b = 4$, $b' = 2$; therefore from (24), $h = 1$, $k = 1$, $k' = 2$, and $\dfrac{a'}{k} = -\dfrac{b'}{k'} = -1$, then the second of (27) gives

$$x = 2y - 1,$$

that is, all hexagonal numbers are also triangular numbers.

The doubtful cases of the question may now be included in the formula

$$2P = hk^2x^2 - (hk^2 - 2)x = hk'^2y^2 - (hk'^2 - 2)y \quad . \quad . \quad (30),$$

where $m-2=hk^2$ and $n-2=hk'^2$; and this is dependent on the equation

$$k'^2p^2-k^2q^2=(k^2-k'^2)(h^2k^2k'^2-4)\ . \ . \ . \ . \ (31),$$

where p and q must be such that

$$x=\tfrac{1}{2}+\frac{p-2}{2hk^2} \text{ and } y=\tfrac{1}{2}+\frac{q-2}{2hk'^2}\ . \ . \ . \ . \ . \ (32),$$

may be positive integers. Now all the solutions of (31) may be derived from the two equations

$$k'p-kq=d_1, \text{ and } k'p+kq=d_2\ . \ . \ . \ . \ . \ . \ (33),$$

where $$d_1d_2=(k^2-k'^2)(h^2k^2k'^2-4)\ . \ . \ . \ . \ . \ . \ (34),$$

$$\therefore p=\frac{d_2+d_1}{2k'},\ q=\frac{d_2-d_1}{2k}\ . \ . \ . \ . \ . \ (35),$$

which must be whole numbers; but the only general *literal* values of d_1 and d_2 that answer these conditions are $d_1=(k-k')(hkk'-2)$, $d_2=(k-k')(hkk'+2)$, which give $p=hk^2+2$, and $q=hk'^2+2$, therefore $x=1$, $y=1$; or the number is 1, which is a polygonal number of all orders. Hence, if there are other values than $x=1$, $y=1$ which solve equation (30) they can only be found from particular values of h, k, k', by taking for d_1 all the separate divisors of $(k^2-k'^2)(h^2k^2k'^2-4)$.

In this way the solutions of numerical cases may always be detected if they exist. But there are classes of possible cases, for a knowledge of the existence of which I am indebted to the very elegant solution of Professor Peirce, and which may be found thus:

Let $d_1=hkk'+2$, then will $d_2=(k^2-k'^2)(hkk'-2)$, and from (35),

$$p=\tfrac{1}{2}hk(k^2-k'^2+1)+k'-\frac{k^2-1}{k'},\ q=\tfrac{1}{2}hk'(k^2-k'^2-1)-k+\frac{k'^2-1}{k}\ (36).$$

If in these equations $k'=1$, then

$$p=\tfrac{1}{2}hk^3-k^2+2,\ q=\tfrac{1}{2}h(k^2-2)-k\ . \ . \ . \ . \ . \ (37),$$

and by (32), $$x=\tfrac{1}{2}-\frac{1}{2h}+\frac{k}{4},\ y=\frac{k^2}{4}-\frac{k+2}{2h}\ . \ . \ . \ . \ . \ . \ . \ (38).$$

These will be integral, 1°. when $h=1$, and $k=4r$, for then

$$x=r,\ y=4r^2-2r-1,\ \text{P}=8r^4-8r^3+r\ . \ . \ . \ . \ (39),$$

which is a polygonal number of the orders 3 and $16r^2+2$. 2°. when $h=2$, and $k=4r-1$, for then

$$x=r,\ y=4r^2-3r,\ \text{P}=16r^4-24r^3+9r^2\ . \ . \ . \ . \ (40),$$

which is a square and a polygonal number of the order $32r^2-16r+4$.

Again, let $d_1=2(k+k')$, and $d_2=\tfrac{1}{2}(k-k')(h^2k^2k'^2-4)$, then will

$$p=\tfrac{1}{4}h^2k^2k'(k-k')+2,\ q=\tfrac{1}{4}h^2kk'^2(k-k')-2,\ . \ . \ . \ (41);$$

$$\therefore x=\tfrac{1}{8}hk(k'-k')+\tfrac{1}{2},\ y=\tfrac{1}{8}hk(k-k')-\frac{2}{hk'^2}+\tfrac{1}{2}\ . \ . \ (42).$$

which are integral, 1°. when $h=4$, $k'=1$, $k=2r$, which gives P as in (39), therefore that number is also hexagonal.

2°. When $h=2$, $k'=2$, $k=2r+1$,

$$x=r,\ y=r^2,\ \text{P}=4r^4-3r^2\ . \ . \ . \ . \ . \ . \ (43),$$

which is a polygonal number of the orders 10 and $8r^2+8r+4$.

3°. When $h=1$, $k'=1$, $k=8r-3$, then

$$x=r,\ y=8r^2-7r,\ \text{P}=32r^4-56r^3+\tfrac{1}{2}(57r^2-7r)\ . \ (44),$$

a polygonal number of the orders 3 and $(8r-3)^2+2$.

—— The solutions of Professors Catlin, Peirce and Strong are very

general and complete, and should have been inserted had room permitted. We trust they will continue their investigations on this interesting class of problems. They and our readers generally, will perhaps be glad to learn that the celebrated theorem of Fermat, that "Every integral number is either a polygonal number of the order m, or the sum of 2, 3, 4, ... or m, such numbers," has been at last demonstrated by M. Cauchy. We propose in some future number either to give a translation of his demonstration, or an original investigation of the subject by some one of our own contributors.

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(44). QUESTION IX. *By* ——.

Having given a series of polygonal numbers, of the n^{th} order; to find two terms in that series, when there are such, whose sum and difference shall be equal to two other terms in the same series. Or, to solve, when it is possible, the two indeterminate equations

$$(n-2)x^2-(n-4)x+(n-2)y^2-(n-4)n=(n-2)z^2-(n-4)z,$$
$$(n-2)x^2-(n-4)x-(n-2)y^2+(n-4)y=(n-2)v^2-(n-4)v.$$

n, x, y, z, and v, being positive integers, of which n is given.

SOLUTION. *By Mr. C. Gill.*

Let the equation be

$$mx^2 - m'x + my^2 - m'y = mz^2 - m'z \quad . \; . \; . \; . \; . \quad (1),$$
$$mx^2 - m'x - my^2 + m'y = mv^2 - m'v \quad . \; . \; . \; . \; . \quad (2),$$

where m and m' are any given integers prime to each other.

I. Equation (1) may be put in the form

$$x(mx - m') = (z - y)\{m(z + y) - m'\} \; . \; . \; . \; . \; . \; . \quad (3),$$

and therefore we may take

$$z - y = \frac{b}{a}(mx - m'), \text{ and } m(z + y) - m' = \frac{a}{b}.\, x \quad . \; . \; . \quad (4);$$

or,
$$mb\,x - a(z - y) = m'b \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5),$$
$$mb\,(z + y) - ax = m'b \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6).$$

We can always find two numbers r and s, such that

$$mb\,r - as = 1 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7),$$

provided mb and a are prime to each other, then (5) and (6) give

$$x = a\theta + m'br, \; z - y = mb\theta + m'bs \; . \; . \; . \; . \; . \; . \; . \quad (8),$$

and
$$z + y = a\varphi + m'br, \qquad x = mb\varphi + m'bs \; . \; . \; . \; . \; . \; . \; . \quad (9),$$

θ and φ being any whole numbers; but since

$$x = a\theta + m'br = mb\varphi + m'bs,$$

or
$$mb\varphi - a\theta = m'b(r - s);$$

we must have $\varphi = aw + m'br\,(r - s)$, $\theta = mbw + m'bs\,(r - s)$, w being an integer; and these substituted in (8) and (9) give

$$x = mabw + \alpha,\; y = \tfrac{1}{2}(a^2 - m^2b^2)w + \beta,\; z = \tfrac{1}{2}(a^2 + m^2b^2)w + \gamma \quad (10);$$

where
$$\left.\begin{aligned} \alpha &= m'b(mbr^2 - as^2) \\ \beta &= \tfrac{1}{2}m'b\,(a + mb)\,(r - s)^2 \\ \gamma &= \tfrac{1}{2}m'b\,\{(a - mb)(r - s)^2 + 2mbr^2 - 2as^2\} \\ &= \alpha + \frac{a - mb}{a + mb}.\,\beta \end{aligned}\right\} \; . \quad (11).$$

The numbers α, β, γ will themselves solve equation (1), a and b being assumed at pleasure and r, s determined by (7).

Equation (2) may also be put in the form

$$v(mv - m') = (x - y)\{m(x + y) - m'\} \quad . \quad . \quad . \quad . \quad . \quad (12);$$

and therefore it may be solved, like (3), by taking as in (10),

$$v = mcdw' + \alpha',\ y = \tfrac{1}{2}(c^2 - m^2d^2)w' + \beta',\ x = \tfrac{1}{2}(c^2 + m^2d^2)w' + \gamma' \quad (13),$$

where

$$\left.\begin{aligned} \alpha' &= m'd\,(mdr'^2 - cs'^2), \\ \beta' &= \tfrac{1}{2}m'd\,(c + md)\,(r' - s')^2, \\ \gamma' &= \tfrac{1}{2}m'd\,\{2mdr'^2 - 2cs'^2 + (c - md)\,(r' - s')^2\} \\ &= \alpha' + \frac{c - md}{c + md}.\,\beta', \end{aligned}\right\} \quad . \quad (14),$$

c and d being any integers, and r', s' determined by the equation

$$mdr' - cs' = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15).$$

From equations (10) and (13), we have now

$$\left.\begin{aligned} x &= \quad mabw \quad + \alpha = \tfrac{1}{2}(c^2 + m^2d^2)\,w' + \gamma' \\ y &= \tfrac{1}{2}(a^2 - m^2b^2)w + \beta = \tfrac{1}{2}(c^2 - m^2d^2)\,w' + \beta' \end{aligned}\right\} \quad . \quad . \quad (16).$$

We shall only use w, and we find it from these equations

$$w = \frac{2(\alpha + \beta)m^2d^2 - 2(\alpha - \beta)c^2 - 2(\gamma' + \beta')m^2d^2 + 2(\gamma' - \beta')c^2}{(m^2b^2 + 2mab - a^2)c^2 - (a^2 + 2mab - m^2b^2)m^2d^2} \quad (17),$$

or substituting the values of γ' and β' from (14) and reducing by (15),

$$w = \frac{2}{\text{D}}\,\{(\alpha + \beta)\,m^2d^2 - (\alpha - \beta)c^2 - m'd(c + md)\} \quad . \quad . \quad (18).$$

where $\text{D} = (m^2b^2 + 2mab - a^2)c^2 - (a^2 + 2mab - m^2b^2)m^2d^2$ (19).

Substituting this in (10) and reducing by (7) and (11) we find

$$x = \frac{m'b}{\text{D}}\,\{(a + mb)c^2 - 2macd - (a - mb)\,m^2d^2\} \quad . \quad . \quad . \quad (20),$$

$$y = \frac{m'}{\text{D}}.\,(a + mb)\,(c + md)\,(bc - ad) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21),$$

$$z = \frac{m'}{\text{D}}.\,\{2abc^2 - (a^2 + m^2b^2)cd - (a^2 - m^2b^2)md^2\} \quad (22),$$

$$v = \frac{m'd}{\text{D}}\,\{(a^2 + m^2b^2)c - (a^2 + 2mab - m^2b^2)md\} \quad . \quad (23);$$

v being found by the relation $(c + md)\,(x - v) = (c - md)y$, deduced from (13). Where it is evidently necessary that a, b, c, d and w must be positive whole numbers. In particular cases, different artifices will present themselves to make w a positive integer, we shall only notice the two following general methods

1°. Let $m^2b^2 + 2mab - a^2 = (a + mb)^2 - 2a^2 = 1$ . . . (24).

And for this purpose we shall take the two series

$$\left.\begin{matrix} 1,\ 1,\ 3,\ 7,\ 17,\ 41,\ 99,\ 239,\ 577,\ 1393,\ 3363,\ \ldots \\ 0,\ 1,\ 2,\ 5,\ 12,\ 29,\ 70,\ 169,\ 408,\ 985,\ 2378,\ \ldots \end{matrix}\right\} \quad . \quad . \quad (25),$$

the terms of which are represented respectively by

$$t_0,\ t_1,\ t_2,\ t_3,\ \ldots\ldots\ t_i\ \ldots\ldots,$$
$$u_0,\ u_1,\ u_2,\ u_3,\ \ldots\ldots\ u_i\ \ldots\ldots$$

and which possess the following, among other, remarkable properties,

1. $2t_i = (1 + \sqrt{2})^i + (1 - \sqrt{2})^i$, $2\sqrt{2}.\,u_i = (1 + \sqrt{2})^i - (1 - \sqrt{2})^i$;
2. $t_i = 2t_{i-1} + t_{i-2}$, $u_i = 2u_{i-1} + u_{i-2}$,
3. $t_i^2 - 2u_i^2 = (-1)^i$, $u_i^2 - u_{i+1}\,u_{i+1} = t_iu_i - t_{i+1}u_{i+1} = (-1)^{i-1}$;
4. $u_i + u_{i-1} = t_i$, $u_i - u_{i-1} = t_{i-1}$, $2t_i = u_{i+1} + u_{i-1}$;
5. $2t_it_{i+1} - (-1)^i = 4u_iu_{i+1} + (-1)^i = t_{2i+1}$, $t_i^2 + t^2_{i+1} = 2u_{2i+1}$; &c.

(26).

Then we must have $a + mb = t_{2i}, a = u_{2i}$; therefore $mb = t_{2i} - u_{2i} = u_{2i-1}$, which must be a term in the second series, standing in an even place, and divisible by m. Then, by 3 of (26), equation (7) will be satisfied by taking $r = u_{2i-1}$, and $s = u_{2i-2}$; also $\alpha - \beta = \frac{1}{2}m'bt_{2i-3}$, $D = c^2 - t_{4i-1}m^2d^2$, and (18) becomes

$$w = \frac{2m'd\{(a^2 + m^2b^2)md - (m^2b^2 + 2mab - a^2)c\}}{(m^2b^2 + 2mab - a^2)D} - \frac{2(\alpha - \beta)}{m^2b^2 + 2mab - a^2}$$

$$= \frac{2m'd(u_{4i-1}md - c)}{c^2 - t_{4i-1}m^2d^2} - m'bt_{2i-3} \quad \dots \dots \dots \quad (27).$$

Now we can always find, as in the solution to the last question, a series of values for c and d, such that

$$c^2 - t_{4i-1}m^2d^2 = 1 \text{ or } = N \quad \dots \dots \quad (28),$$

N being some number by which $2m'(u_{4i-1}md - c)$ will divide without a remainder, and for such values, since $u_{4i-1} > \sqrt{t_{4i-1}}$, w will be a positive whole number, and we shall thus have a series of positive integral values for x, y, z, v.

2°. Let $\quad m^2b^2 + 2mab - a^2 = (a + mb)^2 - 2a^2 = 2 \quad . \; . \; . \quad (29),$

which may be done by taking

$$a + mb = 2u_{2i+1}, \; a = t_{2i+1}, \; mb = 2u_{2i+1} - t_{2i+1} = t_{2i} \quad . \quad (30)$$

t_{2i} being a term of the first series in (25), standing in an odd place, and divisible by m. Then will $r = -u_{2i}$, $s = -u_{2i-1}$, $\alpha - \beta = m'bu_{2i-2}$, and $D = 2(c^2 - m^2t_{4i+1}d^2)$, and instead of (27) we shall have

$$w = \frac{m'd(u_{4i+1}md - c)}{c^2 - m^2t_{4i+1}d^2} - m'bu_{2i-2} \quad \dots \dots \dots \quad (31),$$

and since $u_{4i+1} > \sqrt{t_{4i+1}}$, if we give to c and d a series of integral values fulfilling the equation

$$c^2 - m^2t_{4i+1}d^2 = 1 \text{ or } = N \quad \dots \dots \quad (32),$$

N being a divisor of $m'(u_{4i+1}md - c)$, we shall have a series of positive integral values for w, and consequently for x, y, z, v.

Then if m be a divisor of any term in either of the two series

$$\left.\begin{array}{l} 1, 5, 29, 169, 985, 5741, \text{\&c.} \\ 1, 3, 17, \;\; 99, 577, 3363, \text{\&c.} \end{array}\right\} \dots \dots \dots \quad (33).$$

The equations (1) and (2) can always be resolved generally in positive integers. It is also evident, from (27), that whatever be the value of m, if we can take a and b so that $2(\alpha - \beta)$ will divide by $m^2b^2 + 2mab - a^2$, we can take $d = (m^2b^2 + 2mab - a^2)d'$, and then

$$w = \frac{2m'd'\{(a^2 + m^2b^2)md' - c\}}{c^2 - \{4m^2b^2 - (a^2 - m^2b^2)^2\}m^2d'^2} - \frac{2(\alpha - \beta)}{m^2b^2 + 2mab - a^2};$$

which can always be made a positive integer, provided $a > mb < mb \times (1 + \sqrt{2})$.

II. x and y may change places in (1) and (3), and therefore instead of (10), we may take

$$x = \tfrac{1}{2}(a^2 - m^2b^2)w + \beta, \; y = mabw + \alpha, \; z = \tfrac{1}{2}(a^2 + m^2b^2)w + \gamma \; . \; (34),$$

all other things remaining the same. Then equating the values of x and y in (13) and (34) we get as before

$$w = \frac{2}{D}\{(\alpha + \beta)m^2d^2 + (\alpha - \beta)c^2 - m'd(c + md)\}$$

$$= -\frac{2m'd\{(a^2 + m^2b^2)md + (a^2 - 2mab - m^2b^2)c\}}{(a^2 - 2mab - m^2b^2)D} + \frac{2(\alpha - \beta)}{a^2 - 2mab - m^2b^2} \; (35),$$

where $\text{D} = (a^2 - 2mab - m^2b^2)c^2 - (a^2 + 2mab - m^2b_2)m^2d_2$. (36).
To render w a whole number we can

1°. Make $a^2 - 2mab - m^2b^2 = (a - mb)^2 - 2m^2b^2 = 1$ . . . . (37),
by taking $a - mb = t_{2i},\ mb = u_{2i},\ a = t_{2i} + u_{2i} = u_{2i+1}$ . . . . . (38),
that is, mb must be some term, standing in an odd place, of the second series of (25), which is divisible by m; and it is well known, from the theory of these numbers, that such terms always occur whatever be the number m. Then for (7), let $r = -u_{2i},\ s = -u_{2i-1}$; therefore $\alpha - \beta = -\frac{1}{2}m'bt_{2i-2},\ \text{D} = c^2 - m^2t_{4i+1}d^2$, and (35) becomes

$$w = \frac{-2m'd(u_{4i+1}md + c)}{c^2 - m^2t_{4i+1}d^2} - m'bt_{2i-2} \quad . \ . \ . \ . \ . \ . \ . \quad (39);$$

where, in order to have w a positive integer, we must find

$$c^2 - m^2t_{4i+1}d^2 = -1 \text{ or } = -\text{N} \quad . \ . \ . \ . \ . \ . \quad (40),$$

N being a divisor of $2m'(mu_{4i+1}d + c)$. This is not always possible, but if one solution can be found, a general one may be deduced from it, as in the solution to the last question. Equation (32) is, however, always possible, and therefore if m' were negative, this would for all values of m be a satisfactory solution. In other words it would give positive integral values of x, y, z, v that would fulfil the conditions

$$\left.\begin{aligned} mx^2 + m'x + my^2 + m'y &= mz^2 + m'z, \\ mx^2 + m'x - my^2 - m'y &= mv^2 + m'v, \end{aligned}\right\} \quad . \ . \ . \ . \quad (41),$$

m, m' being any integers > 0.

2°. Make $a^2 - 2mab - m^2b^2 = 2a^2 - (a + mb)^2 = 2$ . . . . (42),
by taking $a = t_{2i},\ a + mb = 2u_{2i}$, and $mb = 2u_{2i} - t_{2i} = t_{2i-1}$ . . . . (43),
or, mb must be some term, standing in an even place, of the first series of (25), which is divisible by m. Then for (7), let $r = u_{2i-1},\ s = u_{2i-2}$; therefore $\alpha - \beta = -m'bu_{2i-3},\ \text{D} = 2(c^2 - m^2t_{4i-1}d^2)$, and (35) becomes

$$w = \frac{-m'd(mu_{4i-1}d + c)}{c^2 - m^2t_{4i-1}d^2} - m'bu_{2i-3} \quad . \ . \ . \ . \ . \ . \quad (44),$$

where, in order to have w a positive integer, we must find

$$c^2 - m^2t_{4i-1}d^2 = -1 \text{ or } = -\text{N} \quad . \ . \ . \ . \ . \ . \ . \quad (45),$$

N being a divisor of $m'(mu_{4i-1}d + c)$, and if one such value of c and d can be found, a general solution may be obtained.

As it is possible that convenient numbers cannot be found from either of the assumptions (37) or (43), in the cases that do not fall under solution (I), I shall give a third method, which may perhaps apply when the others do not.

III. Assume $x = aw - h,\ y = bw,\ z = cw - h$ . . . . (46),
then the equations (1) and (2) become

$$mw(a^2 + b^2 - c^2) + 2mh(c - a) - m'(a + b - c) = 0 \quad . \quad (47),$$

$$m(a^2 - b^2)w^2 - \{2mah + m'(a - b)\}w + mh^2 + m'h = mv^2 - m'v \quad (48).$$

Make $a^2 + b^2 - c^2 = 0$, and $2mh(c - a) - m'(a + b - c) = 0$ which solve (47); that is, put $a = 2kl,\ b = k^2 - l^2,\ c = k^2 + l^2$ . . . . (49),
then $2mh(c - a) - m'(a + b - c) = 2(k - l)\{mh(k - l) - m'l\} = 0$;
therefore put

$$k - l = m',\ l = mh, \text{ or } k = mh + m' \quad . \ . \ . \ . \ . \ . \quad (50),$$

then $a = 2mh(mh + m'),\ b = m'(2mh + m'),\ c = 2m^2h^2 + 2mm'h + m'^2$ (51);
also $a^2 - b^2 = (a - m'^2)^2 - 2m'^4 = d^2 - 2m'^4,\ 2mah + m'(a - b) = (2mh + m')d$
where $d = a - m'^2 = 2m^2h^2 + 2m'mh - m'^2$ . . . . . (52).

Making these substitutions in (48), multiplying by $4m$, and adding m'^2,

$$4m^2(d^2-2m'^2)w^2-4m(2mh+m')dw+(2mh+m')^2=(2mv-m')^2 \quad . \; (53).$$

Let
$$2mv-m'=\frac{2t}{u}w+2mh+m' \quad \ldots\ldots\ldots \quad (54),$$

and this, substituted in (53), will give

$$w=\frac{-(2mh+m')(t+mdu)u}{t^2-(d^2-2m'^2)m^2u^2} \quad \ldots\ldots\ldots \quad (55),$$

and, by (54),
$$v=\frac{-(2mh+m')\{dt+(d^2-2m'^2)mu\}u}{t^2-(d^2-2m'^2)m^2u^2}-h \quad . \quad (56).$$

Where h may be taken at pleasure, d is determined from (52), and t and u must be such that

$$t^2-(d^2-2m'^2)m^2u^2=-1 \text{ or } -\text{N} \quad \ldots\ldots \quad (57),$$

N being a divisor of $(2mh+m')(t+mdu)$. If $h=km'$, $d=m'^2(2m^2k^2+2mk-1)=em'^2$, and if we put $t=t'm'^2$, (55) becomes

$$w=\frac{1}{m'}\cdot\frac{-(2mk+1)(t'+meu)u}{t'^2-(e^2-2)m^2u^2} \quad \ldots\ldots\ldots \quad (58),$$

where it is only necessary for the second factor to be a positive integer.

We may conclude therefore that, except in the case of $m'=0$, which is known to be impossible, we can always obtain positive integers that fulfil either equations (1) and (2), or equations (41). The solution is defective in not showing when (1) and (2) are possible in the cases where m is not a divisor of any term in either series (33); and although the defect is shared in common with all that has yet been done on such subjects, for there has never been any general test given for the possibility or impossibility of the equation $t^2-\text{A}u^2=-\text{N}$, yet I am inclined to believe that there may be determined general criteria for the possibility or impossibility of this question. As it is not my own intention to resume the subject, I have thought it my duty to put mathematicians in possession of these limited results which have laid by me for upwards of ten years.

In applying this analysis to find two polygonal numbers, X, Y, of the order n, whose sum and difference are two polygonal numbers, Z, V, of the same order. We may take, when n is an odd number,

$$m=n-2,\ m'=n-4 \quad \ldots\ldots\ldots \quad (59),$$

then, when x, y, z, v are determined from the preceding formulas,

$$2\text{X}=mx^2-m'x,\ 2\text{Y}=my^2-m'y,\ 2\text{Z}=mz^2-m'z,\ 2\text{V}=mv^2-m'v \quad (60).$$

But when n is an even number, we must take

$$m=\tfrac{1}{2}n-1,\ m'=\tfrac{1}{2}n-2 \quad \ldots\ldots\ldots \quad (61),$$

and, after finding x, y, z, v from the preceding formulas,

$$\text{X}=mx^2-m'x,\ \text{Y}=my^2-m'y,\ \text{Z}=mz^2-m'z,\ \text{V}=mv^2-m'v \quad (62).$$

Example 1. Let $n=3$, $m=1$, $m'=-1$. Using (II, 2°) and making $i=1$, in (43) we have $a=t_2=3$, $b=t_1=1$, and (44) becomes

$$w=\frac{d(5d+c)}{c^2-7d^2}+1 \quad \ldots\ldots\ldots\ldots \quad (62),$$

a particular solution is $c=4$, $d=1$; $\therefore w=2$, also $r=u_1=1$, $s=u_0=0$, therefore $\alpha=-1$, $\beta=-2$, $\gamma=-2$, so that, by (34), $x=6$, $y=5$, $z=8$, $v=3$, $\therefore \text{X}=21$, $\text{Y}=15$, $\text{Z}=36$, $\text{V}=6$; another is $c=3$, $d=1$; $\therefore w=5$, so that $x=18$, $y=14$, $z=23$, $v=11$, and $\text{X}=171$, $\text{Y}=105$, $\text{Z}=276$, $\text{V}=66$. Hence from each of the series of values that render $c^2-7d^2=1$,

we can get three solutions, from the several values
$w=d(5d+c)+1$, $w=(c+4d)(c+3d)+1$, $w=(c+3d)(4c+11d)+1$ (63).

Example 2. Let $n=5$, $m=3$, $m'=1$. Using (I, 2°), and making $i=1$ in (30), we have $a=t_3=7$, $3b=t_2=3$, or $b=1$, $r=-u_2=-2$, $s=-u_1=-1$; $\alpha=5$, $\beta=5$, $\gamma=7$, and by (31),

$$w=\frac{d(87d-c)}{c^2-369d^2} \quad \ldots\ldots\ldots\ldots (64).$$

While finding the fractions approximating to $\sqrt{369}$, we shall find the numbers $c=1364557$, $d=71036$, which make $c^2-369d^2=25$, and $w=13683167428$, therefore $y=273663348565$, $x=287346515993$, $z=396811855419$, $v=87615021004$, and $\mathrm{x}=1238520303798294339060 77$, $\mathrm{y}=112337442521576199664555$, whence a general solution may be had.

Example 3. Let $n=6$, $m=2$, $m'=1$. Using (II., 1°), and making $i=2$ in (38), we have $a=u_5=29$, $2b=u_4=12$, or $b=6$, $r=-u_4=-12$, $s=-u_3=-5$; $\alpha=6018$, $\beta=6027$, $\gamma=8517$,

and $$w=\frac{-2d(c+1970d)}{c^2-5572d^2}-18 \quad \ldots\ldots (65),$$

where w must be even. A particular case is $c=149$, $d=2$; then $w=170$, $x=65272$, $y=65178$, $z=92242$, $v=3502$; $\mathrm{x}=8520802696$, $\mathrm{y}=8496278190$; and from these a series of sets is derived. In this example also x, y, z, v, are triangular numbers, having all odd roots.

Example 4. Let $n=7$, $m=5$, $m'=3$. Using (I., 1°), and making $i=2$ in (27), we have $a=u_4=12$, $5b=u_3=5$, or $b=1$; $r=u_3=5$, $s=u_2=2$; $\alpha=231$, $\beta=229\frac{1}{2}$, $\gamma=325\frac{1}{2}$,

and $$w=\frac{6d(845d-c)}{c^2-5975d^2}-3 \quad \ldots\ldots\ldots (66)$$

where w must be odd. If $c=232$, $d=3$, which render $c^2-5975d^2=49$, then $w=843$, $x=50811$, $y=50388$, $z=71559$, $v=6543$, $\mathrm{x}=6454318086$, and $\mathrm{y}=6347300778$, and a series of such numbers may be had.

Example 5. Let $n=8$, $m=3$, $m'=2$. We might use (I., 2°), as in Ex. 2, but since $\mathrm{x}=3x^2-2x$, if we put $x=2x'$, we have $\mathrm{x}=4(3x'^2-x')=8\mathrm{x}'$, x' being a pentagonal number, and so for y, z, and v; hence we may multiply any set of numbers found from example 2 by 8, and the products will be octagonals, answering the required conditions. And in the same manner may sets of numbers of the tenth order be deduced from those of the sixth, &c.

Example 6. Let $n=9$, $m=7$, $m'=5$. Using (III.), and putting $h=1$, (52) and (55) give

$$d=143,\quad w=\frac{-19(t+1001u)u}{t^2-940751u^2} \quad \ldots\ldots\ldots (67),$$

if we take $t=12609$, $u=13$, which make $t^2-940751u^2=-38$, we find $w=166543$, $x=27979223$, $y=15821585$, $z=32142798$, $v=23076273$, $\mathrm{x}=2739929148944994$, $\mathrm{y}=876128892138825$; and thence a series of such numbers may be had. Perhaps the numbers t and u which are the lowest that render $t^2-940751u^2=1$, for which I am indebted to a young gentleman of the Institute, are the largest of the kind that have yet been calculated; they are

$t=1052442265723679403769386042332565332655403940191478220799$.
$u=1085077945859876434650947825813724885761762667300102720$.

(46). QUESTION X. *By P.*

Let lines be drawn from a given point, meeting the tangents of any curve on the same plane, and making with them a constant angle α, the points of intersection will be found in another curve. Then if v, φ be the polar co-ordinates of any point in the first curve, v, θ those of the corresponding point in the second curve, and s the length of that curve; the pole being in the given point, and the angular axis taken at pleasure; prove that

$$ds = \frac{vd\theta}{\sin\alpha}.$$

FIRST SOLUTION. *By Professor C. Avery, Hamilton College.*

Let u and u' be the first and second curves, ω and φ the angles which u and u' make with v and v, and the remaining notation as in the question. Then

$$ds = \frac{\mathrm{v}d\theta}{\sin\varphi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

But, by similar triangles, $\frac{\mathrm{v}}{v} = \frac{\sin\omega}{\sin\alpha}$, and, at the limit, the angles ω and φ are equal to each other, therefore

$$\mathrm{v} = \frac{v\sin\omega}{\sin\alpha} = \frac{\mathrm{v}\sin\varphi}{\sin\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

and writing this in (1), we have

$$ds = \frac{vd\theta}{\sin\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

SECOND SOLUTION. *By Dr. Strong.*

Let p denote the perpendicular from the given point, or origin, to the tangent to the first curve, then $\mathrm{v} = \frac{p}{\sin\alpha}$, but

$$ds^2 = \mathrm{v}^2 d\theta^2 + d\mathrm{v}^2 = \frac{p^2 d\theta^2 + dp^2}{\sin^2\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Let A denote the angle at which v cuts the tangent to the first curve, drawn through the extremity of v; then we have

$$\mathrm{A} = \alpha - \theta + \varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

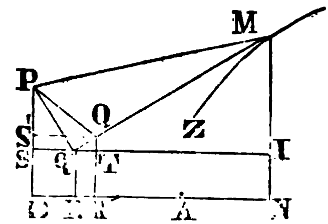
then supposing the tangent to be in contact with the element ds' of the first curve, and to turn about the extremity of v, until it comes in contact with the consecutive element ds'' of the same curve, we must regard v and φ each as constant; but α is constant, therefore by (2) we get $-d\mathrm{A} = d\theta$, but $v\sin\mathrm{A} = p$, therefore since v is to be regarded as constant, $v\cos\mathrm{A}d\mathrm{A} = -v\cos\mathrm{A}d\theta = dp$, and since $\sin^2\mathrm{A} + \cos^2\mathrm{A} = 1$, we get by (1)

$$ds = \frac{vd\theta}{\sin\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

as required.

THIRD SOLUTION. *By Professor Benedict.*

Let MQQ′ be a tangent to any point M, of a curve MZ, whose rectangular co-ordinates are AN and MN. From any given point P, draw the lines PQ, PQ′ making $\angle$ PQ′M$=90^\circ$, and PQQ′$=\alpha$. Draw IS parallel to AN, meeting MN at I; and from P, Q′ let fall the perpendiculars PSD and Q′R. Join PM and put PQ′ $=$ V′, PQ $=$ V, PM $= v$, AN $= x$, MN $= y$, PD $= b$, AD $= e$, PS $= x'$, SQ′ $= y'$, $\angle$ SPQ′ $= \theta$ and let s, s′ be the lengths of the curves which Q and Q′ have described from PD. Comparing the elementary triangle at M, with the similar triangles MIQ′ and PSQ′, we have



$$dy : dx :: \text{MN} = y - b + x' : \text{IQ}' = \frac{dx}{dy}(y - b + x');$$

and therefore $\text{DN} = e + x = \text{SQ}' + \text{IQ}' = \dfrac{dx}{dy}(y - b + x') + y' \quad . \ . \quad (1).$

Also from PSQ′ we have

$$\frac{dx}{dy} = \frac{x'}{y'} \ . \ . \ (2),\ x' = \text{V}' \cos\theta \ . \ . \ (3),\ y' = \text{V}' \sin\theta \ . \ . \ (4), \text{ and } \therefore \frac{dx}{dy} = \cot\theta \ (5).$$

From (1) eliminate x', y' and $\dfrac{dx}{dy}$, by means of (3), (4), (5), we get

$$e + x = \cot\theta(y - b + \text{V}'\cos\theta) + \text{V}'\sin\theta;$$

rom which are derived

$$\text{V}' = \frac{e + x - (y - b)\cot\theta}{\cot\theta\cos\theta + \sin\theta} = (e + x)\sin\theta - (y - b)\cos\theta \ . \quad (6),$$

and $\quad d\text{V}' = dx\sin\theta + (e + x)\cos\theta d\theta - dy\cos\theta + (y - b)\sin\theta d\theta;$

and since from (5) $dx\sin\theta = dy\cos\theta$,

$$d\text{V}' = (e + x)\cos\theta d\theta + (y - b)\sin\theta d\theta \quad . \ . \ . \ . \ . \quad (7).$$

But we have evidently $ds'^2 = \text{V}'^2 d\theta^2 + d\text{V}'^2$, which becomes after eliminating V'^2 and $d\text{V}'^2$, and reducing, by means of (6) and (7),

$$ds' = \sqrt{(e + x)^2 + (u - b)^2} \ . \ d\theta.$$

But $\text{PM} = v = \sqrt{(e + x)^2 + (y - b)^2}$, and $\therefore ds' = v d\theta$. But since the angle Q′PQ is constant, being the complement of α, we have

$$ds' : ds :: \text{PQ}' : \text{PQ},$$

and therefore $\qquad ds = \dfrac{\text{PQ} \ . \ ds'}{\text{PQ}'} = \dfrac{v d\theta}{\sin\alpha} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (8).$

Remarks. If the equation of MZ is $y = \text{F}(x)$, the second number being any function of x, the equations

$$e + x = \frac{dx}{dy}(y - b + x') + y',\ \frac{dx}{dy} = \frac{x'}{y'},\ y = \text{F}(x) \quad . \ . \ . \ . \ . \quad (9)$$

are adequate to the determination of the curve that Q′ traces, referred to the co-ordinates x', y'.

If PS′ $= x''$, QS′ $= y''$, then QQ′ $= \sqrt{x'^2 + y'^2} \ . \cot\alpha$, Q′T $= x'\cot\alpha$, QT $= y'\cot\alpha$, and therefore $y'' = y' + x'\cot\alpha$, and $x'' = x' - y'\cot\alpha$; from which we obtain $y' = \dfrac{y'' - x''\cot\alpha}{1 + \cot^2\alpha}$, $x' = \dfrac{x'' + y''\cot\alpha}{1 + \cot^2\alpha}$.

Substituting these expressions for x', y' in (9), we obtain the equations

$$\left.\begin{array}{l} e+x=\dfrac{x''+y''\cot\alpha}{y''-x''\cot\alpha}\left(y-b+\dfrac{x''+y''\cot\alpha}{1+\cot^2\alpha}\right)+\dfrac{y''-x''\cot\alpha}{1+\cot^2\alpha}, \\ \dfrac{dx}{dy}=\dfrac{x''+y''\cot\alpha}{y''-x''\cot\alpha}, \text{ and } y=\mathrm{F}(x) \end{array}\right\} (10),$$

which are adequate to the determination of the curve that Q traces, referred to the rectangular co-ordinates x'', y''.

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(46), QUESTION XI. *By Mr. James F. Macully.*

It is required to divide a given paraboloid of revolution into two equal parts, by a plane passing through a given line on its base.

FIRST SOLUTION *By Professor Peirce.*

Let $y^2=2px$ be the equation of the revolving parabola,
$h=$ the height of the paraboloid,
$a=$ distance of the given line from the centre of the base,
$\theta=$ the required inclination of the plane to the base,
$n=\cot\theta$,
$\mathrm{V}=$ volume of that part of the paraboloid cut off by the plane, which does not include the vertex.

Let the paraboloid be cut by a plane parallel to its base, and let
$\mathrm{Y}=$ the distance of the intersection of this plane with the sought plane, from the centre of the section thus formed,
$2\varphi=$ the angle included by the two lines drawn from the centre of this section to the points in which its circumference is cut by the above line of intersection;

and we have

$$\begin{aligned} \mathrm{Y}&=nh-a-nx=y\cos\varphi, \\ d\mathrm{V}&=\tfrac{1}{2}y^2(2\varphi-\sin 2\varphi)\,dx \\ &=pxdx(2\varphi-\sin 2\varphi); \\ \mathrm{V}&=\int pxdx\,(2\varphi-\sin 2\varphi) \\ &=\tfrac{1}{2}px^2\ (2\varphi-\sin 2\varphi)-\int px^2\ (1-\cos 2\varphi)\,d\varphi \\ &=\tfrac{1}{2}ph^2\ (2\varphi'-\sin 2\varphi')-\int 2px^2\sin^2\varphi d\varphi, \end{aligned}$$

the integral being taken from $\varphi=0$ to $\varphi=\varphi'$, φ' being determined by the equation

$$-a=\sqrt{2ph}\,.\cos\varphi'.$$

But $\quad y\sin\varphi=\sqrt{\{-(nh-a)^2+2(nh-a)nx+2px-n^2x^2\}}$,
and

$$2px^2\sin^2\varphi d\varphi=\tfrac{1}{2}(nh-a+nx)\sqrt{\{-(nh-a)^2+2(nh-a)nx+2px-n^2x^2\}}dx,$$

whence

$$\begin{aligned} \int 2px^2\sin^2\varphi d\varphi&=\left(nh-a+\frac{p}{n}\right).\frac{na-p}{4n^2}.\sqrt{2ph-a^2}-\frac{1}{8n}(2ph-a^2)^{\frac{3}{2}} \\ &\quad+\frac{2pn(nh-a)-p^2}{4n^3}\times\left(\tfrac{1}{2}\pi-\text{arc. tang.}\frac{an-p}{n\sqrt{2ph-a^2}}\right) \\ &=\mathrm{V}-\tfrac{1}{2}ph^2(2\varphi'-\sin 2\varphi') \\ &=\tfrac{1}{2}ph^2(\pi-2\varphi'+\sin 2\varphi'), \end{aligned}$$

and the value of n is to be determined from this equation, which gives that of θ, or of the inclination of the plane.

SECOND SOLUTION. *By the Proposer.*

Let a = height of the paraboloid, b = radius of its base, $2p$ = the parameter of the generating parabola, then will

$$b^2 = 2pa \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Let $2d$ be the given line on the base, c its distance from the centre of the base, so that

$$c^2 + d^2 = b^2 = 2pa \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Let the cutting plane (C), passing through $2d$, make an angle θ with the base; it will cut the axis of the paraboloid, and the principal axis of the section will pierce the paraboloid at a distance, k, from the axis, such that

$$k^2 = 2p\,\{a - (c+k)\tan\theta\},$$

or

$$k = -p\tan\theta + \sqrt{2pa - 2pc\tan\theta + p^2\tan^2\theta}$$
$$= -p\tan\theta + \sqrt{2ph} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

where

$$h = a - c\tan\theta + \tfrac{1}{2}p\tan^2\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Let a parabolic section (A) be drawn through $2d$, perpendicular to the base, it will be similar to the generating parabola, and therefore its height will be $= \dfrac{d^2}{2p}$; the solid standing on the lesser segment of the base has been calculated by Hutton, it is $a^2p.\tan^{-1}\dfrac{d}{c} - \tfrac{1}{2}acd - \dfrac{cd^3}{6p}$, and the volume of the paraboloid $= a^2p\pi$, therefore, by the question, the solid P contained between the planes (A), (C) and the surface of the paraboloid, is

$$\text{P} = \tfrac{1}{2}a^2p\pi - a^2p.\tan^{-1}\frac{d}{c} + \tfrac{1}{2}acd + \frac{cd^3}{5p} = a^2p\tan^{-1}\frac{c}{d} + \tfrac{1}{2}acd + \frac{cd^3}{6p} \quad (5).$$

To find this solid in terms of θ, let a parabolic section (B) be drawn parallel to (A) at the distance of z from it, it is bounded by the paraboloid and the cutting plane (C), and therefore its height is

$$\text{H} = a - \frac{(c-z)^2}{2p} - z\tan\theta = h - \frac{(p\tan\theta - c + z)^2}{2p}$$

or if we put $\quad p\tan\theta - c + z = \sqrt{2ph}.\sin\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$

Then $\quad \text{H} = h\cos^2\varphi,$

and its area, $\quad \text{B} = \tfrac{4}{3}\text{H}\sqrt{2p\text{H}} = \tfrac{4}{3}h\sqrt{2ph}.\cos^3\varphi$;

hence the solid $\quad \text{P} = \int \text{B}dz = \tfrac{8}{3}.ph^2\int\cos^4\varphi d\varphi$

$$= ph\ (\varphi + \tfrac{2}{3}\sin 2\varphi + \tfrac{1}{12}\sin 4\varphi) \quad . \quad . \quad . \quad . \quad . \quad (7).$$

This integral must be taken from the limit $z = 0$, and φ = an angle φ', such that, by (6),

$$p\tan\theta - c = \sqrt{2ph}.\sin\varphi', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

to the limit $z = c + k = c - p\tan\theta + \sqrt{2ph}$, when, by (6), $\varphi = \tfrac{1}{2}\pi$, and therefore $\quad \text{P} = ph^2(\tfrac{1}{2}\pi - \varphi' - \tfrac{2}{3}\sin 2\varphi' - \tfrac{1}{12}\sin 4\varphi') \quad . \quad . \quad . \quad . \quad (9).$

But by eliminating θ between the equations (4) and (8), we find

$$h = \frac{d^2}{2p}\sec^2\varphi' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

$$\therefore \text{P} = \frac{d^4}{4p}\,(\tfrac{1}{2}\pi - \varphi' - \tfrac{2}{3}\sin 2\varphi' - \tfrac{1}{12}\sin 4\varphi')\sec^4\varphi',$$

and $$\varphi' + \tfrac{2}{3}\sin 2\varphi' + \tfrac{1}{12}\sin 4\varphi' + \frac{4p\text{P}}{d^4}.\cos^4\varphi' = \tfrac{1}{2}\pi. \quad . \quad . \quad (11),$$

P having the given value in (5), and φ' being found from this equation, θ is determined by the relation, obtained by substituting (9) in (8)

$$\tan\theta = \frac{c}{p} + \frac{d}{p}\tan\varphi' \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (12).$$

Cor. When the section passes through one extremity of the base, or when $c = b$, $d = 0$; then $\text{P} = \frac{1}{2}a^2p\pi$, $h = \dfrac{(b - p\tan\theta)^2}{2p}$, $k = b - 2p\tan\theta$, then, by (8), $\sin\varphi' = -1$, and $\varphi' = -\frac{1}{2}\pi$, and, by (9), $\text{P} = \frac{1}{2}a^2p\pi = h^2p\pi$, or $h = a\sqrt{\frac{1}{2}}$, and $(b - p\tan\theta)^2 = 2ap\sqrt{\frac{1}{2}} = b^2\sqrt{\frac{1}{2}}$, and $\tan\theta = \dfrac{b}{p}(1 \pm \sqrt[4]{\tfrac{1}{2}})$.

The sign — must be used for the cutting plane in the question since the tangent plane at the given point makes an angle with the base whose tangent is $\dfrac{b}{p}$; if the sign + be used the resulting plane will also cut off a solid equal half the given paraboloid, but on the opposite side of the base.

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(47). QUESTION XII. *By Professor Catlin.*

Two straight lines revolve in parallel planes with given velocities about given points in a common axis. Required the locus of the apparent intersection of these lines, when viewed from a given point.

FIRST SOLUTION. *By Dr. Strong.*

We shall imagine the eye to be on the same side of the planes, and that L$''$ denotes the line in the nearer plane, L that in the other plane; also put P and p for the perpendiculars from the points of intersection of the axis and planes, to L and L$''$ respectively.

Imagine a plane, supposed horizontal, to pass through the eye and axis; also that two other planes pass through the eye and p, L$''$ respectively; the common sections of these planes with the nearer plane are at right angles to each other, ∴ also their lines of common section with the remoter plane are at right angles to each other; put O for the origin of P, then evidently, the plane of the eye and p, will intersect a horizontal line through O, at a given point A, and AO is given in length. Put L$'$ for the common section of the plane of the eye and L$''$ with the remoter plane, and it is evident that L$'$ is parallel to L$''$, also put p' for the perpendicular from A to L$'$ and it is manifestly parallel to p, and in the plane of the eye and p. Hence supposing the intersection to be viewed on the remoter plane, we have to find the locus of the intersection of the lines L and L$'$; and it is evident that the angular motion of L$'$ is the same as that of L$''$, also that the angular motions of L, L$'$ are the same as as those of P, p', and that P, p, p' are given in length.

Put AO $= l$, $r =$ the radius vector, its origin being at O; $\varphi =$ the angle made by r and l at any time t from the origin of motion; A$'$ and A$''$ the initial angles made by P and p' with l, reckoned in the same direction as

φ; V and V′ the given angular velocities of P and p' also put $n = \frac{V'}{V}$. If the angles made by P and p' with AO increase, we have

$$r\cos\{\varphi - (Vt + A')\} = P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

$$r\cos\{\varphi - (V't + A'')\} = l\cos(V't + A'') + p' \quad . \quad . \quad . \quad (2),$$

which enable us to construct the locus by points, for assuming t, they will give the corresponding value of r and φ.

Put $\frac{P}{r} = \cos Z$, therefore, by (1), $t = \frac{\varphi - A' - Z}{V}$, and (2) becomes

$$r\cos\{\varphi - A'' + n(A' + Z - \varphi)\} = l\cos\{A'' + n(\varphi - A' - Z)\} + p' \ (3),$$

which is the equation of the required locus; if the lines revolve in contrary directions, then regarding V as positive, V′ must be considered as negative, and its sign must be changed.

If $p' = 0$, $P = 0$, and therefore $Z = \frac{1}{2}\pi$, (3) will be changed to

$$r\cos\{\varphi - A'' + n(A' - \varphi + \tfrac{1}{2}\pi)\} = l\cos\{A'' + n(\varphi - A' - \tfrac{1}{2}\pi)\} \ (4),$$

and the lines revolve like radii about the axis. If $n = 1$, (4) becomes

$$r = l\{\cos\varphi + \sin\varphi\cot(A'' - A')\},$$

which is an equation of the circle. If $P = 0$, but p' not $= 0$, and if $n = 1$, (3) will be changed to

$$r = l\{\cos\varphi + \sin\varphi\cot(A'' - A')\} + \frac{p'}{\sin(A'' - A')},$$

which shows the locus to be a line of the fourth order.

SECOND SOLUTION. *By Professor C. Avery.*

Let a be the line joining the centres of two circles on the same plane, which line is the projection of the common axis on the plane, and the centres of the two circles are the points in space projected on the above-mentioned plane. Let φ' be the angle the radius of one circle makes with a at the beginning of motion, $\varphi + \varphi'$ the angle it makes with a at any time, then if $n =$ the ratio of the angular velocities, $n\varphi$ will be the angle the revolving radius of the second circle makes with a; then the angle opposite a made by the two intersecting radii, one from each circle, is

$$\varphi' + \varphi - n\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

which shows that when $n = 1$, or when the angular velocities are equal, the locus is a circle, since φ' is constant. Again, since the side of triangles are as the sines of the opposite angles, we have

$$r\sin(\varphi' + \varphi - n\varphi) = a\sin n\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where r is the distance from the centre of the first circle to any point in the required locus. If (2) be reduced

$$r = \frac{a\operatorname{cosec}(\varphi' + \varphi)}{\cot n\varphi - \cot(\varphi' + \varphi)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

When $\varphi + \varphi' = 180°$, $r = a$; when $n\varphi = \varphi' + \varphi$, $r = \infty$, and when $\varphi' + \varphi = 90°$, $r = a\tan n\varphi$.

—— The solutions of Professor Catlin, and of Messrs. Abbot and Root, were essentially the same as Professor Avery's.

(46). QUESTION XIII. *By Professor Catlin.*

A point oscillates with a given uniform motion between the centre of suspension and the centre of oscillation of a given pendulum. Required the locus of the oscillating point during a complete vibration of the pendulum.

SOLUTION. *By the Proposer.*

Let $r =$ the radius vector, the origin being at the point of suspension; $m =$ the velocity of the moving point, and $t =$ the time. And we may suppose $r = 0$, when $t = 0$, $\therefore$ we shall have

$$r = mt \quad \ldots \ldots \ldots \ldots \quad (1),$$

$$\therefore t = \frac{r}{m} \quad \ldots \ldots \ldots \ldots \quad (2).$$

But using the notation of Legendre in "*Exercices de Calcul Integral,*" we have

$$t = h \,.\, \mathrm{F} \,.\, (c, \varphi) \quad \ldots \ldots \ldots \ldots \quad (3).$$

Equations (2) and (3) give

$$\frac{r}{m} = h .\ \mathrm{F}(c, \varphi \quad \ldots \ldots \ldots \ldots \quad (4),$$

which is the equation of the locus required. By assuming different values of r, we may find the corresponding values of φ, by means of the tables of integrals of the first species; and hence the curve can easily be constructed.

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(47). QUESTION XIV. (*Communicated by Dr. Strong.*)

The axes of a given cone and cylinder of revolution intersect at right angles; to find the portion of the solid common to both, the surface of the cone included by the cylinder, and the surface of the cylinder included by the cone.

⁂ This question was published in "Marratt's Scientific Journal," but no solution to it has yet been published.

FIRST SOLUTION. *By Dr. Strong.*

Imagine the base of the cone to be horizontal, and its axis vertical; then the plane of the axes bisects the sought solid and surfaces; also a vertical plane through the axis of the cone at right angles to the axis of the cylinder bisects the solid, its section with the cylinder being a circle, whose centre is the intersection of the axes, its radius we will call R, and the distance of its centre from the cone's vertex, c. Let r denote any right line drawn from the vertex of the cone to any point in the circle, $\varphi =$ the angle made by r and c, and $z =$ the perpendicular through the extremity of r to the plane of the circle, limited by the circle itself and the surface of the cone, ; put $n =$ tangent of half the vertical angle of the cone and $m = \sqrt{1 + n^2} =$ its secant. Then will

$$r^2 + z^2 = r^2 m^2 \cos^2 \varphi, \text{ or } z^2 = r^2 (m^2 \cos^2 \varphi - 1) \quad . \ . \ . \quad (1)$$

be the equation of the conical surface, and

$$(r \sin \varphi)^2 + (c - r \cos \varphi)^2 = \mathrm{R}^2, \text{ or } r^2 + c^2 - 2rc \cos \varphi = \mathrm{R}^2 \quad (2),$$

that of the circle. Let s denote the portion of the solid which is between the plane of the axes and that of r and z; we shall have

$$d^2s = 2rzdrd\varphi = 2r^2 drd\varphi\sqrt{m^2 \cos^2\varphi - 1} \quad . \quad . \quad . \quad . \quad (3);$$

$$\therefore ds = \tfrac{1}{3}(r''^3 - r'^3)\, d\varphi \sqrt{m^2 \cos^2\varphi - 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Put $c^2 - \text{R}^2 = \text{T}^2$, then $r^2 - 2cr\cos\varphi = -\text{T}^2$, or $r = c\cos\varphi \pm \sqrt{c^2\cos^2\varphi - \text{T}^2}$; hence $r'' = c\cos\varphi + \sqrt{c^2\cos^2\varphi - \text{T}^2}$, $r' = c\cos\varphi - \sqrt{c^2\cos^2\varphi - \text{T}^2}$, and $r''^3 - r'^3 = 2(4c^2\cos^2\varphi - \text{T}^2)\sqrt{c^2\cos^2\varphi - \text{T}^2}$, and (4) becomes

$$2ds = \tfrac{2}{3}(4c^2\cos^2\varphi - \text{T}^2)\sqrt{c^2\cos^2\varphi - \text{T}^2} \times \sqrt{m^2\cos^2\varphi - 1} \times d\varphi \quad (5),$$

the integral of which from $\cos\varphi = 1$ to $\cos\varphi = \dfrac{\text{T}}{c}$, will be the solid required.

Denote $2ds$ by ds, then since $d\varphi = \cos^2\varphi \times d\tan\varphi$, put

$$\tan\varphi = ny,\ \cos^2\varphi = \frac{1}{1+n^2y^2},\ 4c^2 - \text{T}^2 = a^2,\ \sqrt{1-y^2}.\sqrt{\text{R}^2 - \text{T}^2n^2y^2} = \text{R}' \quad . \quad (6),$$

and by these substitutions we shall give to (5) the form

$$ds = \tfrac{2}{3}n^2.\frac{(a^2 - \text{T}^2n^2y^2)\,\text{R}dy}{(1+n^2y^2)^3}$$

$$= \frac{8dy}{3\text{R}'}.\left\{\frac{4m^2c^4}{(1+n^2y^2)^3} - \frac{5m^2c^2\text{T}^2 + 4c^4}{(1+n^2y^2)^2} + \frac{m^2\text{T}^4 + 5\text{T}^2c^2}{1+n^2y^2} - \text{T}^4\right\} \quad (7).$$

Or, by help of the formulas, easily obtained by differentiating,

$$\frac{4m^2c^2dy}{\text{R}'(1+n^2y^2)^3} = n^2d\left\{\frac{y\text{R}'}{(1+n^2y^2)^2}\right\} + \frac{3\{(m^2+1)c^2 + m^2\text{T}^2\}dy}{\text{R}'(1+n^2y^2)^2}$$
$$- \frac{2\{(m^2+1)\text{T}^2 + c^2\}dy}{\text{R}'(1+n^2y^2)} + \text{T}^2.\frac{dy}{\text{R}'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a),$$

$$\frac{2m^2c^2dy}{\text{R}'(1+n^2y^2)^2} = n^2d.\left(\frac{y\text{R}'}{1+n^2y^2}\right) + \frac{\{(m^2+1)c^2 + m^2\text{T}^2\}dy}{\text{R}'(1+n^2y^2)} - \frac{\text{T}^2(1+n^2y^2)dy}{\text{R}'}(b),$$

we shall change (7) into

$$ds = \tfrac{8}{3}n^2c^2.\,d\left\{\frac{y\text{R}'}{(1+n^2y^2)^2}\right\} + \frac{4n^2}{3m^2}(n^2c^2 + 2m^2\text{R}^2).d\left\{\frac{y\text{R}'}{1+n^2y^2}\right\}$$
$$+ \frac{4n^2}{3m^2}\left\{\frac{c^2(c^2+3m^2\text{R}^2)dy}{\text{R}'(1+n^2y^2)} - \frac{c^2\text{T}^2dy}{\text{R}'} - \text{T}^2(n^2c^2 + 2m^2\text{R}^2).\frac{y^2dy}{\text{R}'}\right\} \text{(A)},$$

which must be integrated from $y = 0$ to $y = \dfrac{\text{R}}{n\text{T}}$.

To find the surfaces, s', cut from the conical surface by the sides of the cylinder, we shall put $\sin\psi = \dfrac{\tan\varphi}{n} = y$, then $d\psi = \dfrac{dy}{\sqrt{1-y^2}}$, and

$$ds' = \tfrac{1}{2}(r''^2 - r'^2)\, mn\cos^2\varphi d\psi$$
$$= \tfrac{1}{2}mn\cos^2\varphi(r''^2 - r'^2).\frac{dy}{\sqrt{1-y^2}}.$$

But $r''^2 - r'^2 = 4c\cos\varphi\sqrt{c^2\cos^2\varphi - \text{T}^2}$; by substituting this denoting $4ds'$ by ds', and using the transformations and reductions contained in (6), and (b) we find

$$ds' = 8nmc \cos^2\varphi\sqrt{c^2\cos^2\varphi - \mathrm{T}^2} \times \frac{dy}{\sqrt{1-y^2}} = 8nmc. \frac{dy\sqrt{\mathrm{R}^2 - n^2\mathrm{T}^2y^2}}{(1+n^2y^2)^2\sqrt{1-y^2}}$$

$$= 8nmc\, dy \left\{ \frac{c^2}{\mathrm{R}'(1+n^2y^2)^2} - \frac{\mathrm{T}^2}{\mathrm{R}'(1+n^2y^2)} \right\}$$

$$= \frac{4nc}{m}. \left[n^2 d. \left\{ \frac{y\mathrm{R}'}{1+n^2y^2} \right\} + \frac{(c^2+m^2\mathrm{R}^2)dy}{\mathrm{R}'(1+n^2y^2)} - \frac{\mathrm{T}^2}{\mathrm{R}'}(1+n^2y^2)dy \right] \quad \text{(B)}$$

which must be integrated between the limits $y = 0$ and $y = \frac{\mathrm{R}}{n\mathrm{T}}$.

To find the surface of the cylinder which is within that of the cone, we shall put $z' = r'\sqrt{m^2\cos^2\varphi - 1}$, $z'' = r''\sqrt{m^2\cos^2\varphi - 1}$, and we have

$$ds'' = 4z'\sqrt{r'^2 d\varphi^2 + dr'^2} + 4z''\sqrt{r''^2 d\varphi^2 + dr''^2} \quad . \quad . \quad (1');$$

but $$dr' = \frac{r'c \sin\varphi d\varphi}{\sqrt{c^2\cos^2\varphi - \mathrm{T}^2}},\ dr'' = \frac{-r''c\sin\varphi d\varphi}{\sqrt{c^2\cos^2\varphi - \mathrm{T}^2}};$$

also, $r'^2d\varphi^2 + dr'^2 = \frac{\mathrm{R}^2r'^2d\varphi^2}{c^2\cos^2\varphi - \mathrm{T}^2}$, $r''^2\varphi^2 + dr''^2 = \frac{\mathrm{R}^2r''^2d\varphi^2}{c^2\cos^2\varphi - \mathrm{T}^2}$.

Substituting these in (1), and applying the transformations in (6), (*b*)

$$ds'' = 4\mathrm{R}d\varphi(r'^2 + r''^2). \sqrt{\frac{m^2\cos^2\varphi - 1}{c^2\cos^2\varphi - \mathrm{T}^2}}$$

$$= 8\mathrm{R}\cos^2\varphi d.\tan\varphi(2c^2\cos^2\varphi - \mathrm{T}^2). \sqrt{\frac{m^2\cos^2\varphi - 1}{c^2\cos^2\varphi - \mathrm{T}^2}}$$

$$= 8\mathrm{R}n^2dy. \frac{c^2 + \mathrm{R}^2 - n^2\mathrm{T}^2y^2}{(1+n^2y^2)^2}. \sqrt{\frac{1-y^2}{\mathrm{R}^2 - n^2\mathrm{T}^2y^2}}$$

$$= 8\mathrm{R}\left\{ \frac{2c^2m^2dy}{\mathrm{R}'(1+n^2y^2)^2} - \frac{(2c^2+m^2\mathrm{T}^2)dy}{\mathrm{R}'(1+n^2y^2)} + \frac{\mathrm{T}^2dy}{\mathrm{R}'} \right\}$$

$$= 8n^2\mathrm{R}\left\{ d.\left(\frac{y\mathrm{R}'}{1+n^2y^2}\right) + \frac{c^2dy}{\mathrm{R}'(1+n^2y^2)} - \frac{\mathrm{T}^2y^2dy}{\mathrm{R}'} \right\} \quad . \quad \text{(C)},$$

which must be integrated between the same limits as before.

If any circular section of the cone, whose radius $= b$, remain invariable, while n is supposed to be indefinitely diminished, the cone will approach to a cylinder as its limit, which it will attain when $n = 0$. At this limit also, $\mathrm{T}^2 = c^2$, $a^2 = 3c^2$, $n\mathrm{T} = b$, $m = 1$, $ny = 0$, and $\mathrm{R}' = \sqrt{\mathrm{R}^2 - b^2y^2}$, $\sqrt{1-y^2}$. By making these substitutions in (A), (B), (C), they become, for two intersecting cylinders,

$$ds = \tfrac{8}{3}b^2\left\{ d.(y\mathrm{R}') + \frac{2\mathrm{R}^2dy}{\mathrm{R}'} - \frac{(b^2+\mathrm{R}^2)y^2dy}{\mathrm{R}'} \right\} \quad . \quad . \quad . \quad . \quad \text{(A')},$$

$$ds' = 8b\left\{ \frac{\mathrm{R}^2dy}{\mathrm{R}'} - \frac{b^2y^2dy}{\mathrm{R}'} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(B')},$$

$$ds'' = 8b^2\mathrm{R}\left\{ \frac{dy}{\mathrm{R}'} - \frac{y^2dy}{\mathrm{R}'} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(C')}.$$

The algebraic integrals in (A), (B), (C), (A') may be omitted, since they vanish at the two limits; at the first because $y=0$, and at the second because $\mathrm{R}' = 0$ at that limit.

If $\frac{n\text{T}}{\text{R}} > 1$, that is if the cylinder pierces the cone, we shall put $\frac{n\text{T}}{\text{R}}.\ y = \sin\theta$ and $\frac{\text{R}}{n\text{T}} = e$, which give $dy = \frac{\text{R}}{n\text{T}}.\cos\theta d\theta$, $\text{R}' = \text{R}\cos\theta\sqrt{1-e^2\sin^2\theta}$, and (A), (B), (C), give

$$ds = \frac{4n}{3m^4}\left\{\frac{(c^2+3m^2\text{R}^2)c^2 d\theta}{\text{T}(1+n^2e^2\sin^2\theta)\sqrt{1-e^2\sin^2\theta}} + (n^2c^2+2m^2\text{R}^2)\text{T}d\theta\sqrt{1-e^2\sin^2\theta}\right.$$
$$\left. -m^2\text{T}(c^2+2\text{R}^2)\times\frac{d\theta}{\sqrt{1-e^2\sin^2\theta}}\right\} \quad \ldots\ldots \quad (\text{A}''),$$

$$ds' = \frac{4c}{m\text{T}}\left\{\frac{c^2+m^2\text{R}^2}{1+n^2e^2\sin^2\theta}\times\frac{d\theta}{\sqrt{1-e^2\sin^2\theta}} + n^2\text{T}^2 d\theta\sqrt{1-e^2\sin^2\theta},\right.$$
$$\left. -\frac{m^2\text{T}^2 d\theta}{\sqrt{1-e^2\sin^2\theta}}\right\} \quad \ldots\ldots \quad (\text{B}''),$$

$$ds'' = \frac{8n\text{R}}{m\text{T}}\left\{\frac{c^2}{1+n^2e^2\sin^2\theta}.\frac{d\theta}{\sqrt{1-e^2\sin^2\theta}} + \text{T}^2 d\theta\sqrt{1-e^2\sin^2\theta}\right.$$
$$\left. -\frac{\text{T}^2 d\theta}{\sqrt{1-e^2\sin^2\theta}}\right\} \quad \ldots\ldots \quad (\text{C}'').$$

But if $\frac{n\text{T}}{\text{R}} < 1$, that is if the cone pierces the cylinder we put $\frac{n\text{T}}{\text{R}} = e$ and $y = \sin\theta$, $\therefore dy = \cos\theta d\theta$, $\text{R}' = \text{R}\cos\theta\sqrt{1-e^2\sin^2\theta}$, $\frac{dy}{\text{R}'} = \frac{d\theta}{\text{R}\sqrt{1-e^2\sin^2\theta}}$, hence (A), (B), (C) give

$$ds = \frac{4}{3m^4\text{R}}\left\{\frac{c^2+3m^2\text{R}^2}{1+n^2\sin^2\theta}.\frac{n^2c^2 d\theta}{\sqrt{1-e^2\sin^2\theta}} + \text{R}^2(n^2c^2+2m^2\text{R}^2)d\theta\sqrt{1-e^2\sin^2\theta}\right.$$
$$\left. -\frac{(n^2c^4+2m^2\text{R}^4)d\theta}{\sqrt{1-e^2\sin^2\theta}}\right\} \quad \ldots\ldots \quad (\text{A}'''),$$

$$ds' = \frac{4cn}{\text{R}m}\left\{\frac{c^2+m^2\text{R}^2}{1+n^2\sin^2\theta}\times\frac{d\theta}{\sqrt{1-e^2\sin^2\theta}} + \text{R}^2 d\theta\sqrt{1-e^2\sin^2\theta}\right.$$
$$\left. -\frac{c^2 d\theta}{\sqrt{1-e^2\sin^2\theta}}\right\} \quad \ldots\ldots \quad (\text{B}'''),$$

$$ds'' = 8\left\{\frac{n^2c^2}{1+n^2\sin^2\theta}.\frac{d\theta}{\sqrt{1-e^2\sin^2\theta}} + \text{R}^2 d\theta\sqrt{1-e^2\sin^2\theta}\right.$$
$$\left. -\frac{\text{R}^2 d\theta}{\sqrt{1-e^2\sin^2\theta}}\right\} \quad \ldots\ldots \quad (\text{C}''');$$

the integrals, in both cases, being taken from $\theta = 0$ to $\theta = \frac{1}{2}\pi$.

It is remarkable that each of the above equations involves the three kinds of elliptic functions treated of by Le Gendre in his "*Functions Elliptiques*," see p. 19 of that work, to which the reader is referred for the methods of obtaining the integral; we would observe however that if we substitute for $\sin^2\theta$ its equal $\frac{1}{2}(1-\cos 2\theta)$ the integrals may be had by the ordinary methods of converging series, a method particularly applicable when e^2 is very small, for then the terms into which the higher

powers of e enter may be neglected. In the case where $e=1$, that is, when the surfaces of the cone and cylinder touch each other, the integrals will not depend on elliptic functions, but on circular and logarithmic functions.

For the integrals of (A′), (B′), (C′); if we imagine b to be the radius of the larger cylinder, we shall always have $\frac{b}{R}>1$, and we may put $\frac{b}{R}y=\sin\theta$, $\frac{R}{b}=e$, which give $dy=\frac{R}{b}\cos\theta d\theta$, $R'=R\cos\theta\sqrt{1-e^2\sin^2\theta}$, and we have

$$ds=\tfrac{4}{3}b\left\{\frac{(R^2-b^2)d\theta}{\sqrt{1-e^2\sin^2\theta}}+(R^2+b^2)d\theta\sqrt{1-e^2\sin^2\theta}\right\} \quad (a'),$$

$$ds'=8\left\{\frac{(R^2-b^2)d\theta}{\sqrt{1-e^2\sin^2\theta}}+b^2d\theta\sqrt{1-e^2\sin^2\theta}\right\} \quad . \; . \; . \quad (b'),$$

$$ds''=8Rbd\theta\sqrt{1-e^2\sin^2\theta} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (c');$$

which are to be integrated from $\theta=0$ to $\theta=\frac{1}{2}\pi$. They depend on elliptic functions of the first and second species, and may be found as before. We would observe that in any of the above cases $s'+s''$ will be the whole surface of the solid cut out of the perforated solid by the sides of the perforating solid. Finally, we observe that if in (a'), (b'), (c') we put $b=R$, we get $s=\frac{16}{3}R^3$, $s'=8R^2$, $s''=8R^2$, $s'+s''=16R^2$, as we evidently ought to have; for as the cylinders are equal, it is manifest that the solid common to both cylinders is a double *groin*.

Second Solution. *By Professor Catlin.*

(I.) Let a represent any point in the common intersection of the surfaces of the cone and cylinder, $R=$ the radius of the circular section of the cone passing through the point (a), $2z=$ the common intersection of the cylindric surface and the circle whose radius $=R$, $r=$ radius of the base of the cylinder, and $r\varphi=$ the arc (of a circular section of the cylinder) intercepted between the point (a) and a plane passing through the vertex of the cone and axis of the cylinder. Then we shall obviously have

$$z=\sqrt{(R^2-r^2\sin^2\varphi)} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1).$$

Put $p=$ the distance of the vertex of the cone from the axis of the cylinder, and $m=$ the ratio of the radius of the base of the cone to its height. Then

$$R=mp-mr\cos\varphi \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2),$$

$$\therefore z=(m^2p^2-2m^2rp\cos\varphi+m^2r^2\cos^2\varphi-r^2+r^2\cos^2\varphi)^{\frac{1}{2}} \quad . \quad (3).$$

But $2rzd\varphi=ds=$ the differential of the cylindric surface,

$$\therefore ds=(4r^2m^2p^2-8r^3pm^2\cos\varphi+4m^2r^4\cos^2\varphi-4r^4+4r^4\cos^2\varphi)^{\frac{1}{2}}d\varphi (4).$$

Put $$x=\cos\varphi \therefore d\varphi=\frac{-dx}{\sqrt{(1-x^2)}} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (5),$$

$$\therefore ds=\frac{\{4r^4-4r^2m^2p^2+8r^3pm^2x-(4m^2r^4+4r^4)x^2\}^{\frac{1}{2}}dx}{\sqrt{(1-x^2)}} \quad . \quad (6).$$

Put $a=4r^4-4r^2m^2p^2$, $b=8r^3m^2$, $c=8r^4+4m^2r^4-4r^2m^2p^2$,
$d=4r^4m^2+4r^4$. (7).

Then (6) is easily changed to

$$ds = \frac{(a + bx + (a - c)x^2)dx}{(a + bx - cx^2 - bx^3 + dx^4)^{\frac{1}{2}}} \quad \ldots\ldots \quad (8).$$

We readily integrate (8) by Le Gendre's Elliptic Functions.

Hence the cylindric surface is determined.

(II.) Again, let us determine the required solid.

It is obvious that the required solid is the difference between two solids having the vertex of the cone for a common vertex, and the convex and concave surfaces of the cylinder for their bases. In either of these two solids we have

$$2z = \text{the base of a parabolic section} \quad \ldots . \quad (9).$$

Put $\text{H} =$ the height of this section, $\beta = \tan \frac{1}{2}$ the vertical angle of the cone. Then

$$\tfrac{4}{3}\beta r \text{H} z \cos \varphi d\varphi = ds' = \text{the differential of the solid} \quad \ldots \quad (10).$$

But
$$\text{H} = \frac{\sqrt{(m^2 + 1)}}{2m} \times (\text{R} + r \sin \varphi) \quad \ldots\ldots \quad (11).$$

Put $k = \dfrac{\sqrt{(m^2+1)}}{2m}$, and substituting for R and $\cos \varphi$ their values in (2) and (5) we get

$$ds' = \frac{\frac{4}{3}z(\beta k r^2 m x^2 - \beta r k m p x)dx}{\sqrt{(1 - x^2)}} - \tfrac{4}{3}z \times \beta r^2 k \times dx \quad \ldots \quad (12).$$

But the value of z in (3) may be put in the form

$$z = \frac{(m^2 p^2 - r^2) - 2m^2 r x + (m^2 r^2 + r^2)x^2}{\{(m^2 p^2 - r^2) - 2m^2 r x + (m^2 r^2 + r^2)x^2\}^{\frac{1}{2}}} \quad . . \quad (13).$$

Substituting (13) in (12) and putting a', b', c', &c. for the known co-efficients we shall have

$$ds' = \frac{(a'x + b'x^2 + c'x^3 + d'x^4)dx}{(e' + f'x + g'x^2 + h'x^3 + k'x^4)^{\frac{1}{2}}} - \frac{(l'x + m'x^2 + n'x^3)dx}{(p' + q'x + r'x^2)^{\frac{1}{2}}} \quad . \quad (14).$$

Equation (14) is integrated by *Le Gendre's Tables of Elliptic Functions*. By integrating between the necessary limits we get the two solids whose difference is the solid required.

(III.) *To find the required conic surface.* Put $\text{R}' =$ the distance of the point (a) from the vertex. Then we shall evidently have

$$\tfrac{1}{2}\text{R}' d\theta = ds'' \quad \ldots\ldots\ldots \quad (15),$$

where s'' = the cone's surface between the vertex of the cone and the convex or concave surface of the cylinder, and $\text{R}\theta$ = the arc of the *circular section of the cone* referred to above, intercepted between the point (a) and the *circular section of the cylinder* passing through the axis of the cone. Also we have

$$\text{R}' = \frac{\sqrt{m^2 + 1}}{m} . \text{R}, \text{ and } \text{R} \cos \theta = r \sin \varphi \quad \ldots . \quad (16),$$

$$\therefore d\theta = \frac{\text{R} r \cos \varphi d\varphi - r \sin \varphi d\text{R}}{\text{R}^2 \sin \theta} \quad \ldots\ldots\ldots \quad (17).$$

But
$$\beta \sin \theta = z, \text{ and } d\text{R} = - mr dx \quad \ldots\ldots \quad (18).$$

Substituting (5), (16), (17), and (18) in (15)

$$ds'' = \frac{\frac{1}{2}(\text{R}\sqrt{\frac{(m^2+1)}{m}}\, rx - \sqrt{(m^2+1)}\, r^2 + r^2\sqrt{(m^2+1)x^2})dx}{z\sqrt{(1-x^2)}} \quad (19).$$

Substituting for R and z their values, and putting a'', b'', c'', &c., for the known co-efficients we get

$$ds'' = \frac{(a'' + b''x + c''x^2)dx}{(d'' + e''x + f''x^2 + g''x^3 + h''x^4)^{\frac{1}{2}}} \quad . \; . \; . \quad (20),$$

which is integrated by Le Gendre's Tables of Elliptic Functions.

If we integrate between the necessary limits, we shall have two surfaces whose difference is the surface required.

THIRD SOLUTION. *By Professor C. Avery.*

In the first place let the cone pierce the cylinder. Let h = distance of its vertex from the convex surface of the cylinder, measured on the cone's axis, a = sine of the semi-vertical angle of the cone, a' = sine of the base angle, r = the radius of the cylinder, and φ the variable angle made by the axis of the cone and r drawn to a point made by the intersection of a plane cutting the cone parallel to one of its slant sides, and that circumference of the cylinder situated in a plane passing through the axis of the cone at right angles to the first mentioned plane. We designate the surface of the cylinder nearest to the vertex of cone, *convex*, that the most remote, *concave*. It is evident that the plane parallel to the side of the cone cuts from it a parabola, whose area multiplied by the differential of its distance from the side, and integrated between proper limits will give two solids, the smaller resting on the convex surface of the cylinder, and the larger on the concave, whose difference will be "the portion of the solid common to both" cone and cylinder. It is also evident that the base of the parabola, resting on the *convex* surface $= 2\sqrt{\frac{a^2}{a'^2}\{h + r(1 - \cos\varphi)\}^2 - r^2\sin^2\varphi}$, and its height $= \frac{1}{2a'}\{h + r(1 - \cos\varphi)\} - \frac{r}{2a}\sin\varphi$. If s = the solid, we have

$$ds = d\varphi(g - e\cos\varphi + f\cos^2\varphi)^{\frac{1}{2}}.\,(e'\cos\varphi + g'\cos^2\varphi \pm f'\cos\varphi\sin\varphi) \quad (1),$$

where

$$g = 2a^2r^2h + h^2ra^2 + a^2r^2 - a'^2r^2,\; e = \frac{2a^2r^2 + 2a^2r^2h}{a'^2},\; f = \frac{r^2}{a'^2},$$

$$e' = \tfrac{2}{3}(h + r),\; f' = \frac{2ra'}{3a}, \text{ and } g' = -\tfrac{2}{3}r. \text{ If } \cos\varphi = x, \text{ (1) becomes}$$

$$ds = \mp\frac{f'x - ef'x^2 + ff'x^3}{(g - ex + fx^2)^{\frac{1}{2}}}.\,dx$$
$$-\frac{ge'x + (gg' - ee')x^2 + (fe' - eg')x^3 + fg'x^4}{\{g - ex + (f - g)x^2 + ex^3 - fx^4\}^{\frac{1}{2}}}.\,dx \quad (2).$$

Of the two parts of (2), the integral of the *first part* is

$$\mp \left\{ \frac{f'}{f}.\sqrt{\mathrm{X}} + \frac{e}{2f}.\mathrm{X}' - ef' \left(\frac{x}{2f} + \frac{3e}{4f^2}\right)\sqrt{\mathrm{X}} + \left(\frac{3e^2}{8f^2} - \frac{g}{2f}\right)\mathrm{X}' \right.$$
$$\left. + f' \left(\tfrac{1}{3}x^2 + \frac{5ex}{12f} + \frac{5e^2}{3f^2} - \frac{2g}{3f}\right)\sqrt{\mathrm{X}} - \left(\frac{3ge}{4f^2} - \frac{5e^3}{16f^3}\right)\mathrm{X}' \right\} \quad (3),$$

where $\mathrm{X} = g - ex + fx^2$, and $\mathrm{X}' = \frac{\log\,(2fx - e + 2\sqrt{f\mathrm{X}})}{\sqrt{f}}$. The integral of the *last part* of (2) can be found by elliptic functions, since it is of the form $\frac{\mathrm{P}dx}{\mathrm{R}}$. (See *Le Gendre's Traité des Functions Elliptiques*, page 4.)

Suppose the constants to be given so as to bring $\frac{\mathrm{P}dx}{\mathrm{R}}$ under the first species, we shall have four integrals of the form $m\mathrm{F}\,(c,\ \varphi')$ the sum of which, added to (3), and limits taken will $= \mathrm{S}$. The integral of (2) must be taken between the limits $x = 1$ and $x = s$, where $s = \cos.$ of the angle made by the axis of the cone, and r drawn to the point where the slant side of the cone meets the convex surface of the cylinder, both when the signs plus and the minus are used; the sum of these two integrals is the part of the cone resting on the convex surface of the cylinder. If we take the integral of (2) in the same manner within the limits $x = s'$ to $x = -1$, we have that part of the cone which rests on the concave surface of the cylinder, which diminished by the other part will give the part included by the cylinder. When the cylinder pierces the cone, the parabola must move until it becomes a tangent to the cylindric surface; as the limits at this point are known, and attended with no difficulty, I shall not particularize them.

The differential of the surface of the cylinder included by the cone $= \frac{-(g - ex + fx^2)dx}{\{g - ex + (f - g)\,x^2 + ex^3 - fx^4\}^{\frac{1}{2}}}$, which is integrated as before. The limits of the *convex* surface are $\varphi = \cos^{-1}s$ to $\varphi = -\cos^{-1}s$. When the cylinder pierces the cone the limits are $\varphi = 0$ to $\varphi = 360^\circ$.

Again, the portion of the conic surface included by the cylinder is the difference of the surfaces of the two solids above named, resting on the concave and convex surfaces of the cylinder. The differential of this surface is

$$\frac{h + r(1 - x)}{2a'} \times \frac{-rxdx}{\sqrt{\{(1 - x^2)(\mathrm{R}^2 - r^2 + r^2x^2)\}}} \quad . \quad . \quad . \quad . \quad (4),$$

when $\mathrm{R} = \frac{a}{a'}\{h + r(1 - \cos\varphi)\}$, and $x = \cos\varphi$, φ being the same as before; hence (4) becomes by reduction,

$$\frac{r}{2a'}.\ \frac{(-hx - rx + rx^2)dx}{(g - ex + fx^2 + bx^3 - cx^4)^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

where $g = \frac{a^2}{a'^2}(h + r)^2 - r^2$, $e = \frac{2a^2r}{a'^2}(h + r)$, $f = 3r^2 - \frac{a^2}{a'^2}\,(h + r)^2$,

$$b = \frac{2a^2r}{a'^2}\,(h + r) \text{ and } c = -\frac{r^2}{a'^2}.$$

This is the same form as for the cylindric surface, and must be integrated by elliptic functions, between the same limits as for the solid.

(50). QUESTION XV. *By* ——.

Having given the magnitude of two circles; it is required to place them in such a position upon a plane, that of any given number (n) of circles, having placed the first one in any assigned position in contact with the two given ones, the second in contact with the first and also the two given ones, the third in contact with the second and also the two given ones, &c.; the last, or n^{th}, shall not only have like contact with the last but one, and the two given ones, but shall also touch the first one. To find also the position and magnitude of these tangent circles.

FIRST SOLUTION. *By Mr. C. Gill.*

Let R, r be the given radii of the circles, R being $> r$, and d the required distance of their centres. Let r_x, r_{x+1} be the radii of any two contiguous tangent circles, and φ_x, φ_{x+1} the angles made by lines drawn from the centre of r to the centres of r_x, r_{x+1} with the angular axis d. When the circle r_x touches the circles R and r, the lines $d, r+r_x$, $\text{R}-r_x$ which join their centres form a triangle, of which the angle included by the two first is φ_x, and similarly for the circle r_{x+1}; hence

$$(\text{R}-r_x)^2 = d^2 + (r+r_x)^2 - 2d(r+r_x)\cos\varphi_x \quad . \quad . \quad . \quad (1),$$

$$(\text{R}-r_{x+1})^2 = d^2 + (r+r_{x+1})^2 - 2d(r+r_{x+1})\cos\varphi_{x+1} \quad (2).$$

And since the circles r_x, r_{x+1} also touch each other, the lines $r+r_x$, $r+r_{x+1}$, r_x+r_{x+1} which join their centres and the centre of r, form a triangle, of which the angle included by the two first is $\varphi_{x+1}-\varphi_x$; hence

$$(r_x+r_{x+1})^2 = (r+r_x)^2 + (r+r_{x+1})^2 - 2(r+r_x)(r+r_{x+1})\cos(\varphi_{x+1}-\varphi_x)(3).$$

This scheme is formed for the case where the circle r is within the circle R, or $d < \text{R} - r$, and the tangent circles come between the two given ones, having convex tangency with r, and concave tangency with R; but when r is placed without R, or $d > \text{R} + r$, the tangent circles will touch the convexity of both R and r, and the distance of their centres from the centre of R will be $\text{R} + r_x$ and $\text{R} + r_{x+1}$, so that equations (1) and (2), and those resulting from them, may be adapted from this case, by writing in them $-$R for R, equation (3) remaining the same.

Develope the squares in equation (3) and divide by 2,

$$r_x r_{x+1} = r^2 + r(r_x + r_{x+1}) - (r+r_x)(r+r_{x+1})\cos(\varphi_{x+1}-\varphi_x);$$

add $r_x r_{x+1}$ to each member, and divide by 2,

$$r_x r_{x+1} = \tfrac{1}{2}(r+r_x)(r+r_{x+1})\{1-\cos(\varphi_{x+1}-\varphi_x)\}$$
$$= (r+r_x)(r+r_{x+1})\sin^2\tfrac{1}{2}(\varphi_{x+1}-\varphi_x) \quad . \quad . \quad . \quad . \quad (4).$$

Develope the squares in equation (1) and transpose,

$$2(\text{R}+r)r_x - 2d(r+r_x)\cos\varphi_x = \text{R}^2 - d^2 - r^2,$$

and adding $2(\text{R}+r)r$ to each member

$$2(\text{R}+r-d\cos\varphi_x)(r+r_x) = (\text{R}+r)^2 - d^2 \quad . \quad . \quad . \quad (5);$$

or, if we put

$$\text{R}+r+d = 2s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

this is easily transformed into

$$(s-d\cos^2\tfrac{1}{2}\varphi_x)(r+r_x) = s(s-d) \quad . \quad . \quad . \quad . \quad (7),$$

and this again into

$$s(\text{R}-s) + rd\cos^2\tfrac{1}{2}\varphi_x = (s-d\cos^2\tfrac{1}{2}\varphi_x)r_x \quad . \quad . \quad . \quad (8).$$

Multiply together equations (7) and (8), then

$$\{s(R-s)+rd\cos^2\tfrac{1}{2}\varphi_x\}(r+r_x)=s(s-d)\,r_x;$$

or, dividing by $\cos^2\frac{1}{2}\varphi_x$, and reducing

$$\{s(R-s)\tan^2\tfrac{1}{2}\varphi_x+(s-r)(s-d)\}\,(r+r_x)=s(s-d)\sec^2\tfrac{1}{2}\varphi_x\,.\,r_x,$$

or if we put, for brevity,

$$c^2=\frac{(s-r)(s-d)}{s(R-s)}=\frac{R^2-(r-d)^2}{R^2-(r+d)^2}\quad\ldots\quad(9),$$

this may be written

$$(R-s)(\tan^2\tfrac{1}{2}\varphi_x+c^2)(r+r_x)=(s-d)r_x\sec^2\tfrac{1}{2}\varphi_x\quad.\quad(10).$$

In like manner from equation (2) we find

$$(R-s)(\tan^2\tfrac{1}{2}\varphi_{x+1}+c^2)(r+r_{x+1})=(s-d)\,r_{x+1}\sec^2\tfrac{1}{2}\varphi_{x+1}\;(11);$$

and multiplying together the three equations (4), (10), (11), we find

$$(R-s)^2\{\tan^2\tfrac{1}{2}\varphi_x\tan^2\tfrac{1}{2}\varphi_{x+1}+c^2(\tan^2\tfrac{1}{2}\varphi_x+\tan^2\tfrac{1}{2}\varphi_{x+1})+c^4\}$$
$$=(s-d)^2(\tan\tfrac{1}{2}\varphi_{x+1}-\tan\tfrac{1}{2}\varphi_x)^2.$$

Multiply this equation by s and take $c^2s(R-s)^2(\tan\frac{1}{2}\varphi_{x+1}-\tan\frac{1}{2}\varphi_x)^2$ from each member, then because

$$s(s-d)^2-c^2s(R-s)^2=s(s-d)^2-(R-s)(s-r)(s-d)=Rr(s-d),$$

it becomes

$$s(R-s)^2\{\tan^2\tfrac{1}{2}\varphi_x\tan^2\tfrac{1}{2}\varphi_{x+1}+2c^2\tan\tfrac{1}{2}\varphi_{x+1}\tan\tfrac{1}{2}\varphi_x+c^4\}$$
$$=Rr\,(s-d)(\tan\tfrac{1}{2}\varphi_{x+1}-\tan\tfrac{1}{2}\varphi_x)^2;$$

and extracting the square root,

$$(R-s)\sqrt{s}.\,(\tan\tfrac{1}{2}\varphi_{x+1}\tan\tfrac{1}{2}\varphi_x+c^2)=\pm(\tan\tfrac{1}{2}\varphi_{x+1}-\tan\tfrac{1}{2}\varphi_x)\sqrt{Rr(s-d)}\;(12),$$

In this equation the two signs belong to the two circles which may be made to touch the circle r_x, one on each side of it; but as we shall make our angular magnitude increase in the same direction from the angular axis d, we need only take one of the two signs; thus if the angles (φ) increase in the order of the numbers x, $x+1$, &c., the upper one must be taken. Then assuming

$$\tan\tfrac{1}{2}\varphi_{x+1}=c\tan P_{x+1},\ \tan\tfrac{1}{2}\varphi_x=c\tan P_x\quad.\ .\ .\quad(13),$$

equation (12), after dividing it by $c\sqrt{s-d}$, and observing that $c(R-s)\sqrt{\dfrac{s}{s-d}}=\sqrt{(R-s)(s-r)}$, becomes

$$(\tan P_{x+1}\tan P_x+1)\sqrt{(R-s)(s-r)}=(\tan P_{x+1}-\tan P_x)\sqrt{Rr},$$

$$\text{or,}\quad\frac{\tan P_{x+1}-\tan P_x}{1+\tan P_{x+1}\tan P_x}=\tan(P_{x+1}-P_x)=\sqrt{\frac{(R-s)(s-r)}{Rr}}\quad.\ .\ (14).$$

If we assume again an auxiliary angle, β, such that

$$\tan\beta=\sqrt{\frac{(R-s)(s-r)}{Rr}}=\sqrt{\frac{(R-r)^2-d^2}{4Rr}}\quad.\ .\ .\ (15),$$

equation (14) will give

$$P_{x+1}-P_x=\Delta P_x=\beta\quad\ldots\ldots\quad(16),$$

Δ being the symbol of finite differences; and the integral of this equation is

$$P_x=x\beta+\theta\quad\ldots\ldots\ldots\quad(17),$$

and therefore, from equation (13),

$$\tan\tfrac{1}{2}\varphi_x=c\tan(x\beta+\theta)\quad\ldots\ldots\quad(18),$$

where θ is a constant quantity to be determined from the position assigned to the first of the tangent circles, and as this circle must touch the two given circles after they are placed as required, its position may be made

to depend wholly on the angle φ_1 which the radius vector of its centre makes with d; if we put $x=1$ in equation (18), we have

$$\tan \tfrac{1}{2}\varphi_1 = c \tan (\beta + \theta),$$

therefore,
$$\tan(\beta+\theta) = \frac{\tan \frac{1}{2}\varphi_1}{c} = \tan \tfrac{1}{2}\varphi_1 . \sqrt{\frac{\text{R}^2-(r+d)^2}{\text{R}^2-(r-d)^2}} \quad (19),$$

from which θ may be found.

By substituting (18) in (8) we find

$$\begin{aligned} r_x &= \frac{s(\text{R}-s)+rd\cos^2\frac{1}{2}\varphi_x}{s-d\cos^2\frac{1}{2}\varphi_x} \\ &= \frac{s(\text{R}-s)(\tan^2\frac{1}{2}\varphi_x + c^2)}{s\tan^2\frac{1}{2}\varphi_x + s - d} \\ &= \frac{c^2 s(\text{R}-s)\sec^2(x\beta+\theta)}{c^2 s\tan^2(x\beta+\theta)+s-d} \\ &= \frac{(s-r)(\text{R}-s)}{(s-r)\sin^2(x\beta+\theta)+(\text{R}-s)\cos^2(x\beta+\theta)} \\ &= \frac{2(s-r)(\text{R}-s)}{\text{R}-r-d\cos 2(x\beta+\theta)} \quad \ldots\ldots \quad (20). \end{aligned}$$

Moreover, if we put $r+r_x = v_x$ the radius vector of the centre of the x^{th} circle, we shall have

$$v_x = r + \frac{2(s-r)(\text{R}-c)}{\text{R}-r-d\cos 2(x\beta+\theta)} \quad \ldots\ldots \quad (21).$$

Equations (18), (20) and (21) completely determine the magnitude and position of a series of circles successively touching each other, and also two given circles placed in any position, either wholly within or wholly without each other, requiring merely a change in the sign of R in the latter case. When they intersect, that is, when $d > \text{R} - r$ and $< \text{R} + r$, the expressions for c and $\tan\beta$ in (11) and (15) become imaginary; hence, by transforming the circular functions in this investigation into analogous exponential ones, we should get a law for the angles and radii quite as covenient as the preceding: as, however, from simple geometrical considerations, the case could not be applied to the present question, it would burthen this solution too much to insert the extremely neat results I have obtained.

Now if, in equation (18), we imagine x to be a continuous quantity, varying through all magnitudes from 1 to ∞, it is evident that the angles $\frac{1}{2}\varphi_x$ and $x\beta+\theta$ would describe the same number of right angles, by the variation of x, because their tangents would pass through the magnitudes 0 and ∞ at the same time; hence when the angle φ_x has described i complete revolutions, i being any whole number, that is when

$$\varphi_x - \varphi_1 = i.2\pi, \text{ or } \tfrac{1}{2}\varphi_x = i\pi + \tfrac{1}{2}\varphi_1 \quad \ldots\ldots \quad (22),$$
$$\therefore \tan\tfrac{1}{2}\varphi_x = \tan\tfrac{1}{2}\varphi_1, \text{ and } \tan(x\beta+\theta) = \tan(\beta+\theta),$$

we shall also have

$$x\beta+\theta = i\pi+\beta+\theta,$$
or
$$(x-1)\beta = i\pi \quad \ldots\ldots\ldots \quad (23).$$

If, at the same time, $\dfrac{i\pi}{\beta}$ be a whole number, x will be a whole number, and φ_x will be an angle corresponding to the centre of one of the tangent

circles, the x^{th} in order from the first, and by (20), (21) and (22), coincident with it in all respects. Now whatever be the angle β, i may be so assumed that $\frac{i\pi}{\beta}$ will differ from a whole number by less than any assignable difference, and therefore in whatever position we place the two given circles, if we continue to draw circles in the manner described in the question, their surfaces spreading round and round between the two given circumferences, we shall at last find one precisely coincident with the first circle. Moreover, as this is true, independently of the angle φ_1 or θ, the coincidence will take place, for the same position of the given circles, after the same number of circles are described, and after the same number of revolutions of their surfaces, wherever the first tangent circle is placed.

The question requires that the $(n+1)^{th}$ circle shall be coincident with the first, or that $\varphi_{n+1} - \varphi_1 = i.2\pi$; therefore writing $n+1$ for x in equation (23), we find

$$n\beta = i\pi, \text{ or } \beta = \frac{i\pi}{n}, \text{ and } \tan\beta = \tan\frac{i\pi}{n} \quad . \; . \; . \; . \; . \; . \quad (24).$$

By substituting this in (15), squaring and reducing

$$(\text{R} - r)^2 - d^2 = 4\text{R}r\tan^2\frac{i\pi}{n},$$

$$\therefore\; d^2 = (\text{R} - r)^2 - 4\text{R}r\tan^2\frac{i\pi}{n} \quad . \; . \; . \; . \; . \; . \quad (25),$$

which determines the position of the given circles.

We know that $\tan^2\frac{n+i}{n}.\pi = \tan^2\frac{n-i}{n}.\pi = \tan^2\frac{i}{n}.\pi$, and therefore we shall find all the independent solutions that can be had from (25), by taking $i=1, 2, 3,$ &c. to $\frac{1}{2}n-1$ if n be even, or to $\frac{1}{2}(n-1)$ if n be odd; that is, there are $\frac{1}{2}n-1$ or $\frac{1}{2}(n-1)$ particular solutions comprised in (25). These solutions however may not all be possible, for from (25) we must have

$$(\text{R} - r)^2 - 4\text{R}r\tan^2\frac{i\pi}{n} > 0,$$

$$\therefore\; \frac{\text{R}}{r} > \cot^2\frac{n-2i}{4n}.\pi \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (26).$$

Hence if $\frac{\text{R}}{r} > \cot^2\frac{\pi}{2n}$ when n is even, or $> \cot^2\frac{\pi}{4n}$ when n is odd, all the solutions will be possible, and if $\frac{\text{R}}{r} < \cot^2\frac{n-2i}{n}.\pi$, none of them are possible. If $\frac{\text{R}}{r} = \cot^2\frac{n-2k}{4n}.\pi$, k being any whole number, the solution arising from taking $i = k$ in (25) will give $d = 0$, or the given circles must be concentric, and for this case equation (20) becomes

$$r_x = \tfrac{1}{2}(\text{R} - r) = \frac{\text{R}\sin\frac{k\pi}{n}}{1+\sin\frac{k\pi}{n}} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (27),$$

or the tangent circles are all equal.

I have hitherto supposed the circle r to be within the circle R, but if we suppose it to be without, by changing the sign of R in (25),

$$d^2 = (R + r)^2 + 4Rr \tan^2 \frac{i\pi}{n} \quad \text{. . . . .} \quad (28),$$

which is possible for every real value of R and r, and therefore it always comprises $\frac{1}{2}n - 1$ or $\frac{1}{2}(n - 1)$ particular solutions, according as n is even or odd; so that whatever be the relative magnitude of two given circles, they can always be placed in $\frac{1}{2}n - 1$ or $\frac{1}{2}(n - 1)$ positions, and their relative magnitude may be such that they can be placed in $n - 2$ or $n - 1$ different positions, according as n is even or odd, in any of which they will fulfil the conditions as the question.

But there may be one or more circles before the n^{th} which touches the first circle. For if $n = km$, k and m being whole numbers, then when $i = i'k$, i' being a whole number, $\beta = \frac{i'\pi}{m}$, or every m^{th} circle will touch the first one, and when $i = i'm$, $\beta = \frac{i'\pi}{k}$, or every k^{th} circle will touch the first one; in the first case every period of m circles, and in the second every period of k circles would be entirely coincident. But if n be a prime number, or if i be prime to n, then the n^{th} circle is the first in order that touches the first circle.

In the solutions which are had from (25), the n tangent circles touch each other's convexity, and their surfaces, for any value of i within the proper limits, spread i times round between the two given circumferences; but in the solutions deduced from (28) this is not necessarily the case. The angle $\varphi_{n+1} - \varphi_1$ is still $= i \, . \, 2\pi$, or the radius vectors of the centres of the n successive tangent circles embrace an angular space of i times four right angles, but equation (20) becomes for this case

$$r_x = \frac{1}{2} . \frac{d^2 - (R + r)^2}{R + r + d \cos 2(x\beta + \theta)} \quad \text{. . . . .} \quad (29);$$

and since $d > R + r$, x may be such that

$$R + r + d \cos 2(x\beta + \theta) < 0 \quad \text{. . . . . . .} \quad (30),$$

and then r_x will be negative, and therefore that circle will be touched by the given ones and the two tangent circles in contact with it on its concave side. In order the better to examine how this circumstance takes place, since $d > R + r$, put

$$\frac{R + r}{d} = \cos \omega \quad \text{. . . . .} \quad (31);$$

then substituting this in (15), which becomes for this case

$$\tan^2 \beta = \frac{d^2 - (R + r)^2}{4Rr},$$

we find $\tan^2 \beta = \frac{d^2 \sin^2 \omega}{4Rr}$, and $\sin^2 \omega = \frac{4Rr}{d^2} . \tan^2 \beta$,

But $\frac{4Rr}{d^2} < \frac{(R+r)^2}{d^2} < \cos^2 \omega$,

$$\therefore \sin^2 \omega < \cos^2 \omega \tan^2 \beta, \text{ or } \tan^2 \omega < \tan^2 \beta$$

$$\therefore \omega < \beta \quad \text{. . . . .} \quad (32)$$

Equation (29) also becomes

$$r_x = \frac{\frac{1}{2}d \sin^2 \omega}{\cos \omega + \cos 2(x\beta + \theta)} \quad \ldots \ldots \ldots \quad (33),$$

and the denominator of this fraction will be negative when

$$2x\beta + 2\theta \text{ is} > (2k+1)\pi - \omega \text{ and} < (2k+1)\pi + \omega \quad . \quad . \quad (34),$$

k being any whole number. Now the angle $2x\beta + 2\theta$, between two integral values of x comprehends the magnitude 2β, while the two limits in (34) contain between them only the angle 2ω, which is less than the former one, by (32), and therefore there can but be one integral value of x fulfilling the condition in (34) for every separate value of k, that is for every magnitude 2π that the angle $2x\beta + 2\theta$ passes through; hence, for every particular value of i in (28), there may be i tangent circles, touched on their concave side by the given circles, and no more. But there may not be so many as i; for if, for any integral value of x, $2x\beta + 2\theta$ be less than $(2k+1)\pi - \omega$ and yet differing from it by a quantity less than $2\beta - 2\omega$, then for the next integral value of x, $2(x+1)\beta + 2\theta$ will be $> (2k+1)\pi + \omega$, and the two radii corresponding to these angles will be both positive. Two circles posited in this manner have not their point of mutual contact situated between the two given circles, like those which have the angles $2x\beta + 2\theta$ and $2(x+1)\beta + 2\theta$ corresponding to them between the magnitudes $(2k+1)\pi + \omega$ and $(2k+3)\pi - \omega$, but they embrace the two given circles between their convexities, on the same side of their point of contact.

EXAMPLE 1.

Let $n = 3$, then i can only $= 1$, or $\beta = \frac{1}{3}\pi$, and $\tan \frac{1}{3}\pi = 3$; by (25) and (28)

$$1. \quad d^2 = \text{R}^2 - 14\text{R}r + r^2,$$
$$2. \quad d^2 = \text{R}^2 + 14\text{R}r + r^2,$$

which give two different positions in which the circles may be placed to fulfil the required conditions. In the first the circle r will be within the circle R, and it will only be possible when

$$\frac{\text{R}}{r} = \text{or} > \cot^2 \tfrac{1}{12}\pi = \text{or} > 7 + 4\sqrt{3}.$$

In the second the circle r will be without the circle R, and it will be possible for all values of R and r. The position and magnitude of the second and third tangent circles are had from (18) and (20), after determining θ from equation (19), and writing in it the values of β and d.

EXAMPLE 2.

Let $n = 5$, then $i = 1$ or $i = 2$, and $\beta = \frac{1}{5}\pi$ or $\frac{2}{5}\pi$; $\tan^2 \frac{1}{5}\pi = 5 - 2\sqrt{5}$ and $\tan^2 \frac{2}{5}\pi = 5 + 2\sqrt{5}$, therefore (25) and (28) become

$$1. \quad d^2 = \text{R}^2 + r^2 - 2(11 - 4\sqrt{5})\,\text{R}r,$$
$$2. \quad d^2 = \text{R}^2 + r^2 - 2(11 + 4\sqrt{5})\,\text{R}r,$$
$$3. \quad d^2 = \text{R}^2 + r^2 + 2(11 - 4\sqrt{5})\,\text{R}r,$$
$$4. \quad d^2 = \text{R}^2 + r^2 + 2(11 + 4\sqrt{5})\,\text{R}r.$$

which give four different positions in which the circles may be placed to fulfil the required conditions. In the first and second the circle R will envelope r, the first being possible only when

$$\frac{R}{r} = \text{or} > 11 - 4\sqrt{5} + 2\sqrt{50 - 22\sqrt{5}},$$

and both, only when

$$\frac{R}{r} = \text{or} > 11 + 4\sqrt{5} + 2\sqrt{50 + 22\sqrt{5}}.$$

In the third and fourth r will be without R, and they will both be possible in all cases. In the first and third solutions, the five tangent circles will go once round before touching; in the second and fourth, they will spread twice round between the two given circumferences, the third, for instance, intersecting the first, the fourth the second and third, and the fifth touching the first.

Remark 1. I have thus made the solution of the question to depend in all respects on the division of the semicircumference of a circle into n equal parts, and when this known problem can be performed geometrically, the question itself can be done so. The mode of applying one solution to the other, is sufficiently obvious from equations (25) and (28).

Remark 2. In order to make n circles touch two circles, given in magnitude and position, there would be n equations, such as (1) and (2) between their radii $r_1, r_2, \ldots\ldots r_n$ and the angles $\varphi_1, \varphi_2, \ldots\ldots \varphi_n$; and to make each of these circles touch two of the others, there would be n more equations such as (3) between these quantities. Hence, to find the $2n$ unknown quantities determining the position and magnitude of n such circles, we should have $2n$ equations, and these would be sufficient to determine them were the equations altogether independent. But such is not the case, since by eliminating the unknown $2n$ quantities $r_1, r_2 \ldots r_n$, $\varphi_1, \varphi_2 \ldots \varphi_n$ among these equations, there would result the relation (25) or (28), accordingly as $d < R - r$ or $d > R + r$, among the three given things R r and d. Therefore, unless such a relation has place among R, r, d, the n circles could not be made to touch the two given ones and each other, two and two. Moreover, if this relation existed, the $2n$ equations, not being independent of each other, would not fix the magnitude of the $2n$ unknown quantities, although, as we have seen, $2n - 1$ of them can be made to depend upon the $(2n)^{th}$. If this $2n^{th}$ quantity is the one determining the position of the centre of the first tangent circle as we have made it, and the problem can be solved for one such position; then, wherever that circle may be placed, touching the two given circumferences, the n^{th} circle will still touch the first; in other words, the condition of the n^{th} circle touching the first one, does not depend on the position of the first tangent circle, but on that of the given ones. It is therefore evident that question 269 of the Mathematical Diary, is either impossible or indeterminate.

SECOND SOLUTION. *By Professor Benjamin Peirce, Cambridge, Mass.*

Let R = the radius of the smaller of the given circles,
R$'$ = that of the other,
r = that of one of the required circles,
r' = that of the succeeding circle,
a = the distance between the centres of the given circles,

$\varphi =$ the angle made by the line joining the centres of r and R with that joining the centres of R and R',

$\varphi' =$ corresponding angle for r';

and we have

$$\left.\begin{array}{r}(R+r)^2+a^2-2a(R+r)\cos\varphi=(R'-r)^2,\\ (R+r')^2+a^2-2a(R+r')\cos\varphi'=(R'-r')^2,\\ (R+r)^2+(R+r')^2-2(R+r)(R+r')\cos(\varphi'-\varphi)=(r+r')^2,\end{array}\right\}\ .\ (1);$$

Hence $\quad (R+r)(R+r')\cos(\varphi'-\varphi)=R^2+R(r+r')-rr'$,

and, by transposition

$$(R+r)(R+r')\sin\varphi\sin\varphi'=R^2+R(r+r')-rr'-(R+r)(R+r')\cos\varphi\cos\varphi'$$
$$=(R+r)(R+r)(1-\cos\varphi\cos\varphi')-2rr';$$

the square of which is

$$(R+r)^2(R+r')^2(1-\cos^2\varphi)(1-\cos^2\varphi')=(R+r)^2(R+r')^2(1-\cos^2\varphi\cos^2\varphi')^2$$
$$-4rr'(R+r)(R+r')(1-\cos\varphi\cos\varphi')+4r^2r'^2,$$

and, by transposition

$$(R+r)^2(R+r')^2(\cos^2\varphi+\cos^2\varphi'-2\cos\varphi\cos\varphi')$$
$$=4(R+r)(R+r')rr'(1-\cos\varphi\cos\varphi')-4r^2r'^2,$$

which multiplied by $4a^2$, becomes,

$$4a^2(R+r)^2(R+r')^2(\cos\varphi-\cos\varphi')^2$$
$$=16a^2(R+r)(R+r')rr'(1-\cos\varphi\cos\varphi')-16a^2r^2r'^2\ (2).$$

But the first of equations (1) being multiplied by $(R+r')$, and the second by $(R+r)$, give by transposition,

$$2a(R+r)(R+r')\cos\varphi=(a^2+R^2-R'^2)(R+r')+2(R+R')(Rr+rr'),$$
$$2a(R+r)(R+r')\cos\varphi'=(a^2+R^2-R'^2)(R+r)+2(R+R')(Rr'+rr'),$$

which become, by putting

$$S=R+R',\ D=R'-R,$$
$$2a(R+r)(R+r')\cos\varphi=(a^2-SD)(R+r')+2S(Rr+rr'),$$
$$2a(R+r)(R+r')\cos\varphi'=(a^2-SD)(R+r)+2S(Rr'+rr'),$$

the difference of which is

$$2a(R+r)(R+r')(\cos\varphi-\cos\varphi')=(-a^2+SD+2SR)(r-r')=(S^2-a^2)(r-r'),$$

and squaring

$$4a^2(R+r)^2(R+r')^2(\cos\varphi-\cos\varphi')^2=(S^2-a^2)^2(r-r')^2,$$

Hence, by equation (2),

$$(S^2-a^2)^2(r-r')^2=16a^2(R+r)(R+r')rr'(1-\cos\varphi\cos\varphi')-16a^2r^2r'^2\ (3).$$

Again,

$$16a^2(R+r)(R+r')rr'-16a^2r^2r'^2=16a^2(R^2+R(r+r')rr'),$$

and from equations (1)

$$-16a^2(R+r)(R+r')rr'\cos\varphi\cos\varphi'=-4rr'(a^2-SD+2Sr)(a^2-SD+2r'S)$$
$$=-4(a^2-SD)^2rr'-8(a^2-SD)S(r+r')rr'-16S\ r^2r'^2.$$

The sum of these three equations is, by the suppression of terms, common to both members,

$$(S^2-a^2)^2(r-r')^2=4(S^2-a^2)(a^2-D^2)rr'+8D(S^2-a^2)(r+r')rr'-16S^2r^2r'^2.$$

Hence, putting

$$\sin^2\tfrac{1}{2}\theta=\frac{D^2-a^2}{S^2-a^2},$$
$$B=\frac{8D}{S^2-a^2},$$
$$C=\frac{4S}{S^2-a^2},$$

the preceding equation becomes by dividing by $s^2 - a^2$

$$(r-r')^2 = -4\sin^2\tfrac{1}{2}\theta rr' + \text{B}(r+r')rr' - \text{C}^2 r^2 r'^2,$$

and dividing by $r^2 r'^2$, and transposing,

$$\left(\frac{1}{r'} - \frac{1}{r}\right)^2 + 4\sin^2\tfrac{1}{2}\theta \cdot \frac{1}{rr'} - \text{B}\left(\frac{1}{r'} + \frac{1}{r}\right) + \text{C}^2 = 0.$$

Hence if we represent the reciprocals of the radii of the successive circles by

$$x_0, x_1, x_2, x_3 \ldots\ldots\ldots x_{n-1}, x_n, x_{n+1}, \text{\&c.},$$

we have

$$(x_n - x_{n-1})^2 + 4\sin^2\tfrac{1}{2}\theta x_n x_{n-1} - \text{B}(x_n + x_{n-1}) + \text{C}^2 = 0,$$

and, in the same way,

$$(x_{n-1} - x_{n-2})^2 + 4\sin^2\tfrac{1}{2}\theta x_{n-1} x_{n-2} - \text{B}(x_{n-1} + x_{n-2}) + \text{C}^2 = 0;$$

the difference between which, gives either

$$x_n - x_{n-2} = 0,$$

or

$$x_n + x_{n-2} - 2\cos\theta x_{n-1} - \text{B} = 0.^*$$

But it is found that the values of x_n, x_{n-1}, &c., which satisfy the equation $x^n - x_{n-2} = 0$, also satisfy this other equation, so that it is universally applicable. Hence x_0, x_1, x_2, &c., are evidently the co-efficients of t, in the development of the fraction,

$$\frac{x_0 + (x_1 - 2\cos\theta x_0)t + \text{B}(t^2 + t^3 + t^4 + \text{\&c.})}{1 - 2\cos\theta t + t^2} = x_0 + x_1 t + x_2 t^2 + \text{\&c.} \quad (4).$$

But $1 - 2\cos\theta t + t^2 = (c^{\theta\sqrt{-1}} - t)(c^{-\theta\sqrt{-1}} - t) = (b-t)(e-t)$,

having, hyp. log. $c = 1, b = c^{\theta\sqrt{-1}}, e = c^{-\theta\sqrt{-1}}$.

Now

$$(b-t)^{-1} = b^{-1} + b^{-2}t + b^{-3}t^2 + \text{\&c},$$
$$(e-t)^{-1} = e^{-1} + e^{-2}t + e^{-3}t^2 + \text{\&c.};$$

And

$$(1 - 2\cos\theta t + t^2)^{-1} = (b-t)^{-1}.(e-t)^{-1}$$
$$= b^{-1}e^{-1} + b^{-2}e^{-2}(b+e)t + b^{-3}e^{-3}(b^2 + be + e^2)t^2 + \text{\&c.}$$
$$\ldots\ldots + b^{-m-1}e^{-m-1}(b^m + b^{m-1}e + \text{\&c.}\ldots)t^m.$$

But since $be = 1$, the general co-efficient of this developement becomes

$$b^m + b^{m-1}e + \text{\&c.}\ldots e^m = \frac{b^{m+1} - e^{m+1}}{b-e} = \frac{\sin(m+1)\theta}{\sin\theta},$$

which substituted in equation (4), gives

$$x_0 + x_1 t + x_2 t^2 + \text{\&c.}$$
$$= \{x_0 + (x_1 - 2\cos\theta x_0)t - \text{B}(t^2 + t^3 + \text{\&c.})\}.\frac{\sin\theta + \sin 2\theta.t + \text{\&c.}}{\sin\theta}$$
$$= \{x_0 - \text{B} + (x_1 - 2\cos\theta x_0 - \text{B})t\}.\frac{\sin\theta + \sin 2\theta t + \text{\&c.}}{\sin\theta}$$
$$+ \text{B}(1 + t + t^2 + t^3 + \text{\&c.}).(\sin\theta)^{-1}.(\sin\theta + \sin 2\theta t + \text{\&c.}) \quad . \quad (5).$$

But the co-efficient of t^n in

$$(1 + t + t^2 + \text{\&c.})(\sin\theta + \sin 2\theta t + \sin 3\theta t^2 + \text{\&c.}),$$

is $\sin\theta + \sin 2\theta + \ldots\ldots + \sin(n+1)\theta = \dfrac{\cos(n+\frac{3}{2})\theta - \cos\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}$,

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* This equation might also be solved, by the usual process, for an equation of finite differences. Professor Peirce will excuse our changing his notation, the printer not having type proper for it.

which substituted in equation (5), gives

$$x_n = \frac{\sin(n+1)\theta(x_0-\text{B})+(x_1-2\cos\theta x_0-\text{B})\sin n\theta}{\sin\theta} + \text{B}.\frac{\cos(n+\frac{1}{2})\theta-\cos\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta\sin\theta}.$$

But if there are only m circles, it must be the case that

$$x_m = x_0,\ x_{m+1} = x_1, \ldots\ldots\ x_{m+n} = x_n.$$

But $0 = x_{m+n} - x_n = \dfrac{2\sin\frac{1}{2}m\theta}{\sin\theta}.\{\cos(\frac{1}{2}m+n+1)\theta(x_0-\text{B})$

$$+\cos(\tfrac{1}{2}m+n)(x_1-2\cos\theta x_0-\text{B})\} - \frac{\sin\frac{1}{2}m\theta\sin(\frac{1}{2}m+n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta\sin\theta}.\ (6),$$

which is always satisfied when $\sin\frac{1}{2}m\theta = 0$, or $\theta = \dfrac{i\pi}{m}$, π being a circumference. Whence, if θ, determined from the equation

$$\sin\tfrac{1}{2}\theta = \sqrt{\frac{\text{D}^2-a^2}{\text{S}^2-a^2}},$$

is such an angle as to be commensurate with the circumference, the problem is possible, and the number of circles is equal to the quotient of the circumference, divided by the greatest common divisor of θ and the circumference.

When $a=0$, we have $\sin\frac{1}{2}\theta = \dfrac{\text{D}}{\text{S}}$, and θ is the angle subtended by the lines joining the centres of two circles, as r and r', to the common centre of R and R'; so that if $\text{R}'=3\text{R}$, we have

$$\sin\tfrac{1}{2}\theta = \tfrac{1}{2},\ \theta = 60^\circ,\ m = 6,$$

or there are six circles. And the other factor gives

$$2\cos(\tfrac{1}{2}m+n+1)\theta(x_0-\text{B}) + 2\cos(\tfrac{1}{2}m+n)(x_1-2\cos\theta x_0-\text{B})$$
$$-\frac{\text{B}\sin(\frac{1}{2}m+n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta} = 0.$$

Or, if x_0 and x_1 are regarded as belonging to the given circles, we have $x_1 - 2\cos\theta x_0 - \text{B} = -x_{-1}$, whence

$$2\cos(\tfrac{1}{2}m+n+1)\theta x_0 - 2\cos(\tfrac{1}{2}m+n)x_{-1} = \text{B}\cot\tfrac{1}{2}\theta\sin(\tfrac{1}{2}m+n+1)\theta$$
$$+3\text{B}\cos(\tfrac{1}{2}m+n+1)\theta,$$

$$x_0 - \frac{\cos(\frac{1}{2}m+n)\theta}{\cos(\frac{1}{2}m+n+1)\theta}.x_{-1} = \tfrac{1}{2}\text{B}\cot\tfrac{1}{2}\theta\tan(\tfrac{1}{2}m+n+1)\theta + \tfrac{3}{2}\text{B},$$

and from this equation, or the equation

$$x_0^2 + x^2_{-1} - 2\cos\theta x_0 x_{-1} - \text{B}(x_0+x_{-1}) + \text{C}^2 = 0,$$

the values of x_0 and x_{-1} are determined, and their position is obtained from the equations (1).

—— The relation between the angles φ_x and φ_{x+1} of two successive circles which may be stated thus, in the notation of the first solution, $(\text{R}-s)^2(\tan^2\frac{1}{2}\varphi_x+c^2)(\tan^2\frac{1}{2}\varphi_{x+1}+c^2) = (s-d)^2(\tan\frac{1}{2}\varphi_{x+1}-\tan\frac{1}{2}\varphi_x)^2$, is reduced, by Dr. Strong, in the following ingenious manner:

Put $\tan\frac{1}{2}\varphi_x = c\tan\psi_x$, $\tan\frac{1}{2}\varphi_{x+1} = c\tan\psi_{x+1}$, then will

$$\tan^2\tfrac{1}{2}\varphi_x + c^2 = \frac{c^2}{\cos^2\psi_x},\ \tan^2\tfrac{1}{2}\varphi_{x+1} + c^2 = \frac{c^2}{\cos^2\psi_{x+1}},$$

$\tan\frac{1}{2}\varphi_{x+1} - \tan\frac{1}{2}\varphi_x = \dfrac{c\sin(\psi_{x+1}-\psi_x)}{\cos\psi_{x+1}\cos\psi_x}$, so that by substitution and mul-

tiplying by $\cos^2\psi_{x+1}\cos^2\psi_x$, there arises

$$\sin(\psi_{x+1}-\psi_x)=c^2.\frac{(R-s)^2}{(s-d)^2}.$$

It thus appears that

$$\sin(\psi_2-\psi_1)=\sin(\psi_3-\psi_2)=\sin(\psi_4-\psi_3)=\&c.$$

"And if we put $\psi_2-\psi_1=2t'\pi+v$, we shall have $\psi_3-\psi_2=2t''\pi+v$, $\psi_4-\psi_3=2t'''\pi+v$, &c.., $\psi^n{}_{+1}-\psi_n=2t_n\pi+v$, where $t', t''\ldots t_n$ denote integers, 0 being included; by adding these equations $\psi^n{}_{+1}-\psi_1=(t'+t''+t'''+\&c.).2\pi+nv$; but since $\psi_{n+1}=2m'\pi+\psi_1$, we have $\psi_{n+1}-\psi_1=2m'\pi$, $\therefore$ put $m'-(t'+t''+\&c.)=a$, and we have $nv=2a\pi$, or $v=\dfrac{2a\pi}{n}$, where $a=$ an assumed integer." He then proceeds to give conclusions adapted to the different cases of the question. The analysis of Professor Catlin is also very elegant, and he exhibits a final equation in which the only unknown quantity is the required distance between the centres of the given circles.

That nothing may be left in doubt with regard to this interesting question, we would remark that the condition of equality of the radii of two tangent circles, as the x_{th} and y^{th}, will give by using their values in equation (20) of the first solution

$$\frac{1}{r_x}-\frac{1}{r_y}=\frac{d}{2(s-r)(R-s)}\{\cos 2(y\beta+\theta)-\cos 2(x\beta+\theta)\}$$
$$=\frac{d}{(s-r)(R-s)}.\sin(x-y)\beta.\sin(\overline{x+y}.\beta+2\theta),$$

and Professor Peirce's equation (6), may be reduced to this form by substituting in it the relation he has found to exist between x_1 and x_0. This equation is solved by making

1°. $\sin(x-y)\beta=0$, which, when $x-y=n$, is the same as equation (24).

2°. $\sin((x+y)\beta+2\theta)=0$, or $(x-y)\beta+2\theta=i\pi$.

The first of these conditions gives, as we have seen, not only equality between the circles, but absolute coincidence. But if we substitute the second condition in equation (18), we shall find, whatever i may be,

$$\tan\tfrac{1}{2}\varphi_x=-\tan\tfrac{1}{2}\varphi_y,$$

showing that the equal circles are situated on different sides of the angular axis, and therefore not coincident. For the rest, this condition shows that, two circles being given in magnitude and position upon a plane, we can place the first tangent circle so that the others being described as in the question, the x^{th} tangent circle may be equal to the y^{th}; although we could not, generally, under the same circumstances, place it so that the x_{th} should be *coincident* with the y_{th}.

List of Contributors and of Questions answered by each. The figures refer to the number of the Question, as marked in Number II., Art. VIII.

LYMAN ABBOT, jr., Niles, Cayuga Co., N. Y., ans. 1, 2, 3, 4, 5, 6, 7. 10, 11, 12, 13.
Prof. C. AVERY, Hamilton College, Clinton, N. Y., ans. all the Questions.
P. BARTON, jr., South Orange, Mass., ans. 1, 2. 4, 5, 6, 7.
Prof. F. N. BENEDICT, University Vt. Burlington, ans. 2, 3, 4, 5. 10.
B. BIRDSALL, Clinton Liberal Institute, Clinton, N. Y., ans. 1, 2. 4, 5, 6. 8.
Prof. M. CATLIN, Hamilton College, Clinton, N. Y., ans. all the Questions.
J. KETCHUM, Principal of Gaines Academy, Orleans Co., N. Y., ans. 1, 4. 6.
JAMES F. MACULLY, New-York, ans. 1 to 13 inclusive.
P. ans. 10.
Prof. B. PEIRCE, Harvard University, Cambridge, ans. all the Questions.
O. ROOT, Math. Tutor, Hamilton College, ans. all the Questions.
Prof. T. STRONG, L. L. D., Rutgers' College, N. B., ans. all the Questions.
N. VERNON, Frederick, Md., ans. 1. 3, 4, 5, 6. 9.

⁂ All communications for No. V., which will be published on the first of May, 1838, must be post paid, addressed to the Editor, at the Institute, Flushing, L. I.; and must arrive before the first of February, 1838. New Questions must be accompanied with their solutions.

☞ We should have inserted the question proposed by our ingenious correspondent, Mr. L. ABBOT, but were afraid that its solution would involve principles far beyond the present grasp of the mathematical sciences.

⁂ Those of our subscribers, who find it more convenient, may pay their subscriptions to Mr. Wm. Jackson, Bookseller, Broadway, New-York.

NEW BOOKS.

I. "An Elementary Treatise on SOUND; being the second volume of a Course of Natural Philosophy, designed for High Schools and Colleges. Compiled by Benjamin Peirce, A. M., Professor of Mathematics and Natural Philosophy in Harvard University."—Boston: James Munroe & Co., 1836.

II. "An Elementary Treatise on Plane and Solid Geometry." By Professor Peirce. Boston: Munroe & Co., 1837.

III. A Treatise on Algebra, by Professor Peirce, is also nearly through the press.

To those who have long felt, with us, the want of a Course of Mathematics and Natural Philosophy, adapted to the present advanced state of analysis, and to the use of College Classes, we need only say that these books, so far as published, seem to us all that can be desired on the several subjects; and Professor Peirce's eminent attainments in science is a sufficient pledge that the whole will be completed in the same finished taste.

IV. A respected contributor also informs us, that an Elementary Treatise on Geometry, has lately been published by Mr. E. Nulty of Philadelphia.

ARTICLE XV.

NEW QUESTIONS TO BE ANSWERED IN NUMBER VI.

Their Solutions must arrive before August 1st, 1838.

(67). QUESTION I. *By Petrarch, New-York.*

A boat, moving uniformly in a current, performs a mile in t seconds when going with the current, and a mile in T seconds when going against the current. To find the velocity of the current.

(68). QUESTION II. *By* ———.

To find the relation between the parts into which any system of conjugate diameters divides the surface of an ellipse.

(69). QUESTION III. *By Mr. P. Barton, jun., South Orange, Mass.*

Find x so that

$$376x^2 + 114x + 34 = \text{a square number.}$$

(70). QUESTION IV. *By P.*

Find the diameter of the sphere, which placed in a given conical glass full of water, shall cause the greatest quantity of water to overflow.

(71). QUESTION V. *By Mr. O. Root, Hamilton College.*

If through the extremity of the diameter of a semicircle, chords be drawn, and semicircles be described upon them as diameters, their vertices will be in the semicircle described on the chord which passes through the vertex of the given semicircle.

(72). QUESTION VI. *By* ———.

Def. In the parabola, the parameter of any diameter is that chord of the system it bisects, which is equal to four times the distance of its middle point from the vertex of the diameter.

It is required to show that all parameters of the parabola pass through a given point, and to find the locus of their middle points.

(73). QUESTION VII. (*From the Phil. Mag. and Jour., Aug.*, 1836.)

Theorem. The circumference drawn through the points of intersection of any three tangents of a parabola, passes through the focus of that parabola.

(74). QUESTION VIII. *By Wm. Lenhart, Esq., York, Pa.*

It is required to find three consecutive natural members, that are divisible by cube numbers greater than unity.

(75). QUESTION IX. *By P.*

Given the area and vertical angle of a plane triangle, its base being on a straight line given in position, and one extremity of it at a given point of that line. To find the locus of the intersection of perpendiculars from the angles on the opposite sides; to trace the curve, and find its form under every relation of the constants.

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(76). QUESTION X. *By Richard Tinto, Esq., Greenville, Ohio.*

If a given cone of revolution be cut by planes, so that the principal parameter of all the sections shall be equal to a given line; it is required to find the surface to which these planes shall all be tangent.

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(77). QUESTION XI: *By James F. Macully, Esq., New-York.*

Required the sum of the infinite series

$$\frac{\sin^3 \frac{1}{4}\theta}{\cos^2 \theta} + \frac{4 \sin^3 \frac{1}{16}\theta}{\cos^2 \frac{1}{4}\theta} + \frac{16 \sin^2 \frac{1}{64}\theta}{\cos^2 \frac{1}{16}\theta} + \&c. \ldots\ldots$$

(78.) QUESTION XII. *By* ——.

The co-ordinates of five points in space, are

$$\begin{array}{llll} 1. & 2, & -1, & 3; \\ 2. & 3, & -2, & 5; \\ 3. & 1, & 5, & -2; \\ 4. & -3, & 0, & 7; \\ 5. & -7, & 4, & -1. \end{array}$$

It is required to find the volume of the polyedron which has its angles at these points.

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(79). QUESTION XIII. *By* ——.

Find the sum of the reciprocals of the radii, the sum of the radii, and the sum of the areas, of the n tangent circles described as in Question (50). See page 245 of the Mathematical Miscellany, Number IV.

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(80). QUESTION XIV. (*Communicated by Professor Peirce.*)

From Talbot's Researches in the Integral Calculus, Phil. Trans. Lond., 1836.

1°. Find such an equation between x and y that

$$\int dx \sqrt{1+x^n} + \int dy \sqrt{1+y^n}$$

may be expressed algebraically; n being either 3, 4 or 5.

2°. Find two such equations between x, y and z that

$$\int dx \sqrt{1+x^n} + \int dy \sqrt{1+y^n} + \int dz \sqrt{1+z^n}$$

may be expressed algebraically; n being 3, 4, 5, 6, 8 or 10.

(81). QUESTION XV. *By ψ.*

A vertical cylinder is revolving uniformly about its axis, which is fixed; it is required to determine the motion of a particle of matter in the cylindric surface, supposing it to begin to move from a given point in the surface, with a given velocity, and in a given direction. The point is confined to the surface and subjected to the power of gravitation, and the friction varies directly as the pressure and as the square of the velocity of the particle.

ARTICLE XVI.

NOTE,

On Question (35) of the Mathematical Miscellany,

BY DR. STRONG.

"PROFESSOR GILL, *New-Brunswick, May 12th*, 1837.

Dear Sir—I herewith give you a solution of Question (35), of the last number of the Miscellany, which occurred to me at too late a period for insertion in that number. If you have no objection, I should like to have it inserted in the next number, as a supplement to my solution in the last number.

Yours, with great respect,

T. STRONG."

We shall premise the well known formula

$$\sin\frac{r\text{P}}{s}=\frac{r\text{P}}{s}\left(1-\frac{r^2}{s^2}\right)\left(1-\frac{r^2}{4s^2}\right)\left(1-\frac{r^2}{9s^2}\right)\left(1-\frac{r^2}{16s^2}\right)\&\text{c.}\ldots(1),$$

where $\text{P}=3,14159$, &c., which is true for all values of r and s.

By the solution (see page 175 of the Mathematical Miscellany) we have $\dfrac{\text{T}}{1+\cot^2\frac{1}{2}u}$ for the radius of the circle whose centre is given, also

$$\text{S}\,({}_nr+r_n)=\text{S}\left\{\frac{\text{T}}{1+(\cot\frac{1}{2}u+n\sqrt{b})^2}\right\}+\text{S}\left\{\frac{\text{T}}{1+(\cot\frac{1}{2}u-n\sqrt{b})^2}\right\},$$

for the sum of the radii of all the other circles. Put $\text{Q}'\text{T}=$ the sum of the radii of all the circles, then

$$\text{Q}'=\frac{1}{1+\cot^2\frac{1}{2}u}+\text{S}\left\{\frac{1}{1+(\cot\frac{1}{2}u+n\sqrt{b})^2}\right\}+\text{S}\left\{\frac{1}{1+(\cot\frac{1}{2}u-n\sqrt{b})^2}\right\}\ (2)$$

We will now put $\text{Q}''\text{T}^2=$ the sum of the squares of the radii of all the circles, and we shall have

$$\text{Q}''=\{1+\cot^2\tfrac{1}{2}u\}^{-1}+\text{s}\{1+(\cot\tfrac{1}{2}u+n\sqrt{b})^2\}^{-2}+\text{s}\{1+(\cot\tfrac{1}{2}u-n\sqrt{b})^2\}^{-2}(3).$$

Now we evidently have the infinite series

$$\frac{1}{a^2-c^2}+\text{S}\{(nb+a)^2-c^2\}^{-1}+\{(nb-a)^2-c^2\}^{-1}$$

$$=\frac{d\,\text{h. l.}\ [(a^2-c^2)\{(b+a)^2-c^2\}\{(b-a)^2-c^2\}\{(2b+a)^2-c^2\}\ \&\text{c.}]^{-1}}{2cdc}$$

$$=\frac{d\,\text{h. l.}\ [(a^2-c^2)\{b^2-(a+c)^2\}\{b^2-(a-c)^2\}\{4b^2-(a+c)^2\}\ \&\text{c.}]^{-1}}{2cdc}$$

$$=\frac{d\text{h.l.}\left\{\left[\frac{(a+c)\text{P}}{b}.\left\{1-\frac{(a+c)^2}{b^2}\right\}\left\{1-\frac{(a+c)^2}{4b^2}\right\}\&\text{c}\right]\times\left[\frac{(a-c)\text{P}}{b}.\left\{1-\frac{(a-c)^2}{b^2}\right\}\&\text{c}\right]\right\}^{+1}}{2cdc}$$

$$=\frac{d\,\text{h.l.}\left[\sin\frac{a+c}{b}.\,\text{P}\times\sin\frac{a-c}{b}.\,\text{P}\right]}{2cdc}, \text{ by (1)},$$

$$=\frac{\text{P}}{2bc}\left\{\cot\frac{a-c}{b}.\,\text{P}-\cot\frac{a+c}{b}.\,\text{P}\right\}=\frac{\sin\frac{2c\text{P}}{b}}{cb\left\{\cos\frac{2c\text{P}}{b}-\cos\frac{2a\text{P}}{b}\right\}}\quad(4).$$

Now put in (3), $c\sqrt{-1}$ for c, and $\sqrt{b}$ for b, also let e denote the hyperbolic base, and put $v=\frac{2c}{\sqrt{b}}.\,\text{P}$, then since

$$\sin\frac{2c\sqrt{-1}}{\sqrt{b}}.\,\text{P}=\frac{e^{-v}-e^{v}}{2\sqrt{-1}},\ \cos\frac{2c\sqrt{-1}}{\sqrt{b}}.\,\text{P}=\frac{e^{v}+e^{-v}}{2};$$

(4) will become

$$(c^2+a^2)^{-1}+\text{S}\{c^2+(a+n\sqrt{b})^2\}^{-1}+\text{S}\{c^2+(a-n\sqrt{b})^2\}^{-1}$$

$$=\frac{e^{v}-e^{-v}}{c\sqrt{b}\left(e^{v}+e^{-v}-2\cos\frac{2a}{\sqrt{b}}\text{P}\right)}=\frac{e^{2v}-1}{c\sqrt{b}\left(e^{2v}-2e^{v}\cos\frac{2a}{\sqrt{b}}\text{P}+1\right)}\quad(5).$$

If we put $c=1$, $a=\cot\frac{1}{2}u$, and suppose that $\sqrt{b}$ in (5) is the same as in (2), we shall have, by (5) and (2),

$$\text{Q}'=\frac{e^{2v}-1}{\sqrt{b}\left\{e^{2v}-2e^{v}\cos\left(\frac{2\text{P}}{\sqrt{b}}\cot\frac{1}{2}u\right)+1\right\}}$$

$$\therefore\ \text{Q}'\text{TP}=\frac{\text{TP}}{\sqrt{b}}\times\frac{e^{2v}-1}{e^{2v}-2e^{v}\cos\left(\frac{2\text{P}}{\sqrt{b}}\cot\frac{1}{2}u\right)+1},$$

the sum of the circumferences of all the circles. Again, by (5),

$$(c^2+a^2)^{-2}+\text{S}\{c^2+(a+n\sqrt{b})^2\}^{-2}+\text{S}\{c^2+(a-n\sqrt{b})^2\}^{-2}$$

$$=\frac{-1}{2cdc}.\,d\left\{\frac{e^{2v}-1}{c\sqrt{b}\left(e^{2v}-2e^{v}\cos\frac{2a\text{P}}{\sqrt{b}}+1\right)}\right\}.\quad .\quad .\quad (6).$$

$\therefore$ by substituting the value of v, in (6), then taking the differential with regard to c, as indicated by the formula, putting $c=1$, after the differentiation, $a=\cot\frac{1}{2}u$, and $\sqrt{b}$ as before, we shall easily find by (3) the value of Q'', and thence $\text{Q}''\text{T}^2\text{P}=$ the sum of the areas of all the circles becomes known; it is also easy to see how to find the sums of the cubes, fourth powers, &c., of the radii of all the circles.

ARTICLE XVII.

ON FORCES.

By Professor Harnay, Illinois College, South Hanover, Ill.

Let f be any force, v the velocity, s the space described, and t the time. It is proposed to demonstrate the following equations

$$\frac{dv}{dt}=f,\ \frac{ds}{dt}=v.$$

Assume that f, v, and s are functions of t; and that, the time being constant, the velocity is as the force.

We shall suppose, moreover, that the force increases with the time. It will be easy to show by a similar process that the equations obtain if the force remains constant, or diminishes with the time.

Let (1). t become $\quad t+h,$

(2). f becomes . $f+\frac{df}{dt}.h+\frac{d^2f}{2dt^2}.h^2$, &c.

(3). v becomes . $v+\frac{dv}{dt}.h+\frac{d^2v}{2dt^2}.h^2$, &c.

(4). s becomes . $s+\frac{ds}{dt}.h+\frac{d^2s}{2dt^2}.h^2$, &c.

Suppose the force f to remain constant during the time h; the velocity communicated will be

$$(5).\quad \ldots\quad fh<\frac{dv}{dt}h+\frac{d^2v}{2dt^2}.h^2,\ \&c.,$$

the velocity really communicated, since the force increases during the time h.

Suppose the force acquired at the end of the time h, to wit.

$$f+\frac{df}{dt}h+\frac{d^2f}{2dt^2},\ \&c.,$$

expression (1), to remain constant during a time h, the velocity communicated would be,

$$(2).\quad fh+\frac{df}{dt}h^2+\frac{d^2f}{2dt^2}h^3,\ \&c., >\frac{dv}{dt}h+\frac{d^2v}{2dt^2}h^2,\ \&c.,$$

the velocity really communicated, since we suppose the greatest constant force which acts in the time h. We have then these three expressions

$$fh$$

$$\frac{dv}{dt}h+\frac{d^2v}{2dt^2}h^2,\ \&c.,$$

and $$fh+\frac{df}{dt}h^2+\frac{d^2f}{2dt^2}h^3,\ \&c.,$$

the first less than the second, and the second less that the third, then the

difference between the first and the second must be less than the difference between the first and third; that is,

$$(7). \quad \frac{df}{dt}h^2 + \frac{d^2f}{2dt^2}h^3, \&c., > \left(\frac{dv}{dt} - f\right)h + \frac{d^2v}{2dt^2}h^2, \&c.,$$

f then cannot be greater than $\frac{dv}{dt}$, since in that case, after subtraction, there would remain a negative quantity multiplied by the first power of h; and h might be made so small that this negative term would be more than all those multiplied by higher powers of h, and the whole expression on the right of the sign $>$ would be negative, which cannot be, since it is the result of a less subtracted from a greater.

f cannot be less than $\frac{dv}{dt}$; for then, after subtraction, there would remain a positive quantity multiplied by the first power of h, and h might be made so small, that expression (7) could not be true. Therefore

$$\frac{dv}{dt} = f.$$

Again, suppose the velocity v to remain constant during the time h, the space would be vh; but suppose the velocity acquired at the end of the time h, to wit:

$$v + \frac{dv}{dt}h + \frac{d^2v}{2dt^2}h^2, \&c.,$$

to remain constant during a time h, the space described would be,

$$vh + \frac{dv}{dt}h^2 + \frac{d^2v}{2dt^2}h^3, \&c.$$

It is easy to see that the space described in the first of these suppositions, is less than the real space described in the time h, and the space on the second of these suppositions is greater than the real space described, that is

$$vh < \frac{ds}{dt}h + \frac{d^2s}{2dt^2}, h^2, \&c.,$$

and $$vh + \frac{dv}{dt}h^2 + \frac{d^2v}{2dt^2}h^3, \&c. > \left(\frac{ds}{dt} - v\right)h + \frac{d^2s}{2dt^2}h^2, \&c.$$

consequently,

$$\frac{dv}{dt}h^2 + \frac{d^2v}{2dt^2}h^3, \&c. > \left(\frac{ds}{dt} - v\right)h + \frac{d^2s}{2dt^2}h^2, \&c.$$

This, by the same process of reasoning as in the preceding case, gives

$$\frac{ds}{dt} = v.$$

Q. E. D.

ARTICLE XVIII.

DIOPHANTINE SPECULATIONS.

By Wm. Lenhart, Esq., York, Penn.

NUMBER ONE.

Problem 1. To find n cube numbers, such that if from each of them a given number (a) be subtracted, the sum of the remainders shall be a square.

Solution. Let x'^3, x''^3, x'''^3, &c., denote the required cubes, then

$x'^3+x''^3+x'''^3$ &c. $-na=\square=r^2$, or $x'^3+x''^3+x'''^3$&c.$=r^2+na$ (1).

Now, if to the known number na a series of squares be added, we shall have a series of numbers, each term of which may, as we have shown in Number II. of the Miscellany, be divided into n cubes, each greater than a, and which will, therefore, be the cubes required. As an example, let three cubes be required, and let $a=1$, then

$$x'^3+x''^3+x'''^3=r^2+n=r^2+3.$$

Now, if to each square of a series of squares, 3 be added, there will arise the series of numbers,

$$4,\ 7,\ 12,\ 19,\ 28,\ 39,\ 52,\ \&c.,$$

each of which may be divided into 3 cubes, each $>$ than unity. By carrying out the series to some extent, we may frequently, by inspection alone, discover cubes to answer. Thus 3 being added to $(23)^2$ makes 532, which in almost a moment, we perceive to be composed of $(4)^3$, $(5)^3$, and $(7)^3$, which cubes will therefore answer.

Again: by adding 3 to $(16)^2$, we have 259, which is equal to $(5)^3+134$. But 134, by our Table, is composed of $(\frac{7}{2})^3$ and $(\frac{9}{2})^3$, therefore, the 3 required cubes will be $(\frac{7}{2})^3$, $(\frac{9}{2})^3$ and $(\frac{10}{2})^3$. And in the same manner other cubes may be had. As another example, let 4 cubes be required; then

$$x'^3+x''^3+x'''^3+x''''^3=r^2+4.$$

Hence, forming a series as before, we soon find that

$$(17)^2+4=293=189+104,$$

and, by Table, that,

$$189=(4)^3+(5)^3 \text{ and } 104=\left(\frac{4}{3}\right)^3+\left(\frac{14}{3}\right)^3.$$

Consequently the cubes required will be $(\frac{4}{3})^3, (\frac{12}{3})^3, (\frac{14}{3})^3$ and $(\frac{15}{3})^3$.

We also find that $(43)^2+4=1853=1512+341$,

and that $1512=(8)^3+(10)^3$, and $341=(5)^3+(6)_3$;

therefore the cubes are $(5)^3$, $(6)^3$, $(8)^3$ and $(10)^3$, which we presume would be rather difficult to find in any other way. And thus, n cubes of a low denomination may be readily found to answer.

Problem II. To find, n, numbers such that, if each of them be added to the cube of their sum, the respective sums shall be cubes.

Solution Let x, y, z, &c., represent the required numbers, s their sum, and a^3, b^3, c^3, &c., the resulting cubes. Then, by the Problem,

$$s^3+x=a^3,\ s^3+y=b^3,\ s^3+z=c^3,\ \&c.,$$

or, $$x=a^3-s^3,\ y=b^3-s^3,\ z=c^3-s^3,\ \&c.$$

Hence, $$a^3+b^3+c^3,\ \&c.,=ns^3+s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Now, make $s=\frac{1}{r}$, then from (2) we get

$$a^3+b^3+c^3+\&c.=\frac{n}{r^3}+\frac{1}{r}=\frac{n+r^2}{r^3} \text{ or } r^3a^3+r^3b^3+r^3c^3+\&c.=n+r^2 (3),$$

which equation and (1), in the solution to Problem I, being identical, the ra, rb, rc, &c., in (3) being the same as x', x'', x''', &c., in (1), may therefore be resolved in the same manner.

Example I. Let 3 numbers be required.

Application. By Problem I, we have

$$x'=4,\ x''=5,\ x'''=7, \text{ and } r=23,$$

consequently, we have here

$$ra=4, rb=5, rc=7,$$

or dividing by $r=23$,

$$a=\frac{4}{23}, b=\frac{5}{23}, c=\frac{7}{23}, s=\frac{1}{r}=\frac{1}{23},$$

and thence

$$x=\frac{63}{12167}, y=\frac{124}{12167}, \text{ and } z=\frac{342}{12167}.$$

Example II. Let four numbers be required.

Application. By the solution to Problem I,

$$x'=5, x''=6, x'''=8, x''''=10 \text{ and } r=43,$$

therefore in this

$$ra=5, rb=6, rc=8, rd=10,$$

or, dividing by $r=43$,

$$a=\frac{5}{43}, b=\frac{6}{43}, c=\frac{8}{43}, d=\frac{10}{43}, s=\frac{1}{r}=\frac{1}{43},$$

and therefore

$$x=\frac{124}{79507}, y=\frac{215}{79507}, z=\frac{511}{79507}, \text{ and } v=\frac{999}{79507}.$$

Or, if we take the other values in the last Example, in Solution I, viz:

$$x'=\frac{4}{3}, x''=\frac{12}{3}, x'''=\frac{14}{3}, x''''=\frac{15}{3}, r=17,$$

we shall find

$$a=\frac{4}{51}, b=\frac{12}{51}, c=\frac{14}{51}, d=\frac{15}{51}, s=\frac{1}{r}=\frac{1}{17}=\frac{3}{51},$$

and thence

$$x=\frac{37}{132651}, y=\frac{1701}{132651}, z=\frac{2717}{132651}, \text{ and } v=\frac{3348}{132651}.$$

And in the same way n numbers may be found.

Before we proceed to the resolution of the next Problem, which, when $n=3$, is the Nineteenth Question of the Fifth Book of Diophantus' Arithmetic, and the sixty-eighth of the hundred Problems, which Ludol-

phus à Collen or Van Ceulen, proposed to be solved in his Dutch work on the circle, it may not be amiss to mention to the reader that it is also to be found in Bonnycastle's Algebra, and in Leybourn's edition of the Ladies' Diary, vol. i., p. 52, where a brief history of it, and several curious and highly interesting speculations on the same, are recorded. It will be seen that, by our method of solution, any number of numbers may be readily found to fulfil the conditions required; and in the particular case here recited, several sets of a much lower denomination than any that have yet been found. We ought, perhaps, to rejoice too, as it is said Ludolphus did when he discovered his solution, at our success in resolving, in a manner so general and simple, a Problem which, not only occupied the attention of several of the greatest mathematicians, both in England and on the continent, at the close of the sixteenth and the early part of the seventeenth century, but also of many in modern times; and like him, too, conclude our speculations on the subject, by saying, "Constat ergo numeros rite esse inventos. Cujus rei soli Deo debetur gloria."

Problem III. To find n numbers such that each of them being severally subtracted from the cube of their sum, the n remainders may be cubes.

Solution. Observing the same notation as in Problem II., we shall have

$$s^3 - x = a^3,\ s^3 - y = b^3,\ s^3 - z = c^3, \text{ \&c.},$$

or $$x = s^3 - a^3,\ y = s^3 - b^3,\ z = s^3 - c^3, \text{ \&c.};$$

and thence $$a^3 + b^3 + c^3 + \text{\&c.} = ns^3 - s;$$

or putting $s = \dfrac{r}{t}$,

$$t^3a^3 + t^3b^3 + t^3c^3, \text{\&c.} = nr^3 - rt^2 = r(nr^2 - t^2) \quad . \ . \ . \ (3).$$

We may here observe that, when $r = t$, and $n = 2$, the Problem becomes impossible; and when n is $>$ than 2, it resolves itself into the exceedingly difficult Problem of finding n cubes, each $<$ than unity, whose sum shall make $n - 1$; which Problem we have fully discussed in Number II. of the Miscellany. In this solution, therefore, we shall always take t prime to, and $>$ than r, and also such as to render a^3, b^3, c^3, &c., positive in all cases. Now, in order to obtain numbers for a, b, c, &c., with facility, and in a regular manner, we shall place here the following

GENERAL RULE.

From nr^2 deduct successively the terms of a descending series of squares prime to r^2, commencing with the first square less than nr^2, and ending with the square next greater than r^2; multiply each remainder by r, divide the respective products into n cubes, the most convenient way—by inspection, if possible, or by the methods exhibited in our Investigation in Number II. of the Miscellany; and these cubes will be

$$t^3a^3,\ t^3b^3,\ t^3c^3, \text{ \&c.}$$

Example I. Let two numbers be required. Then

$$t^3a^3 + t^3b^3 = r(2r^2 - t^2).$$

Application. In this case the products must be Tabular numbers composed of two cubes, else the supposed numbers for r and t will not an-

swer. If we take $r=4$ and $t=5$, then $r(2r^2-t^2)=28=(3)^3+(1)^3$, therefore $a=\frac{1}{5}$, $b=\frac{3}{5}$, $s=\frac{r}{t}=\frac{4}{5}$, and

$$x=s^3-a^3=\frac{63}{125},\ y=s^3-b^3=\frac{37}{125}.$$

But, in order to exhibit the operation more in full, let us assume for r a larger number, say 13, then according to the Rule, we may take t, 18, 17, 16, 15 or 14, and hence have the following calculation, namely,

$$r(2r^2-t^2)=\left\{\begin{array}{l}(338-324)\times 13= \ \ 182\\ (338-289)\times 13= \ \ 637\\ (338-256)\times 13=1066\\ (338-225)\times 13=1469\\ (338-196)\times 13=1846\end{array}\right\}.$$

Now, 182 and 637 are both Tabular numbers, therefore each of them will furnish a set of numbers to answer. Thus 637 arising from $t=17$, is, by Table, equal to $(5)^3+(8)^3$, consequently, $a=\frac{5}{17}$, $b=\frac{8}{17}$, $s=\frac{r}{t}=\frac{13}{17}$, and thence

$$x=\frac{2072}{4913},\text{ and } y=\frac{1685}{4913}.$$

Example II. Let three numbers be required. Then

$$t^3a^3+t^3b^3+t^3c^3=r(3r^2-t^2).$$

Application. Take $r=7$, then t may be 12, 11, 10, 9 or 8, and the resulting products will be 21, 182, 329, 462, 581, each of which may be divided into 3 cubes, having the necessary properties; and thence may be found as many sets of numbers to answer. Thus, 21, resulting from $t=12$, is equal to $8+13$. Now $8=(2)^3$ and 13 is a tabular number equal to $(\frac{2}{3})^3+(\frac{7}{3})^3$, therefore $21=(\frac{2}{3})^3+(\frac{6}{3})^3+(\frac{7}{3})^3$, and consequently $a=\frac{2}{36}$, $b=\frac{6}{36}$, $c=\frac{7}{36}$, $s=\frac{r}{t}=\frac{21}{36}$, and the numbers required

$$x=\frac{9253}{46656},\ y=\frac{9045}{46656},\text{ and } z=\frac{8918}{46656}.$$

If we take $r=18$ and $t=29$, and proceed as directed in the rule, we shall find $a=\frac{3}{29}$, $b=\frac{10}{29}$, $c=\frac{11}{29}$, and thence

$$x=\frac{4501}{24389},\ y=\frac{4832}{24389},\text{ and } z=\frac{5805}{24389}.$$

A yet less set, and perhaps the least, may be had by taking $r=16$, and $t=25$, for then we shall find $a=\frac{3}{25}$, $b=\frac{4}{25}$, $c=\frac{13}{25}$, and thence the required numbers

$$x=\frac{4069}{15625},\ y=\frac{4032}{15625},\text{ and } z=\frac{1899}{15625}.$$

NOTE. If we assume $r=27$ and $t=44$, we shall find the numbers

$$\frac{13851}{85184},\ \frac{18954}{85184},\ \frac{19467}{85184},$$

which are the same as those in Bonnycastle's Algebra, and those record-

ed in the Ladies' Diary, as being the least that have ever yet been found. The above three sets, however, are much lower.

Example III. Let four numbers be required. Then

$$t^3a^3 + t^3b^3 + t^3c^3 + t^3d^3 = r(4r^2 - t^2).$$

Application. Let $r = 12$, then t may be 23, 19, 17, or 13 and the products will be 564, 2580, 3444 and 4884, each of which may be divided into 4 cubes, having the proper requisites. For instance 2580 arising from $t = 19$ is equal to $1241 + 1339$, tabular numbers equal to $(9)^3 + (8)^3$ and $(2)^3 + (11)^3$ respectively, which give

$$a = \frac{2}{19}, b = \frac{8}{19}, c = \frac{9}{19}, d = \frac{11}{19}, s = \frac{r}{t} = \frac{12}{19},$$

and thence

$$x = \frac{397}{6859}, y = \frac{999}{6859}, z = \frac{1216}{6859}, \text{ and } v = \frac{1720}{6859}.$$

Example IV. Let five numbers be required. Then

$$t^3a^3 + t^3b^3 + \&c. = r(5r^2 - t^2).$$

Application. Take $r=1$, $t=2$, then $r(5r^2 - t^2) = 1$, which is to be divided into 5 cubes. Multiply by $(3)^3 = 27$, and we shall have 27 which is equal to $1 + 7 + 19$. Now, by Table,

$$7 = \left(\frac{4}{3}\right)^3 + \left(\frac{5}{3}\right)^3, \text{ and } 19 = \left(\frac{1}{3}\right)^3 + \left(\frac{8}{3}\right)^3,$$

therefore,

$$27 = 1 + \left(\frac{4}{3}\right)^3 + \left(\frac{5}{3}\right)^3 + \left(\frac{1}{3}\right)^3 + \left(\frac{8}{3}\right)^3,$$

or, dividing by $(3)^3$ and arranging

$$1 = \left(\frac{1}{9}\right)^3 + \left(\frac{3}{9}\right)^3 + \left(\frac{4}{9}\right)^3 + \left(\frac{5}{9}\right)^3 + \left(\frac{8}{9}\right)^3.$$

Hence

$$a = \frac{1}{18}, b = \frac{3}{18}, c = \frac{4}{18}, d = \frac{5}{18}, e = \frac{8}{18}, s = \frac{r}{t} = \frac{1}{2} = \frac{9}{18},$$

and therefore

$$x = \frac{217}{5832}, y = \frac{604}{5832}, z = \frac{665}{5832}, v = \frac{702}{5832}, \text{ and } w = \frac{728}{5832}.$$

And it is obvious that in the same manner n numbers may be found to answer, all of which is respectfully submitted.

Note. In concluding this interesting little speculation in cubes, we would briefly remark that the Problem "to find n numbers such that if from each number the cube of their sum be subtracted, the n remainders shall be cubes," may be solved in the same manner as has been Problem II. The reader may, therefore, exercise and amuse himself by finding numbers to answer.

WM. LENHART.

York, Penn., Feb., 1837.

*** Mr. Lenhart has been kind enough to present a beautiful manuscript copy of his Table of Numbers and their Component Cubes, refer-

red to in this paper, and in that of Number II. together with other interesting Tables, on similar subjects, to the *Institute*, with a permission to publish any, or all of them in the Miscellany. We need not say how happy we shall be to avail ourselves of this permission, and the subscribers to the Miscellany may expect copies of these Tables as soon as we can make arrangements for their publication.

*** It has occurred to us that some of our correspondents may like to see the question proposed by Mr. Abbot, referred to on page 256. Mr. Abbot states that it was suggested to him by seeing the attempt made to blow two cards asunder, by means of a tube inserted in one of them.

Experiment. Let a circular orifice of a given diameter be made in the plane side of a deep cistern, which is kept filled with water, and a circular plate be placed over the orifice, supported at the bottom, so that its weight will not cause it to slide down. A thin sheet of water will issue between the side of the cistern and the plate, on all sides, the pressure of the atmosphere, under ordinary circumstances, causing the sheet to be unbroken between the extremity of the plate and the orifice, and the plate to adhere to the side of the cistern, separated only by the thin sheet of water.

Having given the depth of the orifice below the surface of the water, it is required to find the diameter of the smallest circular plate which can be made to adhere, and also the force with which a plate of any given diameter adheres. The plate and the side of the cistern are supposed perfectly smooth, and the orifice and plate are concentric.

ERRATA.

Page 206, line 12, dele *an*; line 13, for *an* read *a*.
" 207, " 22, dele *the*.
" 211, " 32, equation (2), is $s = 4ab.\dfrac{b'}{y'}$.
" 218, " 6, area lune $= \frac{1}{2}\pi \sin^2 x + \frac{1}{2}\sin 2x - x = u$.
" 219, " 23, for *dxy* read *dy*.
" 257, Quest. VIII., for *members* read *numbers*.
" 258, Quest. XI., for $\sin^2 \frac{1}{64}\theta$, read $\sin^3 \frac{1}{64}\theta$.

METEOROLOGICAL OBSERVATIONS,

MADE AT THE INSTITUTE, FLUSHING, L. I., FOR THIRTY-SEVEN SUCCESSIVE HOURS, COMMENCING AT SIX A. M., OF THE TWENTY-FIRST OF JUNE, EIGHTEEN HUNDRED AND THIRTY-SEVEN, AND ENDING AT SIX P. M., ON THE FOLLOWING DAY.

(*Lat.* 40° 44′ 58″ N., *Long.* 73° 44′ 20″ W. *Height of Barometer above low water mark of Flushing Bay,* 54 *feet.*)

Hour.	Barometer Corrected.	Attached Therm'ter.	External Therm'ter.	Wet Bulb Therm'ter.	Winds —from—	Clouds —to—	Strength of wind.	REMARKS. Heavy rain on the night of the 19th with wind from SSW; wind changed to NW at 9, A. M. of the 20th.
6	29.580	68	62	60¼	WNW	E	Gentle.	Scattered thin clouds.
7	.582	68	63½	61	"	"	"	A curtain of gray clouds.
8	.586	68	65	62	"	"	"	"
9	.587	67	66	63	"	"	"	"
10	.588	66	67¼	64½	W	"	"	Clouds darkening.
11	.586	66	69½	66½	"	NE	"	"
12	.573	68	74½	70	WSW	"	Light.	"
1	.568	68	74½	69½	W	E	Gentle.	"
2	.565	67	68½	64½	"	"	"	"
3	.557	67	67¼	64	NW	SE	"	"
4	.559	66	64¾	63	"	"	"	Dark Clouds and rain.
5	.553	65½	61¼	59½	"	"	"	" Rain began at 3¼
6	.554	65	61½	60	"	E	"	" P. M., and conti-
7	.553	65½	60	59	"	"	"	" nued with little in-
8	.543	66	59	58	"	"	"	" termission until 5¼,
9	.545	67	59	58	"	"	Light.	" A. M.
10	.545	66	58½	58	"	SE	"	"
11	.554	66	58½	58	"	"	"	" At intervals the rain
12	.552	66	57	57	"	"	"	" was very heavy,
1	.552	66	56½	56½	"	"	"	" particularly from
2	.557	66	56¼	56¼	"	"	Gentle,	" 10¼ to 10¾, P. M.
3	.560	65	55¾	55¾	"	"	"	"
4	.566	66	55¾	55¾	"	"	"	"
5	.582	66	56	56	"	"	"	Clouds breaking in the W.
6	.629	66	65	61	"	S	"	"
7	.667	65	69	65	"	"	"	Clouds spread again.
8	.699	63	71	68	"	SE	Brisk.	"
9	.706	65	63½	60	"	"	"	"
10	.711	69	66	61	"		"	A veil of thin gray vapour
11	.745	69	68½	63	"		"	spread uniformly over the
12	.792	69	71¼	65	WNW		Gentle.	heavens, the sun shining
1	.785	74	77	69½	"		"	through it.
2	.783	75	77½	69	"		"	"
3	.781	76½	78½	69½	"		"	"
4	.784	76	78¼	70½	"		"	"
5	.796	74	75	68½	"		"	"
6	.809	73	72½	67	"		"	"
	29.628	68	65½	62¾		Means.		

METEOROLOGICAL OBSERVATIONS,

MADE AT THE INSTITUTE, FLUSHING, L. I., FOR THIRTY-SEVEN SUCCESSIVE HOURS, COMMENCING AT SIX A. M., OF THE TWENTY-FIRST OF SEPTEMBER, EIGHTEEN HUNDRED AND THIRTY-SEVEN, AND ENDING AT SIX P. M., OF THE FOLLOWING DAY.

(*Lat.* 40° 44′ 58″ N., *Long.* 73° 44′ 20″ W. *Height of Barometer above low water mark of Flushing Bay,* 54 *feet.*)

Hour.	Barometer Corrected.	Attached Therm'ter.	External Therm'ter.	Wet Bulb Therm'ter.	Winds —from—	Clouds —to—	Strength of wind.	REMARKS.
6	30.339	59	51	49½	NE		Gentle.	Clear.
7	.348	59	56½	55	"	SW	"	" light clouds in the E.
8	.350	61½	57½	55½	"	"	"	" "
9	.361	62	59¾	56½	"	W	"	" "
10	.376	62	62	57	NNE	"	"	Clouds darkening.
11	.373	62	64	59	SE	"	Fresh.	"
12	.371	62	64	58¾	"	"	"	"
1	.368	62	63	57	"	"	Gentle.	"
2	.362	63	63	56	"	"	"	"
3	.349	63	62	55	E	"	"	Clouds breaking.
4	.346	63	61	55	SE	"	"	Clear.—Banks of clouds on the
5	.342	62	58½	53½	"		"	horizon.
6	.340	61	54¾	51½	"		"	" "
7	.335	61	52	49	"		"	" "
8	.334	61	50½	48	"		"	" Faint auroral appearances.
9	.328	61	50½	48	"		Light.	" in the N.
10	.326	60	51¼	49	"	NW	"	Thin gray clouds spreading.
11	.312	60	50½	48	"		"	Clear.—Aurora.
12	.296	60	51½	49	"	NW	"	Clouds spreading.
1	.288	60	50½	48½	"	"	"	More clear.
2	.278	60	52¼	51	"	"	"	Clouds darker.
3	.265	61	53¼	52	"	"	"	"
4	.247	62	55	54	"	"	"	"
5	.234	61½	54½	53	"	"	"	"
6	.228	61½	55½	54¼	"	"	"	"
7	.230	61	56½	55	"	"	"	"
8	.238	61	60½	59¼	NE	"	"	A few minutes small drizzling
9	.233	62	59½	57¾	"	SW	"	rain about 7½.
10	.194	63	63¼	61	"	"	"	Dark clouds.
11	.166	64	66	62½	SSE	NW	Gentle.	Large light clouds.
12	.136	64	68½	63¾	"	N	"	"
1	.117	64	66	61	"	"	"	Clouds darker.
2	.078	64	66	60½	S	NE	Fresh.	"
3	.065	64	63	58½	"	"	"	"
4	.052	64	62¼	58¼	"	"	"	"
5	.039	64	60½	57	"	"	Brisk.	"
6	.027	64	59¾	56¾	"	"	"	Clearing in the W.
	30.261	61⅞	58¼	55		Means.		

THE

MATHEMATICAL MISCELLANY.

NUMBER V.

JUNIOR DEPARTMENT.

ARTICLE IX.

HINTS TO YOUNG STUDENTS. *(Continued from page 210.)*

21. Elevation of Powers: Extraction of Roots.

Powers and Roots of Numbers: Positive Exponents. To elevate a number A to a *power* indicated by the number B, is to seek a third number which is formed from A by multiplication, as B is formed from unity by addition. The result of this operation on the number A, is called its *power* of the *degree* B. For the better understanding the preceding definition, it is necessary to distinguish three cases, accordingly as the number B is entire, fractional or irrational.

When B designates a whole number, this number is the sum of several units. Then the power of A, of the degree B, should be the product of as many factors equal to A, as there are units in B.

When B represents a fraction $\frac{m}{n}$, (m and n being two whole numbers,) it is necessary, in order to obtain this fraction,

1°. To seek a number which, repeated n times, reproduces unity:

2°. To repeat the number so found m times.

Then it will be necessary, in order to obtain the power of A of the degree $\frac{m}{n}$,

1°. To seek a number such, that the product of n factors equal to this number will reproduce A;

2°. To form a product of m factors, each equal to the same number.

If we suppose, as a particular case, $m=1$, the power of A will be reduced to that of the degree $\frac{1}{n}$, and is determined by the single condition, that the number A is equivalent to the product of n factors each equal to the sought power.

When B is an irrational number, we can obtain, in rational numbers, values approximating nearer and nearer to it. We can easily prove that on the same hypothesis, the powers of A, marked by the rational numbers in question, approach nearer and nearer to a certain limit. This limit is the power of A, of the degree B.

In elevating the number A, to the power of the degree B, the number A is called the *root*, and the number B, which indicates the degree of the power, the *exponent*. The power of A of the degree B is represented by the notation

$$A^B.$$

From the preceding definitions, it follows that the first power of a number is simply the number itself; its second power is the product of two factors equal to this number; its third power of three similar factors, and so on. Geometrical considerations lead us to designate the second power by the name of the *square*, and the third by the name of the *cube*, of the number. With regard to the power of the degree zero, it will be the limit towards which the power of the degree B converges, while the number B is indefinitely decreased. It is easy to show that this limit reduces to unity; from which it follows that we have in general

$$A^0 = 1,$$

supposing always that the value of the number A remains finite and different from zero.

To *extract* the *root* of the number A, marked by the number B, is to seek a third number which, elevated to the power of the degree B, reproduces A. The operation by which we arrive at it, is called *the extraction of the root*, and the result of the operation is the root of A, of the *degree* B. The number B which marks the degree of the root is called the *index*. In order to represent the root, we use the notation

$$\sqrt[B]{A}.$$

The roots of the second and third degrees are usually designated by the name of the *square roots* and *cube roots*. With regard to the square root, the index 2 above the sign $\sqrt{}$ of the root may generally be dispensed with, and the two notations

$$\sqrt[2]{A},\ \sqrt{A}$$

must be considered as equivalent.

NOTE. The extraction of the roots of numbers, being the inverse of their elevation to powers, may always be indicated in two ways. Thus, for example, to express that the number C is equal to the root of A, of the degree B, we may write either

$$A = C^B \quad \text{or} \quad C = \sqrt[B]{A}.$$

We may remark also that by virtue of the definitions, if we designate any whole number by n, $A^{\frac{1}{n}}$ will be a number such that the multiplication of n factors equal to this number will reproduce A. In other words, we shall have

$$(A^{\frac{1}{n}})^n = A,$$

from which we conclude

$$A^{\frac{1}{n}} = \sqrt[n]{A}.$$

Hence, when n is a whole number, the power of A, of the degree $\frac{1}{n}$, and the n^{th} root of A are equivalent expressions. We might easily prove that it is the same in the case where we replace the whole number n by any number whatever.

Powers of Numbers: Negative Exponents. To *elevate* the number A to the *power* marked by the negative *exponent* — B, is to divide unity by A^B. The value of the expression

$$A^{-B}$$

will therefore be found to be determined by the equation

$$A^{-B} = \frac{1}{A^B},$$

which may also be put under the form

$$A^B A^{-B} = 1.$$

Consequently, if we elevate the same number to two powers indicated by two opposite quantities, we shall obtain for results two positive quantities the reciprocals of each other.

22. *Real Powers and Roots of Quantities.* If, in the definitions that we have given of the powers and roots of numbers corresponding to either whole or fractional exponents, we substitute the word quantities instead of numbers, we shall obtain the following definitions of the real powers and roots of quantities.

To *elevate* the quantity a to a *real power of the degree m* (m being a whole number) is to form the product of as many factors equal to a as there are units in m.

To elevate the quantity a to a real power of the degree $\frac{m}{n}$ (m and n being two whole numbers, and supposing the fraction $\frac{m}{n}$ reduced to its simplest expression,) is to form the product of m equal factors such that the n^{th} power of each of them is equivalent to the quantity a.

To *extract* the *real root* of the degree m or $\frac{m}{n}$ of the quantity a, is to seek a new quantity which, elevated to the real power of the degree m or $\frac{m}{n}$ will reproduce a. From this definition, the n^{th} real root of a quantity is evidently the same thing as its real power of the degree $\frac{1}{n}$. Moreover, it is easy to show that the root of the degree $\frac{n}{m}$ is equivalent to the power of the degree $\frac{m}{n}$.

Lastly, to elevate the quantity a to the *real power of the degree* $-m$ or $-\frac{m}{n}$, is to divide unity by this quantity a elevated to the real power of the degree m or $\frac{m}{n}$.

In the operations of which we have spoken, the number or the quantity which marks the degree of a real power of a is called the *exponent* of that power, while the number which marks the degree of a real root is called the *index* of that root.

All powers of a, which correspond to an exponent whose numerical value is integral, that is, to an exponent of the form $+m$ or $-m$ (m representing a whole number) admits of a single and real value which we designate by the notation

$$a^m \text{ or } a^{-m}.$$

With regard to roots, and to powers whose numerical value is fractional, they may admit of two real values, or of only one real value, or of no real value at all. The real values are necessarily either positive or negative quantities. But, besides these quantities, we employ other symbols in algebra which, on account of their properties, receive the names of powers and roots, although they have no signification of themselves. These symbols are of the number of those algebraical expressions to which we give the name of *imaginary expressions*, in opposition to those of *real expressions*, which are only applied to numbers or to quantities.

This settled, it will be sufficient in this place to state that the n^{th} root of any quantity a, and its powers of the degrees $\frac{m}{n}$, $-\frac{m}{n}$ (n being a whole number, and $\frac{m}{n}$ an irreducible fraction) each admit of n distinct values, either real or imaginary. Any one of these values may be represented, if it is the n^{th} root, by the notation

$$\sqrt[n]{a} = a^{\frac{1}{n}},$$

or, if it is the power which has for its exponent $\frac{m}{n}$ or $-\frac{m}{n}$, by the notation

$$^*a^{\frac{m}{n}} \text{ or } a^{-\frac{m}{n}}.$$

We may add that the expression $a^{\frac{1}{n}}$ is comprised as a particular case in the more general expression $a^{\frac{m}{n}}$; and that, by calling A the numerical value of a, we shall find for the real values of the two expressions

$$a^{\frac{m}{n}},\ a^{-\frac{m}{n}},$$

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* M. Cauchy introduces a distinction which, however important in the higher departments of mathematics, will be scarcely appreciated by the *young* student. He designates any one whatever of these n values of the roots or powers by the notations

$$((a))^{\frac{1}{n}},\ ((a))^{\frac{m}{n}},\ ((a))^{-\frac{m}{n}},$$

while the symbols

$$(a)^{\frac{1}{n}},\ (a)^{\frac{m}{n}},\ (a)^{-\frac{m}{n}}, \text{ or } a^{\frac{1}{n}},\ a^{\frac{m}{n}},\ a^{-\frac{m}{n}},$$

are reserved to designate some particular one of the n values of the roots or powers, as for instance, the single positive value, if there is one.

1°., if n designate an odd number,

a being $= +$ A, ... $+\mathrm{A}^{\frac{m}{n}}, +\mathrm{A}^{-\frac{m}{n}}$,

a being $= -$ A, ... $-\mathrm{A}^{\frac{m}{n}}, -\mathrm{A}^{-\frac{m}{n}}$,

2°., if n designate an even number,

a being $= +$ A, ... $\pm\mathrm{A}^{\frac{m}{n}}, \pm\mathrm{A}^{-\frac{m}{n}}$,

when, in the last case, we suppose a negative, all the values of each of the expressions $a^{\frac{m}{n}}, a^{-\frac{m}{n}}$ will become imaginary.

If we make the fraction $\frac{m}{n}$ vary so as to approach indefinitely to the irrational number B, the denominator n will increase beyond all assignable limit, and with it the number of imaginary values of each of the expressions

$$a^{\frac{m}{n}} \text{ and } a^{-\frac{m}{n}};$$

consequently, B being an irrational number, we cannot admit the notations

$$a^{+\mathrm{B}},\quad a^{-\mathrm{B}},$$

or, if we make $b = \pm$ B, the notation

$$a^b,$$

except under the limitation of supposing a to represent a positive number $+$ A, and then only as representing the single positive value

$$+\mathrm{A}^b.$$

23. The powers of numbers and of quantities possess many remarkable properties, which are easily demonstrated. Among others, those comprised in the following formulas, are especially worthy of attention.

Let $a, a', a'', \ldots\ldots b, b', b'', \ldots\ldots$ be any quantities whatever, positive or negative, $\mathrm{A}, \mathrm{A}', \mathrm{A}'', \ldots\ldots$ any numbers whatever, and $m, m', m'', \ldots\ldots$ any whole numbers. We shall have

$$\text{* (3)}\quad \begin{cases} \mathrm{A}^b\mathrm{A}^{b'}\mathrm{A}^{b''} \ldots\ldots\ldots = \mathrm{A}^{b+b'+b''+\cdots}, \\ \mathrm{A}^b\mathrm{A}'^b\mathrm{A}''^b \ldots\ldots\ldots = (\mathrm{A}\mathrm{A}'\mathrm{A}'' \ldots\ldots)^b, \\ (\mathrm{A}^b)^{b'} = \mathrm{A}^{bb'}; \end{cases}$$

$$(4)\quad \begin{cases} a^{\pm m}.\,a^{\pm m}.\,a^{\pm m''} \ldots\ldots = a^{\pm m \pm m' \pm m'' \ldots\ldots}, \\ \text{(each of the numbers } m, m' \ldots \text{ must have the same sign in the two members.)} \\ a^m.\,a'^m.\,a''^m \ldots\ldots\ldots = (aa'a'' \ldots\ldots)^m, \\ a^{-m}.\,a'^{-m}.\,a''^{-m} \ldots\ldots = (aa'a'' \ldots\ldots)^{-m}, \\ (a^m)^{m'} = (a^{-m})^{-m'} \ldots\ldots = a^{mm'}, \\ (a^m)^{-m'} = (a^{m'})^{-m} \ldots\ldots = a^{-mm'}. \end{cases}$$

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* These formulas express, in algebraical language, truths with which the student should make himself familiar in every possible way. For this purpose he should interpret them into ordinary language; thus, the first of (3) signifies that the product of any number of different powers of the same root is equal to a power of the same root whose exponent is the sum of the exponents of the several factors; the second of (3)

The formulas (3) and (4) give birth to a multitude of consequences, among which we shall content ourselves with indicating the following. We draw from the second of formulas (3), by making $A' = \frac{1}{A}$,

$$A^b\left(\frac{1}{A}\right)^b = 1^b = 1;$$

and we conclude that

$$\left(\frac{1}{A}\right)^b = \frac{1}{A^b}.$$

That is, if we elevate, to the same power, two positive quantities which are reciprocals of each other, the result will be two quantities the reciprocals of each other.

24. Formation of Exponentials and of Logarithms.

When in the expression A^x we regard the number A as fixed, and the quantity x as variable, the power A^x takes the name of an *Exponential*. If, on the same hypothesis, we have, for a particular value of x,

$$A^x = B.$$

this particular value of x is called the *Logarithm* of the number B, in the system, the *base* of which is A. We indicate this logarithm by placing before the number the initial letter l or L, as follows:

$$l\,(B) \text{ or } L(B).$$

Besides, as such a notation does not indicate the base of the system in which the logarithm is taken, it is necessary to state the value of this base. This settled, if the characterisric L is used to designate the logarithms taken in the system whose base is A the equation

$$A^x = B,$$

leads to this other one

$$x = L(B).$$

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signifies that the product of similar powers of several different roots, is equal to the same power of the product of these several roots, &c. As particular cases of this last formula most frequently used, the product of the squares of two different numbers is equal to the square of the product of these numbers;—the product of the square roots of two numbers is equal to the square root of their product, &c. The several formulas also receive different, though quite as important, interpretations when their members are interchanged: the first of (3) becomes thus

$$A^{b+b'+b''+\&c.} = A^b \,.\, A^{b'} \,.\, A^{b''} \ldots\ldots,$$

and signifies that any power of a number may be divided into any number of factors, each factor being a power of the same number, such that the sum of the several exponents is equal to the exponent of the original power; thus, since

$$\tfrac{3}{4} = \tfrac{1}{2} + \tfrac{1}{4},$$

$$A^{\frac{3}{4}} = A^{\frac{1}{2}} \,.\, A^{\frac{1}{4}} = \sqrt{A} \,.\, \sqrt[4]{A};$$

also, since

$$1 = \tfrac{1}{2} + \tfrac{1}{2} = \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{6} = \&c.,$$

$$A = A^{\frac{1}{2}} \,.\, A^{\frac{1}{2}} = A^{\frac{1}{2}} \,.\, A^{\frac{1}{3}} \,.\, A^{\frac{1}{6}} = \&c.,$$

$$= \sqrt{A} \,.\, \sqrt{A} = \sqrt{A} \,.\, \sqrt[3]{A} \,.\, \sqrt[6]{A} = \&c.$$

Or, the same thing may be stated again, symbolically, thus:

$$A^{\beta} = A^b \,.\, A^{b'} \,.\, A^{b''} \ldots\ldots,$$

β, b, b', b'', &c., being any quantities whatever, provided only they fulfil the condition

$$\beta = b + b' + b'' + \ldots\ldots$$

Sometimes, when we desire to treat at the same time of logarithms taken in different systems, we distinguish them from each other by the help of one or more accents placed to the right of the letter L; thus the logarithms taken in a first system would be indicated by this letter without an accent; those in a second system by the same letter with a single accent, &c.

By the help of these definitions and the general properties of the powers of numbers, we easily perceive, 1°, that unity has zero for its logarithm in all systems; 2°, that in every system of logarithms the base of which is greater than unity, all numbers greater than unity have positive logarithms, and all numbers less than unity have negative logarithms; 3°, that in every system of logarithms the base of which is less than unity, all numbers less than unity have positive logarithms, and all numbers greater than unity have negative logarithms; 4°, that in two systems the bases of which are reciprocals of each other, the logarithms of the same number are equal but with contrary signs. Moreover, we might demonstrate without trouble the formulas which establish the principal properties of logarithms, and among which the following are the most remarkable.

If we designate by B, B', B'', ... C any numbers whatever, by the characteristics L, L' their logarithms taken in two different systems, the bases of which are A, A', and by k any quantity either positive or negative, we shall have

$$(5).\quad \left\{ \begin{aligned} L(BB'B''\ldots\ldots) &= L(B)+L(B')+L(B'')\ldots\ldots, \\ L(B^k) &= kL(B), \\ B^{L(C)} &= A^{L(B)\cdot L(C)} = C^{L(B)}, \\ \frac{L(C)}{L(B)} &= \frac{L'(C)}{L'(B)}. \end{aligned} \right.$$

We derive from the first of these formulas, by making $B' = \dfrac{1}{B}$,

$$L(B) + L\left(\frac{1}{B}\right) = L(1) = 0,$$

and consequently

$$L\left(\frac{1}{B}\right) = -L(B);$$

that is, two positive quantities which are reciprocals of each other, have equal logarithms with contrary signs. We may add that the fourth formula may easily be deduced from the second. In fact, let us suppose that the quantity k represents the logarithm of the number C in the system whose base is B. We shall have

$$C = B^k,$$

and therefore, by taking the logarithms of each member,

$$L(C) = kL(B),\quad L'(C) = kL'(B);$$

from which we conclude immediately

$$\frac{L\,(C)}{L\,(B)} = \frac{L'\,(C)}{L'(B)} = k.$$

We may further remark that, if we take $B = A$, we shall deduce from the fourth formula, (because $L(A) = 1$),

$$L'(C) = L'(A)\,.\,L(C),$$

or, by making, for abridgment, $L'(A) = \mu$,

$$L'(C) = \mu L(C).$$

Hence, in order to pass from the system of logarithms whose base is A, to that whose base is A', it is sufficient to multiply the logarithms taken in the first system by a certain co-efficient μ equal to the logarithm of A taken in the second system.

The logarithms, of which we have spoken, are those to which are given the name of *real logarithms*, since they always reduce to either positive or negative quantities. But, besides these quantities, there exist imaginary expressions which, because of their properties, have equally received the name of *logarithms*. The theory of these imaginary logarithms cannot, however, be entered upon here.

ARTICLE X.

SOLUTIONS TO THE QUESTIONS PROPOSED IN NUMBER IV.

(13). QUESTION I. *By* ——.

To find x, y, z, there are given the three equations

$$ax + by + cz = p \quad \ldots\ldots\ldots\ldots \quad (1),$$
$$bx + cy + az = q \quad \ldots\ldots\ldots\ldots \quad (2),$$
$$cx + ay + bz = r \quad \ldots\ldots\ldots\ldots \quad (3).$$

FIRST SOLUTION. *By Mr. Robt. S. Howland, St. Paul's College, Long Island.*

Add the three given equations together, and divide by $a + b + c$, then

$$x + y + z = \frac{p + q + r}{a + b + c} = s \quad \ldots\ldots\ldots \quad (4).$$

Multiply (4) by a, and subtract (1) from the product, then

$$(a - b)y + (a - c)z = as - p \quad \ldots\ldots\ldots \quad (5).$$

Multiply (4) by b, and subtract (2) from the product, then

$$(b - c)y - (a - b)z = bs - q \quad \ldots\ldots\ldots \quad (6).$$

By eliminating successively z, and y between these two equations, we find

$$y = \frac{(a^2 - bc)s - p(a - b) - q(a - c)}{a^2 + b^2 + c^2 - ab - ac - bc} \quad \ldots\ldots \quad (7),$$

$$z = \frac{(b^2 - ac)s - p(b - c) + q(a - b)}{a^2 + b^2 + c^2 - ab - ac - bc} \quad \ldots\ldots \quad (8).$$

Adding together (7) and (8),

$$y + z = \frac{(a^2 + b^2 - ac - bc)s - p(a - c) - q(b - c)}{a^2 + b^2 + c^2 - ab - ac - bc}$$

$$= s - \frac{(c^2 - ab)s + p(a - c) + q(b - c)}{a^2 + b^2 + c^2 - ab - ac - bc} \quad \ldots\ldots \quad (9).$$

Subtracting (9) from (4),

$$x = \frac{(c^2 - ab)s + p(a - c) + q(b - c)}{a^2 + b^2 + c^2 - ab - ac - bc} \quad \ldots\ldots \quad (10).$$

SECOND SOLUTION. *By Mr. Wm. R. Biddlecom, Clinton Liberal Institute.*

From the three given equations, we immediately derive

$$x = \frac{p - by - cz}{a} = \frac{q - cy - az}{b} = \frac{r - ay - bz}{c} \quad \ldots \quad (4).$$

The two equations between y and z contained in (4) being reduced, become

$$(b^2 - ac)y - (a^2 - bc)z = bp - aq \quad \ldots \ldots \quad (5),$$
$$(a^2 - bc)y - (c^2 - ab)z = ar - cp \quad \ldots \ldots \quad (6).$$

From these we derive

$$y = \frac{(a^2 - bc)z + bp - aq}{b^2 - ac} = \frac{(c^2 - ab)z + ar - cp}{a^2 - bc} \quad \ldots \quad (7).$$

Clearing this equation of fractions, and transposing,

$$\{(a^2 - bc)^2 - (b^2 - ac)(c^2 - ab)\}\, z = ar(b^2 - ac) + aq(a^2 - bc)$$
$$- cp(b^2 - ac) - bp\,(a^2 - bc),$$

or, by reduction,

$$(a^3 + b^3 + c^3 - 3\,a\,b\,c)\,az = ap\,(c^2 - ab) + aq(a^2 - bc) + ar\,(b^2 - ac);$$

therefore

$$z = \frac{(c^2 - ab)\,p + (a^2 - bc)\,q + (b^2 - ac)\,r,}{a^3 + b^3 + c^3 - 3\,abc}$$

By substituting successively in (7) and (4) we obtain

$$y = \frac{(b^2 - ac)\,p + (c^2 - ab)\,q + (a^2 - bc)\,r,}{a^3 + b^3 + c^3 - 3\,abc}$$

$$x + \frac{(a^2 - bc)\,p + (b^2 - ac)\,q + (c^2 - ab)\,r,}{a^3 + b^3 + c^3 - 3\,abc}$$

—— *Cor.* We easily see from these two solutions that

$$(a + b + c)\,(a^2 + b^2 + c^2 - ab - ac - bc)$$
$$= (a + b + c)\,\{(a + b + c)^2 - 3(ab + ac + bc)\}$$
$$= (a + b + c)\,\{(a^2 - bc) + (b^2 - ac) + (c^2 - ab)\}$$
$$= a\,(a^2 - bc) + b(b^2 - ac) + c(c^2 - ab)$$
$$= a^3 + b^3 + c^3 - 3\,abc$$

Mr. B. Birdsall's solution was on the same principles as the last one.

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(14). QUESTION II. *By θ.*

Let x = logarithm of N to any base,
y = logarithm of N′ to the same base;
prove that $N'^x = N^y$.

FIRST SOLUTION. *By Mr. Isaac A. Saxton, West Winfield, N. Y.*

Let a be the base of the system. Then we have, from the principles of logarithms,

$$a^x = N, \text{ and } a^y = N'.$$

By involving the first of these to the power y, and the second to the power x, we get

$$a^{xy} = N^y, \text{ and } a^{xy} = N'^x;$$

therefore $$N^y = N'^x.$$

Second Solution. *By Mr. Geo. K. Birely, Frederick, Md.*

Let a be the base, then

$$a^x = \text{N}, \text{ and } a^y = \text{N}'$$

therefore $$a = \sqrt[x]{\text{N}} = \sqrt[y]{\text{N}'}.$$

Involving this equation to the power $x\,y$, it becomes

$$\text{N}^y = \text{N}'^x.$$

(15.) Question III. *By* ——.

Divide $a^4 + b^4 - 2\,a^2\,b^2 \cos 2\varphi$ by $a^2 + b^2 - 2\,ab \cos \varphi$.

First Solution. *By Mr. Geo. W. Coaklay, Peekskill Academy, N. Y.*

In the well known formula

$$\cos(\text{A} + \text{B}) = \cos \text{A} \cos \text{B} - \sin \text{A} \sin \text{B}$$

make $$\text{A} = \text{B} = \varphi;$$

then $$\cos 2\varphi = \cos^2 \varphi - \sin^2 \varphi$$
$$= 2 \cos^2 \varphi - 1.$$

Hence, by substitution,

$$a^4 + b^4 - 2\,a^2\,b^2 \cos 2\,\varphi = a^4 + b^4 + 2\,a^2\,b^2 - 4\,a^2\,b^2 \cos^2 \varphi$$
$$= (a^2 + b^2)^2 - (2\,ab \cos \varphi)^2$$
$$= (a^2 + b^2 + 2\,ab \cos \varphi)(a^2\,b^2 - 2\,ab \cos \varphi);$$

therefore

$$\frac{a^4 + b^4 - 2\,a^2 b^2 \cos^2 \varphi}{a^2 + b^2 - 2\,ab \cos \varphi} = \frac{(a^2 + b^2 + 2\,ab \cos \varphi)(a^2 + b^2 - 2\,ab \cos \varphi)}{a^2 + b^2 - 2\,ab \cos \varphi}$$
$$= a^2 + b^2 + 2\,ab \cos \varphi$$

Second Solution. *By Mr. James J. Bowden, St. Paul's College, Flushing, L. I.*

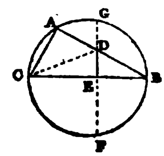
For $\cos 2\,\varphi$ write its value $2 \cos^2 \varphi - 1$, then

$$\frac{a^4 + b^4 - 2\,a^2\,b^2 \cos 2\,\varphi}{a^2 + b^2 - 2\,ab \cos \varphi} = \frac{a^4 + b^4 - 4\,a^2\,b^2 \cos^2 \varphi + 2\,a^2\,b^2}{a^2 + b^2 - 2\,ab \cos \varphi}$$
$$= \frac{(a^2 + b^2)^2 - 4\,a^2\,b^2 \cos^2 \varphi}{a^2 + b^2 - 2\,ab \cos \varphi}$$
$$= \frac{(a^2 + b^2 - 2\,ab \cos \varphi)(a^2 + b^2 + 2\,ab \cos \varphi)}{a^2 + b^2 - 2\,ab \cos \varphi}$$
$$= a^2 + b^2 + 2\,ab \cos \varphi.$$

(16.) QUESTION IV. *By Mr. Lenhart.*

Theorem. If from any point in either side of a right angled plane triangle, a straight line be drawn perpendicular to the hypothenuse; then shall the rectangle of the segments of the hypothenuse be equal to the rectangle of the segments of the sides containing the point, together with the square of the perpendicular thus drawn.

FIRST SOLUTION. *By Mr. J. Blickensderfer, Jun., near Roscoe, Ohio.*



Let ABC be the given right triangle, right angled at A, and let DE be the perpendicular drawn from any point D in AB, then will

$$CE \cdot EB = AD \cdot DB + DE^2.$$

For, from the properties of the right angled triangle, we have

$$BC^2 = AB^2 + AC^2.$$

$$\text{or, } (BE + CE)^2 = AC^2 + (AD + DB)^2.$$

$$\text{or, } BE^2 + 2\,BE \cdot CE + CE^2 = AC^2 + AD^2 + 2\,AD \cdot BD + BD^2$$

$$\text{But } DE^2 + CE^2 = AC^2 + AD^2 = CD^2,$$

and subtracting these equals,

$$BE^2 + 2\,BE \cdot CE - DE^2 = 2\,AD \cdot BD + BD^2$$

$$\text{But } BE^2 + DE^2 = BD^2,$$

and subtracting these equals,

$$2\,BE \cdot CE - 2\,DE^2 = 2\,AD \cdot BD,$$

$$\text{or } BE \cdot CE = AD \cdot BD + DE^2$$

Cor. When DE is drawn from A the vertex of the right angle, then $AD = 0$, and $CE \cdot EB = AE^2$, a well known property of the right triangle.

SECOND SOLUTION. *By* (W.) *Student in Dickinson College, Carlisle, Pa.*

(After a similar solution to the preceding, this gentleman proceeds): Describe a circle on CB as a diameter, the right angle A will be in its circumference, and prolonging DE to G and F, GF will be bisected in E. then

$$GD \cdot DF = GE^2 - DE^2 = AD \cdot DB,$$

and adding DE^2 to each,

$$GE^2 = AD \cdot DB + DE^2,$$

$$\text{But } CE \cdot EB = GE^2,$$

$$\text{therefore} \quad CE \cdot EB = AD \cdot DB + DE^2$$

THIRD SOLUTION. *By Mr. Warren Colburn, St. Paul's College, Flushing, L. I.*

The two right triangles DEB, CAB are similar, since they have the common angle B, hence

$$BE : BD = AB : CB$$

$$= AD + BD : EC + EB,$$

Multiplying means and extremes

$$BE \cdot EC + BE^2 = AD \cdot BD + BD^2,$$

and subtracting BE^2 from one, and its equal $BD^2 - DE^2$ from the other,

$$BE \cdot EC = AD \cdot BD + DE^2.$$

FOURTH SOLUTION. *By A Lady.*

Let p be the perpendicular which divides the side from which it is

drawn into the segments m and n, and the hypothenuse into the segments r and s. By Euc. I. 47,

$$m^2 - p^2 = r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and, from similar triangles, Euc. VI. 4,

$$m : r = r + s : m + n,$$

$$\text{or } m^2 + mn = r^2 + rs \quad . \quad . \quad . \quad . \quad . \quad . \quad (2);$$

subtract (1) from (2), and we have

$$p^2 + mn = rs \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Q. E. D.

(17.) QUESTION V. *By* ——.

Given $v = \sin^n x \,\{(n+2)\sin nx - n\sin(n+2)x\}$; to find $\dfrac{dv}{dx}$.

FIRST SOLUTION. *By Mr. R. Dewar Bacot, St. Paul's College, Flushing, L. I.*

$$\begin{aligned}\frac{dv}{dx} &= n\sin^{n-1}x\cos x\,\{(n+2)\sin nx - n\sin(n+2)x\}\\ &\quad + \sin^n x\,\{n(n+2)\cos nx - n(n+2)\cos(n+2)x\}\\ &= n\sin^{n-1}x\,\{(n+2)\sin nx\cos x - n\sin(n+2)x\cos x\\ &\quad + (n+2)\cos nx\sin x - (n+2)\cos(n+2)x\sin x\}\\ &= n\sin^{n-1}x\,\{(n+2)[\sin nx\cos x - \sin(n+2)x\cos x\\ &\quad + \cos nx\sin x - \cos(n+2)x\sin x] + 2\sin(n+2)x\cos x\}.\end{aligned}$$

But $\sin nx\cos x + \cos nx\sin x = \sin(n+1)x$,

$\sin(n+2)x\cos x + \cos(n+2)x\sin x = \sin(n+3)x$,

and, $2\sin(n+2)x\cos x = \sin(n+1)x + \sin(n+3)x$;

therefore

$$\begin{aligned}\frac{dv}{dx} &= n\sin^{n-1}x\,\{(n+2)[\sin(n+1)x - \sin(n+3)x]\\ &\quad + \sin(n+1)x + \sin(n+3)x\}\\ &= n\sin^{n-1}x\,\{(n+3)\sin(n+1)x - (n+1)\sin(n+3)x\}.\end{aligned}$$

SECOND SOLUTION. *By Mr. B. Birdsall, New Hartford, Oneida Co., N. Y.*

We have, for the differentials of the two separate factors,

$$d(\sin^n x) = n\sin^{n-1}x\,.\,dx,$$

$$\begin{aligned}d.\{(n+2)\sin nx - n\sin(n+2)x\} &= n(n+2)\cos nx\,.\,dx\\ &\quad - n(n+2)\cos(n+2)x\,.\,dx;\end{aligned}$$

hence, we readily get

$$\begin{aligned}\frac{dv}{dx} &= n\sin^{n-1}x\cos x\,\{(n+2)\sin x - n\sin(n+2)x\}\\ &\quad + n(n+2)\sin^n x\,\{\cos nx - \cos(n+2)x\}.\end{aligned}$$

THIRD SOLUTION. *By Mr. Geo. K. Birely.*

Let $w = \sin^n x$, and $\qquad dw = n\sin^{n-1}x.\,dx$,

$y = (n+2)\sin nx$, and $dy = n(n+2)\cos nx\,.\,dx$,

$z = n\sin(n+2)x$, and $dz = n(n+2)\cos(n+2)x.\,dx$,

$$\text{Then } v = w(y - z),$$

$$\text{and } \frac{dv}{dx} = \frac{dw}{dx}(y-z) + w\left(\frac{dy}{dx} - \frac{dz}{dx}\right)$$
$$= n \sin^{n-1} x \cos x \{(n+2) \sin nx - n \sin (n+2)x\}$$
$$+ n(n+2) \sin^n x \{\cos nx - \cos (n+2)x\}.$$

(18.) QUESTION VI. *By a Lady.*

Three ladies purchase a ball of exceedingly fine thread, for which they pay equally. Allowing the radius of the ball to be three inches, and the quality of the thread in each layer to vary as its distance from the centre, how much will she diminish the radius who winds off the first portion?

FIRST SOLUTION. *By Mr. P. Barton, Jun., Athol, Mass.*

Put $r = 3$ = radius of the ball;
q = value of an unit's mass of the thread at an unit's distance from the centre;
x = any variable distance from the centre;
$a = 3{,}14159$ &c.
Then qx = value of an unit's mass at the distance x from the centre;
$4\,ax^2$ = surface of the ball at the distance x from the centre;
$4\,ax^2\,dx$ = magnitude of a shell, at the distance x, and thickness dx,
$4aqx^3\,dx$ = value of that shell, or the differential of the worth of the ball;
$\int_0^x 4\,aqx^3\,dx = aqx^4$ = value of a ball, radius x;
$\int_0^r 4\,aqx^3\,dx = aqr^4$ = value of the whole ball.

Hence, if the first lady leaves a ball, radius x, and worth $\frac{2}{3}$ of the price of the ball, we must have

$$aqx^4 = \tfrac{2}{3}\,aqr^4,$$
$$x = r\sqrt[4]{\tfrac{2}{3}} = 3\sqrt[4]{\tfrac{2}{3}} = \sqrt[4]{54},$$

and she diminishes the radius by

$$r - x = 3 - \sqrt[4]{54} = ,2892 \text{ inches}.$$

SECOND SOLUTION. *By a Lady, the proposer.*

Let r = radius of the given ball, and x = radius of the ball after the first lady has wound off her portion. Then, since the quantity of the thread in any ball varies as the third power of its radius, and the quality varies as the radius, it is manifest that the value will vary as the fourth power of the radius; hence, by the question,

$$x^4 = \tfrac{2}{3}\,r^4,$$
$$\therefore x = r\sqrt[4]{\tfrac{2}{3}};$$

in the present question $r = 3$ inches, and $x = 2{,}7108$ inches, and therefore she diminishes the radius by

$$3 - 2{,}7108 = ,2892 \text{ inches}.$$

List of Contributors to the Junior Department, and of Questions answered by each. The figures refer to the number of the Questions, as marked in Number IV., Article VII., page 203.

A Lady, ans. 1, 2, 3, 4, 5, 6.
R. Dewar Bacot, St. Paul's College, Long Island, ans. 2, 3, 4, 5.
P. Barton, Jun., Athol, Mass., ans. 1, 2, 3, 4, 6.
Wm. R. Biddlecom, Clinton Liberal Institute, ans. 1, 2, 3, 4.
B. Birdsall, New Hartford, Oneida Co., N. Y., ans. 1, 2, 3, 4, 5, 6.
Geo. K. Birely, Frederick, Md., ans. 1, 2, 3, 4, 5.
J. Blickensderfer, Jun., near Roscoe, Ohio., ans. 2, 3, 4.
James J. Bowden, St. Paul's College, Long Island, ans. 1, 2, 3, 4, 5.
Geo. W. Coaklay, Peekskill Academy, N. Y., ans. 1, 2, 3, 4.
Warren Colburn, St. Paul's College, Long Island, ans. 1, 4.
Robt. S. Howland, St. Paul's College, Long Island, ans. 1, 2, 3, 4.
Isaac A. Saxton, W. Winfield, Herkimer Co., N. Y., ans. 1, 2, 3, 4, 5.
(W.), Student in Dickinson College, Carlisle, Pa., ans. 1, 2, 4, 5.

*** The increase of our correspondents in this department, and the evident improvement in many of their communications, has determined us to offer a greater variety of questions for solution, and a wider arena for the display of their talents. We would at the same time caution our young correspondents against indulging an anxiety to answer a great number of questions, rather than the more commendable one of giving *good* solutions to those questions they do attempt. One or two good solutions will be of more service to the student, and will do him more credit, than ten bad ones. Let him not then pass over a problem, contented with knowing enough on the subject to give a meagre sketch of a solution; but let him study the subject in all its phases, and when he has possessed himself with all the properties of figure or of quantity that bear upon the question, he will be properly qualified to write upon it. His communications may then contain a great deal of matter in a little space.

ARTICLE XI.

QUESTIONS TO BE ANSWERED IN NUMBER VI.

Their Solutions must arrive before August 1st, 1838.

(19.) QUESTION I. *By* ——.

1°. Reduce $19^\circ\ 43'\ 27''$ to time, at the rate of 15° to the hour.
2°. Reduce 19 hr. 43 m. 27 s. to degrees, at the same rate.

(20.) QUESTION II. *By* ——.

Find a vulgar fraction equivalent to the circulating decimal,
3,8123123123. . . . .

(21.) QUESTION III. *By* ——.

A banker borrows a sum of money at 4 per cent. per annum, and pays the interest at the end of the year. He lends it out at the rate of 5 per cent. per annum, and receives the interest half yearly. By this means he gains $100 a year; how much does he borrow?

(22.) QUESTION IV. *By* ——.

Prove that

$$(m+n)\sqrt{\frac{m}{n}}-(m-n)\sqrt{\frac{n}{m}}=(m+n)\sqrt{\frac{n}{m}}+(m-n)\sqrt{\frac{m}{n}}$$
$$=m\sqrt{\frac{m}{n}}+n\sqrt{\frac{n}{m}}.$$

(23.) QUESTION V. *By* ——.

Determine A and B, so that

$$\frac{cx+d}{(x+a)(x+b)}=\frac{A}{x+a}+\frac{B}{x+b},$$

whatever x may be.
Example. Let $a=2$, $b=-3$, $c=7$, $d=-6$.

(24.) QUESTION VI. *By a Lady.*

Given the equation

$$x^4-3x^3-\frac{35}{4}x^2+\frac{33}{2}x-2075\tfrac{1}{4}=0,$$

to find x by quadratics.

(25.) QUESTION VII. *By Mr. Geo. W. Coakley.*

If a and b be two sides of a triangle including the angle c, and l the line bisecting the angle c and terminating in the third side, prove that

$$\cos \tfrac{1}{2} c = \frac{l(a+b)}{2\,ab}$$

(26.) QUESTION VIII. *By* ——.

Within a given sphere two equal ones are inscribed, their radii being each half that of the given one. It is required to prove that there can be six other equal spheres inscribed within the first, each touching the three former ones, and each also touching two of the others.

See Solution to question (50), equation (27), page 248, where k must be taken $= 1$.

(27.) QUESTION IX. *By* ——.

The diagonals of a given regular pentagon form, by their intersection, another regular pentagon. It is required to find its side and area.

(28.) QUESTION X. *By* ——.

In any right angled spherical triangle, prove that the ratio of the cosines of the two sides including the right angle is equal to the ratio of the sines of twice their opposite angles.

(29.) QUESTION XI. *By* ——.

Through a point, given by its rectangular co-ordinates, to draw two straight lines, including a given angle, and intercepting a segment on the axis of y, of a given length.

(30.) QUESTION XII. *By Mr. P. Barton, Jun.*

In a given semicircle, it is required to inscribe the greatest isosceles triangle, having its vertex in the extremity of the diameter, and one of its equal sides coinciding with the diameter.

SENIOR DEPARTMENT.

ARTICLE XIX.

SOLUTIONS TO THE QUESTIONS PROPOSED IN ARTICLE XII. NUMBER III.

(51.) QUESTION I. *By* ——:

Divide $x^4 + ax^2 + b$ into two real quadratic factors.

FIRST SOLUTION. *By J. B. H., Cambridge, Mass.*

Suppose the factors to be

$$x^2 + mx + n \text{ and } x^2 + px + q:$$

that is, let

$$x^4 + ax^2 + b = (x^2 + mx + n)(x^2 + px + q)$$
$$= x^4 + (m + p)x^3 + (n + mp + q)x^2 + (np + mq)x + nq,$$

which must be true independently of x, therefore

$$m + p = o, (1); \quad n + mp + q = a, (2);$$
$$np + mq = o, (3); \quad nq = b, (4).$$

From the first,

$$m = -p;$$

Substituting this value of m in (3), we get

$$np - qp = o, \text{ or } p(n - q) = o;$$

whence $\quad p = o, \quad$ or $n - q = o \quad . \; . \; . \; . \; . \; . \quad (5).$

First, $\quad$ let $p = o$, then also $m = o$;

Then (2) and (4) become

$$n + q = a, \; nq = b;$$

also $\quad n - q = \sqrt{(n + q)^2 - 4nq} = \sqrt{a^2 - 4b},$

whence $\quad n = \frac{1}{2}a + \frac{1}{2}\sqrt{a^2 - 4b}, \; q = \frac{1}{2}a - \frac{1}{2}\sqrt{a^2 - 4b};$

so that the required factors are

$$x^2 + \tfrac{1}{2}a + \tfrac{1}{2}\sqrt{a^2 - 4b} \quad \text{and} \quad x^2 + \tfrac{1}{2}a - \tfrac{1}{2}\sqrt{a^2 - 4b},$$

which are real only when

$$a^2 = \text{or} > 4b.$$

Second, $\quad$ in (5) let $n - q = o$,

then by (4),

$$n = q = \sqrt{b},$$

and (2) becomes

$$2\sqrt{b} - p^2 = a,$$

whence, $\quad p = -\sqrt{2\sqrt{b} - a}, \; m = -p = \sqrt{2\sqrt{b} - a},$

and the required factors are

$$x^2 + x\sqrt{2\sqrt{b} - a} + \sqrt{b} \text{ and } x^2 - x\sqrt{2\sqrt{b} - a} + \sqrt{b},$$

which are real when

$$a^2 = \text{or} < 4b.$$

In the case $a^2 = 4b$, either set become

$$x^2 + \tfrac{1}{2}a \text{ and } x^2 + \tfrac{1}{2}a.$$

——— Mr. Macully's solution was nearly like this.

SECOND SOLUTION. *By Professor Catlin, Hamilton College.*

By solving the equation

$$x^4 + ax^2 + b = o,$$

by the usual method, we get

$$x^2 = -\tfrac{1}{2} a \pm \tfrac{1}{2} \sqrt{a^2 - 4b} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

Hence, by the theory of equations,

$$x^4 + ax^2 + b = (x^2 + \tfrac{1}{2} a - \tfrac{1}{2} \sqrt{a^2 - 4b})\,(x^2 + \tfrac{1}{2} a + \tfrac{1}{2} \sqrt{a^2 - 4b}) \quad . \ (2).$$

Which are the quadratic factors required, when $a^2 > 4b$.

When $a^2 < 4b$, or $\dfrac{a}{2\sqrt{b}} < 1$, we can assume an angle δ, such that

$$\cos 2\delta = -\frac{a}{2\sqrt{b}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

and therefore, $$\cos \delta = \sqrt{\frac{1 + \cos 2\delta}{2}} = \frac{\sqrt{2\sqrt{b} - a}}{2\sqrt[4]{b}} \quad . \quad . \quad . \quad . \quad (4).$$

Then, $$x^4 + ax^2 + b = x^4 + 2 \,.\, \frac{a}{2\sqrt{b}} \,.\, x^2\sqrt{b} + b$$

$$= x^4 - 2\cos 2\delta \,.\, x^2\sqrt{b} + b \quad . \quad . \quad . \quad . \quad (5).$$

Whence, from Young's Diff. Cal., page 31, (or from solutions to (15) Question III., in the Junior Department of the Miscellany,)

$$x^4 + a x^2 + b = (x^2 - 2x\sqrt[4]{b} \,.\, \cos\delta + \sqrt{b})\,(x^2 + 2x\sqrt[4]{b} \,.\, \cos\delta + \sqrt{b})$$

$$= (x^2 - x\sqrt{2\sqrt{b} - a} + \sqrt{b})\,(x^2 + x\sqrt{2\sqrt{b} - a} + \sqrt{b}) \ (6).$$

——— Dr. Strong, after showing that

$$x^4 + ax^2 + b = (x^2 + \tfrac{1}{2} a + \tfrac{1}{2} \sqrt{a^2 - 4b})(x^2 + \tfrac{1}{2} a - \tfrac{1}{2} \sqrt{a^2 - 4b}),$$

which are real factors when $a^2 - 4b > o$, proceeds thus:

When $a^2 - 4b < o$, put

$$m^2 = -\tfrac{1}{2} a - \tfrac{1}{2} \sqrt{a^2 - 4b},\ n^2 = -\tfrac{1}{2} a + \tfrac{1}{2} \sqrt{a^2 - 4b},$$

then $$x^4 + ax^2 + b = (x^2 - m^2)\,(x^2 - n^2),$$

$$= (x - m)\,(x + m)\,(x - n)\,(x + n)$$

$$= (x - m)\,(x - n) \times (x + m)\,(x + n)$$

$$= (x^2 - \overline{m + n} \,.\, x + mn)(x^2 + \overline{m + n} \,.\, x + mn).$$

But $$mn = \sqrt{m^2 n^2} = \sqrt{\tfrac{1}{4} a^2 - \tfrac{1}{4} a^2 + b} = \sqrt{b},$$

and $$m + n = \sqrt{(m + n)^2} = \sqrt{2\,mn + (m^2 + n^2)} = \sqrt{2\sqrt{b} - a};$$

$$\therefore\ x^4 + ax^2 + b = (x^2 - x\sqrt{2\sqrt{b} - a} + \sqrt{b})\,(x^2 + x\sqrt{2\sqrt{b} - a} + \sqrt{b}).$$

(52.) QUESTION II. *By Mr. P. Barton, Jun., Athol, Mass.*

On the base of a given right angled triangle, a series of the greatest squares are constructed, each having an angular point in the hypothenuse: determine the side of the n^{th} square, and the sum of the areas of n squares, or of an infinite number of them.

FIRST SOLUTION. *By Mr. N. Vernon, Frederick, Md.*

Let a = base, b = perpendicular, x, x', x'', &c., = the sides of the quares, consequently the bases of the triangles formed between the

squares and the hypothenuse; and as the triangles are similar to the given one we have

$$a : b :: x : b - x,$$
$$\text{or } a + b : a :: b : x,$$
$$\text{and } x = \frac{ab}{a+b}.$$

Similarly
$$a : b :: x' : x - x'$$
$$\text{or } a + b : a :: x : x'$$
$$\text{and } x' = \frac{ax}{a+b} = \frac{a^2 b}{(a+b)^2}.$$

So
$$x'' = \frac{ax'}{a+b} = \frac{a^3 b}{(a+b)^3},\ x''' = \frac{ax''}{a+b} = \frac{a^4 b}{(a+b)^4},\ \text{\&c.}$$

The side of the n^{th} square being evidently $\frac{a^n b}{(a+b)^n}$.

The series of squares will therefore form a geometrical progression, whose first term is $\frac{a^2 b^2}{(a+b)^2}$, and ratio $\frac{a^2}{(a+b)^2}$. Hence, the sum of n squares will be

$$s = \frac{a^2 b}{2a+b} - \frac{a^{2n+2} b}{(2a+b)(a+b)^{2n}},$$

the second term of which will vanish when n is infinite, and therefore the sum of all the squares so formed is

$$s = \frac{a^2 b}{2a+b}.$$

SECOND SOLUTION. *By Mr. B. Birdsall, New Hartford.*

Let b = base, c = perpendicular, x = side of greatest square, then by similar triangles we get

$$b : c :: x : c - x, \therefore x = \frac{bc}{b+c}.$$

In a similar manner, if z = side of the next square, we get

$$b : c :: z : x - z,\ \therefore z = \frac{b^2 c}{(b+c)^2},\ \text{and so on.}$$

Hence, side of 1st square $= \frac{cb}{b+c}$, and its area $\frac{c^2 b^2}{(b+c)^2}$,

$$\ldots\ 2\text{nd}\ \ldots = \frac{cb^2}{(b+c)^2},\quad \ldots\ \frac{c^2 b^4}{(b+c)^4}$$
$$\ldots\ 3\text{rd}\ \ldots = \frac{cb^3}{(b+c)^3},\quad \ldots\ \frac{c^2 b^6}{(b+c)^6}$$
&c. &c.
$$\ldots\ n^{th}\ \ldots = \frac{cb^n}{(b+c)^n},\quad \ldots\ \frac{c^2 b^{2n}}{(b+c)^n}.$$

And, if s be the sum of this geometrical series of squares,

$$s = c^2\left(\frac{b}{b+c}\right)^2 + c^2\left(\frac{b}{b+c}\right)^4 + c^2\left(\frac{b}{b+c}\right)^6 + \ldots + c^2\left(\frac{b}{b+c}\right)^{2n}$$

$$= c^2\left(\frac{b}{b+c}\right)^2 \times \frac{1-\left(\frac{b}{b+c}\right)^{2n}}{1-\left(\frac{b}{b+c}\right)^2},$$

If n is infinite, then we have

$$s = c^2\left(\frac{b}{b+c}\right)^2 \times \frac{1}{1-\left(\frac{b}{b+c}\right)^2}$$

$$= \frac{cb^2}{2b+c}$$

THIRD SOLUTION. *By Mr. R. S. Howland, St. Paul's College, Flushing, L. I.*

Let A = the angle of the triangle adjacent to the side on which are the sides of the inscribed squares,
a = the side opposite to A,
x_n = side of the n^{th} square,
s_n = sum of n squares,
s = sum of an infinite number.

Then will

$$\tan A = \frac{a - x_1}{x_1} = \frac{a}{x_1} - 1,$$

$$\therefore x_1 = \frac{a}{1+\tan A} = ar,$$

$$\text{where } r = \frac{1}{1+\tan A}.$$

But the square, whose side is x_2 is inscribed in a triangle, in which the side opposite the angle A is x, &c., therefore

$$x_2 = x_1\, r = ar^2,$$
$$x_3 = x_2\, r = ar^3,$$
&c.
$$x_n = x_{n-1}\, r = ar^n.$$

$$\therefore s_n = a^2r^2 + a^2r^4 + a^2r^6 + \ldots\ldots + a^2r^{2n}$$
$$= a^2r^2 . \frac{1-r^{2n}}{1-r^2};$$

$$\text{and } s = \frac{a^2r^2}{1-r^2}$$
$$= \frac{a^2}{2\tan A + \tan^2 A}.$$

—— In a similar manner, if B be the other angle of the triangle, b the side opposite to A, and s' the sum of all the squares having their sides on a, and an angle on the hypothenuse, we should have

$$s' = \frac{b^2}{2\tan B + \tan^2 B}$$

$$= \frac{a^2 \cot^2 A}{2 \cot A + \cot^2 A}$$

$$= \frac{a^2}{2 \tan A + 1},$$

and the sum of all these squares is

$$s + s' - x_1^2 = \frac{a^2}{2 \tan A + \tan^2 A} + \frac{a^2}{2 \tan A + 1} - \frac{a^2}{(1 + \tan A)^2}$$

$$= \frac{ab^2}{2b + a} + \frac{a^2 b}{2a + b} - \frac{a^2 b^2}{(a + b)^2}$$

$$= \frac{ab(a+b)^4 - a^3 b^3}{(2b + a)(2a + b)(a + b)^2}.$$

After these squares are taken from the triangle, there will remain a series of small triangles vanishing towards the angles A and B, and the sum of their areas is

$$\tfrac{1}{2} ab - (s + s' - x_1^2) = \frac{a^2 b^2 (a + b)^2 + 2 a^3 b^3}{2(2b + a)(2a + b)(a + b)^2}.$$

(53.) QUESTION III. *By Professor Chamberlain, Oakland College, Miss.*

The distance from one of the angles of a given triangle to a point within it is d, required the lengths of the two lines drawn from the same point to the other two angles of the triangle, when the given line d is equally inclined to the required lines.

FIRST SOLUTION. *By Professor B. Peirce, Harvard University.*

Let 2 C denote the angle from which d is drawn, let a and a' be the sides of the given triangle which include this angle, let v and v' be the angles which d makes with a and a', and φ the angle which d makes with each of the lines drawn to the two other angles. We have, then,

$$v + v' = 2\,\mathrm{C},$$

$$\sin(\varphi + v) : \sin\varphi = d : a = \sin v \cot\varphi + \cos v : 1,$$

$$\sin(\varphi + v') : \sin\varphi = d : a' = \sin v' \cot\varphi + \cos v' : 1.$$

Hence we readily obtain

$$d - a\cos v = a \sin v \cot\varphi,$$

$$d - a'\cos v' = a' \sin v' \cot\varphi;$$

and by division

$$\frac{d - a\cos v}{d - a'\cos v'} = \frac{a \sin v}{a' \sin v'},$$

$$a' d \sin v' - aa' \cos v \sin v' = ad \sin v - aa' \sin v \cos v',$$

$$a' d \sin v' - ad \sin v = aa' \sin(v' - v);$$

and if we use $\quad 2x = v' - v,$

we have $\quad a' d \sin(\mathrm{C} + x) - ad \sin(\mathrm{C} - x) = aa' \sin 2x,$

from which x is more readily obtained than it would be from the equation of the fourth degree to which this would lead. This approximation is readily obtained as follows: make

$$\tan e = \frac{a' - a}{a' + a} \cdot \tan \mathrm{C}, \quad A = \frac{aa' \cos e}{(a' + a)\cos \mathrm{C}},$$

and the equation becomes, by an easy reduction,

$$d \sin(e + x) = \text{A} \sin 2x.$$

If, then, w is a first approximation to the value of x, and if

$$h = \frac{d \sin(e + w) - \text{A} \sin 2w}{2 \text{A} \cos 2w - d \cos(e + w)},$$

the next approximation is

$$x = w + h.$$

Case. When we have given

$$a = a'$$

the equation for finding x is reduced to

$$2d \cos \text{C} \sin x = a \sin 2x = 2a \sin x \cos x;$$

whence $\sin x = o, \quad x = o$ or $180°$;

or $\cos x = \frac{d}{a} \cos \text{C}.$

Having found x by the preceding process, the lengths of the lines drawn to the two other angles, are obviously the sides of two triangles in which the two other sides and their included angle are known.

SECOND SOLUTION. *By Mr. P. Barton, Jun., Athol, Mass.*

Let the sides of the triangle, a and b, from which the line d_1 is drawn, be the axes of x and y, and let their included angle, or the angle of ordination be denoted by C, the co-ordinates of the extremity of d_1 by x_1 and y_1, the line drawn from (a, o) to (x_1, y_1) by d_2, and that from (o, b) to $(x_1 . y_1)$ by d_3; then

the equation of d_1, is $\quad y = \frac{y_1}{x_1} . x = a_1 x,$

$d_2 \quad y = \frac{y_1}{x_1 - a} . (x - a) = a_2 (x - a),$

$d_3 \quad y - b = \frac{y_1 - b}{x_1} . x = a_3 x.$

But $\quad y_1^2 + x_1^2 + 2 y_1 x_1 \cos \text{C} = d_1^2 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$

Also, tangent of angle between d_1 and $d_2 = \frac{(a_1 - a_2) \sin \text{C}}{1 + (a_1 + a_2) \cos \text{C} + a_1 a_2}$

$$= \frac{- a y_1 \sin \text{C}}{x_1^2 - a x_1 + y_1 (2x_1 - a) \cos \text{C} + y_1^2}$$

$$= \frac{- a y_1 \sin \text{C}}{d_1^2 - a x_1 - a y_1 \cos \text{C}}$$

$$= \frac{y_1 \sin \text{C}}{x_1 + y_1 \cos \text{C} - k}$$

tangent of angle between d_1 and $d_3 = \frac{(a_3 - a_1) \sin \text{C}}{1 + (a_3 + a_1) \cos \text{C} + a_1 a_3}$

$$= \frac{- b x_1 \sin \text{C}}{d_1^2 - b y_1 - b x_1 \cos \text{C}}$$

$$= \frac{x_1 \sin \text{C}}{y_1 + x_1 \cos \text{C} - l}$$

where $k=\frac{d_1^2}{a}$, and $l=\frac{d_1^2}{b}$. Hence, by the question,

$$\frac{y_1}{x_1+y_1\cos c-k}=\frac{x_1}{y_1+x_1\cos c-l},$$

$$\text{or } y_1^2-ly_1=x_1^2-kx_1 \quad \ldots\ldots\ldots\ldots \quad (2)$$

By eliminating y_1 between the equations (1) and (2),

$$4x_1^4\sin^2 c-4kx_1^3\sin^2 c+(k^2+2kl\cos c+l^2-4d_1^2)x_1^2+ \\ 2(k-l\cos c)d_1^2x_1+d_1^4-l^2d_1^2=o.$$

After y_1 and x_1 are determined from these equations, we shall have the required lines from,

$$\begin{aligned} d_2^2 &= y_1^2+(x_1-a)^2+2y_1(x_1-a)\cos c \\ &= d_1^2+a^2-2ax_1-2ay_1\cos c \\ &= a(k+a-2x_1-2y_1\cos c), \\ d_3^2 &= (y_1-b)^2+x_1^2+2x_1(y_1-b)\cos c \\ &= b(l+b-2y_1-2x_1\cos c). \end{aligned}$$

——— The equations (1) and (2) of Mr. Barton's solution, are well adapted for determining the point y_1x_1 by construction. Equation (1) is that of a circle whose centre is in the origin of co-ordinates and radius d_1; equation (2) that of an equilateral hyperbola, the co-ordinates of its centre are $\frac{1}{2}l$ and $\frac{1}{2}k$, one of its assymptotes is parallel to a line bisecting the angle of ordination, and its axis is $=\sqrt{(l^2-k^2)\sin c}$. The intersection of these two loci are the points to which the line is to be drawn.

Several of our correspondents mention that the question has been proposed before.

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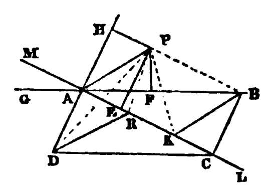
(54). Question V. *By Professor Marcus Catlin.*

If, from a given point in the plane of a given parallelogram, perpendiculars be drawn to the diagonal and to the two sides which contain this diagonal; then the product of the diagonal by its perpendicular is equal to the sum of the products of the two sides into their respective perpendiculars, when the point is taken without the parallelogram, or to their difference when the point is taken within. Required a demonstration.

First Solution. *By Professor Catlin.*



Let ABCD be any parallelogram, P any point in its plane. Draw the perpendiculars PE, PF and PH. Draw BK and DR parallel to AP.

Then the triangles ABP, AKP are obviously equal.

Hence $\quad$ PF $\times$ AB $=$ PE $\times$ AK . . . . (1).

In like manner the equal triangles APD, APR and BCK give

PH $\times$ AD $=$ PE $\times$ AR $=$ PE $\times$ CK . . . . . (2).

Now it is obvious that when P is situated within the angle BAH or its

opposite DAG, the point K will fall between A and C, and consequently the triangle

$$APC = AKP + CKP \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

hence, by (1) and (2)

$$PF \times AB + PH \times AD = PE \times (AK + KC) = PE \times AC \quad . \quad . \quad (4).$$

Again, when P is situated within the angle BAD or GAH, the point K will fall beyond A or C, towards M or L, and consequently the triangle

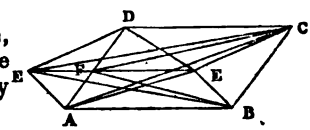
$$APC = \pm (AKP - CKP) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

hence by (1) and (2),

$$PF \times AB - PH \times AD = \pm PE \times AC \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Note. It is obvious that the theorem was not correctly stated. The latter part of the proposition should read thus—"when the point is taken without the angle contained by the two sides including the diagonal, or by the two sides produced; or to their difference when the point is taken within either of said angles."

SECOND SOLUTION. *By Mr. J. B. H., Cambridge.*



Let ABCD be the given parallelogram, and E the given point. Representing the perpendicular from E to AC by p, to AB by p', to AD by p'', we are to prove that

$$AD \times p'' \mp AB \times p' = AC \times p,$$

where the upper sign applies when the point E is within the parallelogram, and the lower when it is without.

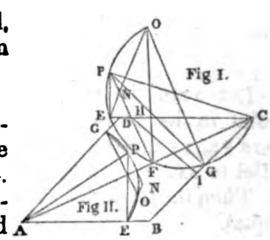
Demonstration. Draw EF parallel to AB, and join EA, EB, EC, ED; join also FB and FC. Then

$$AFC = AFB = AEB.$$

$$\begin{aligned} \text{And} \quad AEC &= AFE + FEC \mp AFC \\ &= AFE + FED \mp AEB \\ &= AED \mp AEB, \end{aligned}$$

$$\text{or } AC \,.\, p = AD \,.\, p'' \mp AB \,.\, p'.$$

THIRD SOLUTION. *By Wm. Lenhart, Esq., York, Penn.*



Let ABCD be a parallelogram, AC a diagonal, P a point without, PE, PF, PG perpendiculars on DC, AC, BC respectively: then

$$AC \times PF = DC \times PE + BC \times PG.$$

Demonstration. On PC as a diameter, describe an arc, which, as the angles at E, F, G are right, will pass through the points E, F, G. Draw GO parallel to PF, and join OF. Join also EO, cutting PF in N (fig. 1) or PF produced (fig. 2). Now the angles EPN = NOF = ACD, standing on the same arc EF, and the angles PEN = NFO = FOG = ACB, standing on equal arcs PO, FG: therefore the triangles PEN, NFO, ACB are all similar, consequently

$$AC : DC = PE : PN, \text{ and } AC \times PN = DC \times PE,$$

$$AC : BC = PG : FN, \text{ and } AC \times FN = BC \times PG \;;$$
$$\therefore AC\,(PN + FN) = AC \times PF = DC \times PE + BC \times PG,$$
which was to be demonstrated.

In the same manner the triangles (fig. 2) PEN, OFN, ACB are proved to be similar, therefore

$$AC : AB = PE : PN, \text{ and } AC \times PN = AB \times PE,$$
$$AC : BC = PG : FN, \text{ and } AC \times FN = AD \times PG;$$
$$\therefore AC\,(PN - FP) = AC \times PF - AD \times PG.$$

Q. E. D.

Cor. When PE (fig. 1) vanishes, the points P and G fall on H and I in the sides DC and BC, then

$$AC \times HF = BC \times HI.$$

(18.) QUESTION V. *By* A.

Convert a^x into a series, in a more simple manner than is usually done; and then deduce Rules for finding the logarithms of numbers.

FIRST SOLUTION. *By Dr. Strong.*

Let n denote any arbitrary number, and put $a = 1 + b$, then

$$a^x = (1 + b)^x = [(1 + b)^{\frac{1}{n}}]^{nx} \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Now $(1 + b)^{\frac{1}{n}} = 1 + \frac{1}{n} \cdot \frac{b}{1} + \frac{1}{n}\left(\frac{1}{n} - 1\right) \cdot \frac{b^2}{1\,.\,2} +$

$$\frac{1}{n}\left(\frac{1}{n} - 1\right)\left(\frac{1}{n} - 2\right) \cdot \frac{b^3}{1\,.\,2\,.\,3} + \&c.$$

$$= 1 + \frac{A}{n} + \varphi\left(\frac{1}{n^2}\right),$$

where $A = b - \frac{1}{2} b^2 + \frac{1}{3} b^3 - \frac{1}{4} b^4 + \&c.$,

$\varphi\left(\frac{1}{n^2}\right) =$ an expression in which all the terms involve $\frac{1}{n^2}$, $\frac{1}{n^3}$, &c.

Then $a^x = \left[1 + \frac{A}{n} + \varphi\left(\frac{1}{n^2}\right)\right]^{nx}$

$$= \left(1 + \frac{A}{n}\right)^{nx} + \varphi'\left(\frac{1}{n^2}\right)$$

$$= 1 + \frac{nx}{n} \cdot \frac{A}{1} + \frac{nx}{n} \cdot \frac{nx - 1}{n} \cdot \frac{A^2}{1\,.\,2} +$$

$$\frac{nx}{n} \cdot \frac{nx - 1}{n} \cdot \frac{nx - 2}{n} \cdot \frac{A^3}{1\,.\,2\,.\,3} + \&c. + \varphi^1\left(\frac{1}{n^2}\right)$$

$$= 1 + Ax + \frac{A^2 x^2}{1\,.\,2} + \frac{A^3 x^3}{1\,.\,2\,.\,3} + \&c. \ldots + \psi(n)\,.\ (2).$$

Where $\psi(n)$ denotes quantities that are dependent on n; and since the

first member of (2) is independent of n, the second member must be so likewise, or $\psi\,(n) = 0$, and we have

$$a^x = 1 + \text{A}x + \frac{\text{A}^2x^2}{1\,.\,2} + \frac{\text{A}^3x^3}{1\,.\,2\,.\,3} + \text{\&c.} \quad . \quad . \quad . \quad (3).$$

If in (3) we put $x = \frac{1}{\text{A}}$, we shall have

$$a^{\frac{1}{\text{A}}} = (1+b)^{\frac{1}{\text{A}}} = 1 + 1 + \frac{1}{2} + \frac{1}{2\,.\,3} + \frac{1}{2\,.\,3\,.\,4} + \text{\&c} = 2{,}7182818\ldots,$$

let e denote this number, which is called the hyperbolic base, and we shall have

$$a = 1 + b = e^{\text{A}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Let L' denote the hyperbolic logarithm of $(1 + b)$, and we have by (4),

$$\text{L}'\,(1 + b) = \text{A} = b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \text{\&c.} \quad . \quad . \quad . \quad (5),$$

$$\therefore \text{L}'\,(1 - b) = \; - b - \frac{b^2}{2} - \frac{b^3}{3} - \frac{b^4}{4}\text{\&c.} \quad . \quad . \quad . \quad . \quad (6),$$

and $$\text{L}'\,(1+b) - \text{L}'\,(1-b) = \text{L}'\left(\frac{1+b}{1-b}\right) = 2(b + \frac{b^3}{3} + \frac{b^5}{5}\,\text{\&c.}) \quad . \quad . \quad (7).$$

$$\text{Put } \frac{1+b}{1-b} = \frac{m}{n}, \text{ or } b = \frac{m-n}{m+n};$$

then $$\text{L}'\left(\frac{m}{n}\right) = 2\left[\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \text{\&c}\right] (8).$$

or if we put $m = n + z$, (8) becomes

$$\text{L}'\,(n+z) = \text{L}'n + 2\left[\frac{z}{2n+z} + \frac{1}{3}\left(\frac{z}{2n+z}\right)^3 + \frac{1}{5}\left(\frac{z}{2n+z}\right)^5 + \text{\&c.}\right] \quad (9),$$

Let $$e^x = a'^y = c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

then $$y = \frac{x}{\text{L}'\,a'} = \log c \text{ to the base } a', \quad . \quad . \quad . \quad . \quad . \quad (11)$$

or $$\log c = \frac{x}{\text{L}'\,a'} = \text{M}\,x = \text{M}\,.\,\text{L}'c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where $\text{M} = \frac{1}{\text{L}'\,a'}$ is the modulous of the system to the base a'. Hence (5) and (9) become

$$\log\,(1 + b) = \text{M}\,(b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4}\,\text{\&c.} \quad . \quad . \quad . \quad . \quad (13),$$

$$\log\,(n + z) = \log n + 2\text{M}\left[\frac{z}{2n+z} + \frac{1}{3}\left(\frac{z}{2n+z}\right)^3 + \text{\&c}\right] \quad (14),$$

which enable us to calculate logarithms in any system whatever.

Again, since $1 + b = a = e^{\text{A}}$, we have $\text{A} = \text{L}'a$, and A may be found by a table of hyperbolic logarithms when the numerical value of a is given, and if $a = 2$, $\text{A} = 1$, and (3) becomes

$$e^x = 1 + x + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \text{\&c.} \quad . \quad . \quad . \quad (15),$$

Since $1 + b = a$, or $b = a - 1$, (13) may also be written

$$\log a = M[a - 1 - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \tfrac{1}{4}(a-1)^4 + \&c.]$$

and if we change a in $a^{\frac{1}{m}}$, it becomes

$$\log a = mM\left[a^{\frac{1}{m}} - 1 - \tfrac{1}{2}\left(a^{\frac{1}{m}} - 1\right)^2 + \tfrac{1}{3}\left(a^{\frac{1}{m}} - 1\right)^3 - \&c.\right] \quad (16);$$

similarly, by changing a into $a^{\frac{-1}{m}}$, it gives

$$\log a = mM\left[1 - a^{\frac{-1}{m}} + \tfrac{1}{2}\left(1 - a^{\frac{-1}{m}}\right)^2 + \tfrac{1}{3}\left(1 - a^{\frac{-1}{m}}\right)^3 + \&c.\right] \quad . \quad (17).$$

(17) and (18) were proposed by Lagrange for computing the logarithms of numbers, and they will converge with great rapidity when m is so great that $a^{\frac{1}{m}}$ is nearly equal to unity.

If we resume (5), and change in it b into b^{-1}, it becomes

$$L'(1 + b^{-1}) = L'(1 + b) - L'b = b^{-1} - \tfrac{1}{2}b^{-2} + \tfrac{1}{3}b^{-3} - \&c.,$$

and subtracting this from (5)

$$L'b = b - b^{-1} - \tfrac{1}{2}(b^2 - b^{-2}) + \tfrac{1}{3}(b^3 - b^{-3}) - \&c.,$$

or changing b into $a^{\frac{-1}{m}}$, it becomes

$$L'a = m\left[a^{\frac{1}{m}} - a^{\frac{-1}{m}} - \tfrac{1}{2}\left(a^{\frac{2}{m}} - a^{\frac{-2}{m}}\right) + \tfrac{1}{3}\left(a^{\frac{3}{m}} - a^{\frac{-3}{m}}\right) - \&c.\right] \quad (18).$$

If, again, we put $a = e^{z\sqrt{-1}}$, then $L'a = z\sqrt{-1}$, $a^{\frac{1}{m}} - a^{\frac{-1}{m}} = e^{\frac{z}{m}\sqrt{-1}} - e^{\frac{-z}{m}\sqrt{-1}} = 2\sqrt{-1}\,.\,\sin\frac{z}{m}$, $a^{\frac{2}{m}} - a^{\frac{-2}{m}} = 2\sqrt{-1}\,\sin\frac{2z}{m}$, &c., and (19) will be changed into

$$z = 2m\left(\sin\frac{z}{m} - \tfrac{1}{2}\sin\frac{2z}{m} + \tfrac{1}{3}\sin\frac{3z}{m} - \&c.\right) \quad . \quad . \ (19).$$

SECOND SOLUTION. *By Mr. J. F. Macully, New York.*

Assume $y = a^x = (1 + a - 1)^x = \{[1 + (a-1)]^n\}^{\frac{x}{n}}$, n being any number whatever. Now

$$[1+(a-1)]^n = 1 + n(a-1) + \frac{n(n-1)}{2}(a-1)^2 + \frac{n(n-1)(n-2)}{2\,.\,3}(a-1)^3 + \&c.$$
$$= 1 + An + Bn^2 + Cn^3 + \&c.,$$

by arranging the terms according to the powers of n, where

$$A = (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \tfrac{1}{4}(a-1)^4 + \&c.$$

$$\therefore y = \{[1+(a-1)]^n\}^{\frac{x}{n}} = \{1 + An + Bn^2 + Cn^3 + \&c.\}^{\frac{x}{n}}$$
$$= 1 + \frac{x}{n}(An + Bn^2 + \&c.) + \frac{x(x-n)}{1\,.\,2n^2}(An + Bn^2 + \&c.,)^2 + \&c.$$
$$= 1 + x(A + Bn + Cn^2 + \&c) + \frac{x(x-n)}{1\,.\,2}(A + Bn + \&c)^2 + \&c.$$

Now as the quantity n is arbitrary, and altogether independent of the function y, all the terms involving it should mutually destroy each other in the development of that function, whence we have

$$y = a^x = 1 + Ax + \frac{A^2x^2}{2} + \frac{A^3x^3}{2.3} + \&c.$$

——— Mr. Macully then proceeds to deduce logarithmic formulas, as in the first solution. Although this question was proposed for the purpose of getting the development of the function by a simple algebraic process, yet we shall be excused for inserting the following elegant analysis.

THIRD SOLUTION. *By Professor Peirce.*

Suppose $f(x)$ to denote a function of x such that

$$d \,.\, a^x = f(x)\, dx;$$

we have also $\quad d \,.\, a^{x+e} = f(x+e).\, dx.$

But $\quad a^{x+e} = a^e \,.\, a^x,$

and $\quad d \,.\, a^{x+e} = a^e d \,.\, a^x = a^e f(x)\, dx = f(x+e)\, dx,$

or $\quad a^e f(x) = f(x+e).$

Hence, if $\quad x = e = -b,$

we have $\quad a^{-b} f(b) = f(0) = \text{const.} = A,$

$$f(b) = Aa^b,$$
$$f(x) = Aa^x,$$
$$d \,.\, a^x = Aa^x\, dx,$$
$$d^2 \,.\, a^x = A^2 a^x\, dx^2, \&c.,$$

and, by Maclaurin's Theorem,

$$a^x = 1 + Ax + \frac{1}{1.2} A^2x^2 + \&c.$$

Hence the rules for finding logarithms may be deduced in the usual way.

(56.) QUESTION VI. *By* ——.

Def. A diameter of a curve is the locus of the middle points of a system of parallel chords.

Find the equations of the diameters of the curves represented by the general equation of the second degree between two variables; show that, in general, they all pass through a fixed point; and determine the position of those diameters which bisect their systems of chords perpendicularly.

FIRST SOLUTION. *By Mr. J. F. Macully.*

1°. The general equation of the second degree between two variables is $\quad Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0 \quad . \quad . \quad . \quad . \quad (1).$

The equation $\quad y = ax + b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$

will represent a system of parallel chords, when a is considered constant and b variable. Let $y'\ x'$, $y''\ x''$ be the points of intersection of the loci (1) and (2), or the extremities of the chord (2), and YX the middle point of the chord, then

$$Y = \tfrac{1}{2}(y' + y''),\ X = \tfrac{1}{2}(x' + x'') \quad . \quad . \quad . \quad . \quad . \quad (3).$$

also, since this point is in the line (2),

$$Y = aX + b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

But, by eliminating y between (1) and (2), we find

$$(Aa^2 + Ba + C)x^2 + (2Aab + Bb + Da + E)x + Ab^2 + Db + F = 0 \,.\, (5),$$

therefore, since x', x'' are the roots of this equation,

$$X = \tfrac{1}{2}(x' + x'') = -\frac{(2Aa + B)b + Da + E}{2(Aa^2 + Ba + C)},$$

or $\qquad (2Aa + B)b + 2(Aa^2 + Ba + C)X + Da + E = 0 \quad . \quad . \quad (6),$

and eliminating b between (4) and (6),

$$(2Aa + B)Y + (Ba + 2c)X + Da + e = 0 \quad . \quad . \quad . \quad (7),$$

which is the equation of the locus of the middle points of (2); therefore *all diameters of the locus* (1) *are straght lines.*

2°. The equation of *any* second diameter is

$$(2Aa' + B)Y + (Ba' + 2C)X + Da' + E = 0 \quad . \quad . \quad . \quad (8),$$

and the equations (7) and (8), solved for Y and X will give the point of intersection of the two diameters they represent, we thus find

$$Y = \frac{2CD - BE}{B^2 - 4AC},\ X = \frac{2AE - BD}{B^2 - 4AC} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

which, being independent of a and a', show that *all* diameters intersect each other in this same point, which is thence called the *centre;* and therefore, if the co-ordinates of this point be represented by y', x', the equation of any diameter, (7), may take the form

$$y - y' = c(x - x') \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

$$\text{where } c = -\frac{Ba + 2C}{2Aa + B},$$

$$\text{or} \qquad 2Aac + B(a + c) + 2C = 0 \quad . \quad . \quad . \quad . \quad . \quad (11).$$

This equation, being symmetrical between a and c, shows that they may be interchanged with each other, or that a diameter whose equation is

$$y - y' = a(x - x') \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

which is that parallel chord of the system in (2) that passes through the centre, bisects a system of chords parallel to the diameter (10). Two such diameters, each bisecting a system of chords parallel to the other one, are called *conjugate diameters,* and the angles they make with the axes of co-ordinates have the relation (11).

In the case $B^2 = 4AC$ when, as is well known, (1) represents the parabola, the values of Y and X in (9) become infinite, and therefore the diameters are parallel to each other; their equations may be

$$Y + \frac{B}{2A}X + \frac{Da + E}{2Aa + B} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

3°. When the diameter (10) is perpendicular to the chords (2), or to its conjugate (12), then (if the axes of co-ordinates are rectangular)

$$ac + 1 = 0, \text{ or } ac = -1,$$

and from (11) $\qquad a + c = 2 \,.\, \dfrac{A - C}{B},$

therefore $\qquad a - c = \dfrac{2}{B}\sqrt{(A - C)^2 + B^2}$

and $$a = \frac{A - C + \sqrt{(A-C)^2 + B^2}}{B},\ c = \frac{A - C - \sqrt{(A-C)^2 + B^2}}{B} \quad . \ (14),$$

which determines the position of the only two diameters that bisect their chords perpendicularly; they are called the axes of the curve.
In the case of the parabola,

$$a \cdot \frac{B}{2A} + 1 = 0, \text{ or } a = -\frac{2A}{B}$$

and therefore the equation of the axis of the parabola is

$$Y + \frac{B}{2A} X + \frac{BE - 2AD}{B^2 - 4A^2} = 0 \quad . \quad . \quad . \quad . \quad . \ (15).$$

SECOND SOLUTION. *By Mr. L. Abbot, Niles.*

Let the general equation be

$$y^2 = mx^2 + nx,$$

and let $$y = c(x + a),$$

be the equation of any chord, then for the contact of the chord and its arc we have

$$y^2 = mx^2 + nx = c^2(x+a)^2$$

$$\text{or } x = \frac{2ac^2 - n \pm \sqrt{4c^2a^2(m - c^2) + (n - 2ac^2)^2}}{2(m - c^2)},$$

and consequently if the middle point of the chord be denoted by β, γ

$$\gamma = \frac{2ac^2 - n}{2(m - c^2)} = \text{half the sum of the two values of } x,$$

$$\text{therefore } a = \frac{2(m - c^2)\gamma + n}{2c^2},$$

$$\text{and } \beta = c(\gamma + a) = \frac{m}{c}\left(\gamma + \frac{n}{2m}\right).$$

This is the relation between the co-ordinates of the middle points of all chords that have the same value for c, or of a system of parallel chords; and therefore it is the equation of a diameter. It is a straight line.

The values $\gamma = -\frac{n}{2m}$, $\beta = 0$, are independent of c and fulfil the equation of the diameter, therefore all diameters pass through the point whose co-ordinates are $\beta = 0, \gamma = -\frac{n}{2m}$. If $m = 0$, this point is at an infinite distance, or all diameters of the parabola are parallel to the axis.

The diameter will be perpendicular to the chords it bisects when

$$\frac{m}{c} = -\frac{1}{c}.$$

This equation is satisfied, first when $m = -1$, showing that all diameters of the circle bisect their chords at right angles; second, when $c = \infty$, or when the chords are perpendicular to the axis of x, which is itself one of the diameters required; third, when $c = 0$, and the equation of the diameter is then $\gamma = -\frac{n}{2m}$, or it is perpendicular to the axis of x.

(57.) QUESTION VII. *By Mr. Lenhart.*

Find x, y, z, such that x^2+xy+y^2, x^2+xz+z^2, y^2+yz+z^2 shall be squares.

FIRST SOLUTION. *By the Proposer.*

By substituting the factors abc, dbf, and ncf for x, y and z respectively, and dividing by b^2, c^2, f^2, the formulas become

$$a^2c^2+ac\,df+d^2f^2=\square \quad . \quad . \quad . \quad . \quad (1),$$
$$a^2b^2+ab\,nf+n^2f^2=\square \quad . \quad . \quad . \quad . \quad (2),$$
$$d^2b^2+db\,nc+n^2c^2=\square \quad . \quad . \quad . \quad . \quad (3),$$

which, according to La Grange, in his additions to Euler, art. 90, vol. II, will be effected by putting

$$p^2-q^2=ac, \qquad 2pq+q^2=df,$$
$$p'^2-q'^2=ab, \qquad 2p'q'+q'^2=nf,$$
$$p''^2-q''^2=db, \qquad 2p''q''+q''^2=nc.$$

Now, in order to resolve these equations, put

$$p+q=a, p-q=c, 2p+q=\frac{d}{r}, q=rf,$$

then $$2p=a+c=\frac{d}{r}-rf \text{ and } 2q=a-c=2rf$$

Hence $$c=a-2rf \text{ and } d=r(2a-rf).$$

In the same manner we find from the second set,

$$b=a-2sf \text{ and } n=s(2a-sf).$$

And lastly, in the third set, put

$$p''+q''=d, p''-q''=b, 2p''+q''=tc \text{ and } q''=\frac{n}{t}$$

then $$2p''=d+b=tc-\frac{n}{t}\ .\ .\ .\ (4),\ 2q''=d-b=\frac{2n}{t}\ .\ .\ .\ .\ (5),$$

From (5), $t=\dfrac{2n}{d-b}$, and by (4), $d+b=\dfrac{2nc}{d-b}-\frac{1}{2}(d-b)$, or

$$(3d+b)(d-b)=4nc \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

substituting the values before found for d, b, n and c in (6),

$$\{(6r+1)a-(2s+3r^2)f\}\{(2r-1)a+(2s-r^2)f\}=8s(a-2rf)(a-\tfrac{1}{2}sf)(7).$$

Now all that seems necessary to be done in (7), is to assign such values for r and s, as shall make a factor in a and f common to the two members, and then, by division, the equation will be of the first degree in a and f. Thus, put $\dfrac{2s+3r^2}{6r+1}=\frac{1}{2}s$, then $s=\dfrac{2r^2}{2r-1}$, and the factor $a-\frac{1}{2}sf$ will be common, thence we get

therefore
$$\left.\begin{aligned} a&=12r^3(5-r)+5r^2,\\ f&=12r^2(3-2r)-2r-1,\\ c&=12r^2(3r-1)+9r+2,\\ b&=\frac{1}{2r-1}\{12r^2(6r-1)-42r-1\},\\ d&=12r(7r+1)+1,\\ n&=\left(\frac{4r}{2r-1}\right)^2\{12r(2r-1)-1\}.\end{aligned}\right\}\ .\ .\ .\ (8).$$

Again: put $\frac{2s-r^2}{2r-1} = -\frac{1}{2}s$, then $s = \frac{2r^2}{2r+3}$, and similarly,

$$\left.\begin{aligned} a &= 12r^3(r-1)-13r^2, \\ f &= 12r^2(2r+1)-14r-3, \\ c &= 12r^2(r+1)-5r-2, \\ b &= \frac{1}{2r+3}\{12r^2(2r+1)+2r+9\}, \\ d &= 4r(3r+1)-1, \\ n &= \left(\frac{4r}{2r+3}\right)^2(4r+3), \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (9).$$

In the same manner other factors might be rendered common and separate sets obtained by taking $s = \frac{1}{2}r(9r+2)$, or $s = \frac{1}{2}r(2-3r)$,

We may here observe that the different factors $a, b, c, \ldots$ must all be positive, or all negative; or, if two in one set be negative, then one in another set must be so likewise, otherwise the required numbers will not all have the same sign. Thus, if $a < o$, then d and n or f must be $< o$: or, if a and $b < o$, then $d < o$, or $f < o$, and if $f < o$, $n < o$ also, &c.

Examining the factors in (8) we find that to make a, b and $f > o$, r must be $> \frac{5}{6}$ and $< \frac{3}{4}$, and any value of r between these limits will furnish a set of numbers to answer. It is evident too that if $r > 5$, a and f become negative, and c, b, d and n positive, therefore numbers so found, though large, will answer. Again: in order that a and f be $< o$, and $c > o$ in (9), r must be assumed within the limits $\frac{1}{2}$ and 1; and to make a, f, and c all positive, r must $=$ or $> \frac{5}{8}$, and hence many sets may be found.

Application. Take $r = 1$ in (8), then $a = 53$, $f = 9$, $c = 35$, $b = 17$, $d = 97$, and $n = 176$, and consequently

$$x = abc = 53 \,.\, 17 \,.\, 35 = 31535,$$
$$y = dbf = 97 \,.\, 17 \,.\, 9 = 14841,$$
$$z = ncf = 176 \,.\, 35 \,.\, 9 = 55440.$$

If $r = 2$ in (9), then $a = 44$, $f = 209$, $c = 132$, $b = \frac{506}{7}$, $d = 110$, $n = \frac{64 \,.\, 22}{49}$: or, reducing them to a common denominator and rejecting it, $a = 28$, $f = 19$, $c = 6$, $b = 23$, $d = 35$ and $n = 64$, and thence

$$x = abc = 28 \,.\, 23 \,.\, 6 = 3864,$$
$$y = dbf = 35 \,.\, 23 \,.\, 19 = 15295,$$
$$z = ncf = 64 \,.\, 6 \,.\, 19 = 7296;$$

or, taking the reciprocals of these numbers (which will answer, since the equations are symmetrical with respect to x, y, z) and rejecting the common denominator, we shall have

$$x = 1520, \qquad y = 384, \qquad z = 805.$$

which are probably the least numbers that can found.

Note. The foregoing method will also apply to make the formulus,

(1), $x^2 - xy + y^2$; (2), $x^2 - xz + z^2$; (3), $y^2 - yz + z^2$,

squares. But these admit of a curious and more simple reduction. Thus, equate (2) to A^2, (3) to B^2, and take their difference, so shall

$$(x-y)(x+y-z) = (A+B)(A-B).$$

Now, put $A + B = 2x + 2y - 2z$, and $A - B = \frac{1}{2}(x - y)$, then $B = \frac{1}{4}(3x + 5y) - z$,

$$\text{and } y^2 - yz + z^2 = B^2 = \{\tfrac{1}{4}(3x + 5y) - z\}^2,$$

$$\text{thence} \quad z = \frac{3x^2 + 10xy + 3y^2}{8(x + y)},$$

when x and y must be such as to render (1) a square, and this is well known to be the case when

$$x = p^2 - q^2, \text{ and } y = 2pq - q^2;$$

and therefore we may take, in integers,

$$x = 8(p^2 - q^2)(p^2 + 2pq - 2q^2),$$
$$y = 8(2pq - q^2)(p^2 + 2pq - 2q^2),$$
$$z = 3(p^2 - q^2)^2 + 10(p^2 - q^2)(2pq - q^2) + 3(2pq - q^2)^2;$$

in which we must have $p > 2q$. If $p = 4$, $q = 1$, then $x = 77$, $y = 117$, $z = 165$.

SECOND SOLUTION. *By Dr. Strong.*

Let it be required to make squares of the three expressions

$$(x + y)^2 - A\,xy,\ (x + z)^2 - B\,xz,\ (y + z)^2 - D\,yz,$$

A, B, D, being given numbers, as, in the question, they each $= 1$. Put

$$(x + y)^2 - A\,xy = (x + y - a)^2,$$
$$(x + z)^2 - B\,xz = (x + z - b)^2,$$
$$(y + z)^2 - D\,yz = (y + z - c)^2,$$

$$\text{therefore} \quad y = \frac{a^2 - 2\,ax}{2\,a - Ax},\ z = \frac{b^2 - 2\,bx}{2\,b - Bx},\ y = \frac{c^2 - 2\,cz}{2\,c - Dz},$$

hence, by substituting the two first into the third, and reducing,

$$(a^2 - 2ax)\{4cb - Db^2 + 2(bD - Bc)x\} = (2a - Ax)\{2bc^2 - 2b^2c + (4bc - Bc^2)x\} \quad . \quad . \quad (1).$$

If we equate the co-efficients which do not involve x, we have

$$a = \frac{4c(c - b)}{4c - bD} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

$$\text{and then } x = 2 \times \frac{Abc(c - b) + a^2(bD - cB) + a(Db^2 + Bc^2 - 8bc)}{Ac(Bc - 4b) + 4a(bD - cB)} \quad . \quad . \quad (3).$$

Again, if we equate the co-efficients of x^2 in (1), we have

$$a = \frac{cA(4b - cB)}{4(Db - Bc)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

$$\text{and then } x = \frac{1}{2} \times \frac{a^2 b(Db - 4c) + 4abc(c - b)}{Acb(c - b) + a^2(bD - cB) + a(Db^2 + Bc^2 - 8cb)} \quad . \quad . \quad (5).$$

If $A = B = D = 1$, and if, in (2) and (3), we put $b = 1$, $c = 2$, we find $a = \frac{8}{7}$, and x, y, z being reduced to the same denominator, their numerators are

14841, 55440, 31535.

If $A = B = D = 2$, and if $b = 2$, $c = 1$ in (4) and (5), then $a = \frac{2}{3}$, and we find for x, y, z in like manner

1584, 187, 1020,

which render $x^2 + y^2$, $x^2 + z^2$, $y^2 + z^2$ each a rational square.

THIRD SOLUTION. *By Mr. N. Vernon.*

Let $$x^2+xy+y^2=\frac{(r^2-xy)^2}{4r^2},\ x^2+xz+z^2=\frac{(s^2-xz)^2}{4s^2},$$
then adding xy and xz to each member, and taking the root,
$$x+y=\frac{r^2+xy}{2r},\ x+z=\frac{s^2+xz}{2s},$$
hence $$x=\frac{s^2-2sz}{2s-z},\ y=\frac{2rs(r-s)-(r-4s)rz}{s(4r-s)-2(r-s)z};$$
which substituted in the third expression, and the square denominator rejected gives
$$4(r-s)^2z^4+2(r-s)(r^2-12rs+2s^2)z^3+(r^4-16r^3s+57r^2s^2-16rs^3+s^4)z^2$$
$$-(4r^4s-28r^3s^2+26r^2s^3-2rs^4)z+4r^2s^2(r-s)^2=\square,$$
put it $=\{2(r-s)z^2+\frac{1}{2}(r^2-12rs+2s^2)z+2rs(r-s)\}^2$, and reducing we find
$$z=\frac{8r^2s(r-9s)+8s^3(9r-s)}{r^2(r-24s)+16s^2(3r-s)}.$$
Let $r=1$, $s=2$; then $z=\frac{176}{17}$, $y=\frac{97}{35}$, $x=\frac{53}{9}$; and therefore we may take in integers $x=31535$, $y=14841$, $z=55440$

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(58). QUESTION VIII. *By Professor T. S. Davies.*

If four points on the sphere be taken at pleasure, and all the great circles joining these be drawn to mutually intersect, they will divide one another into segments, such that the sines of the segments are in harmonical proportion.

—— Several gentlemen have proved that no such property exists. In Leybourn's Mathematical Repository, No. 23, there is an Article by Professor Davies on great circle transversals, exhibiting properties of these lines analogous to those which Carnot has established with respect to rectilinear ones. The Editor concluded that the property enunciated was one Professor Davies had met with while engaged in these investigations, and thus admitted it without further investigation, contrary to his usual practice.

(59.) QUESTION IX. *By Professor Catlin.*

A given cone is suspended from a given point, successively by all the points in a line drawn from the vertex to the circumference of the base, while the axis remains in a given plane; required the locus of the vertex, and also the area of the locus.

FIRST SOLUTION. *By Mr. O. Root, Syracuse. N. Y.*

Let $\varphi=$ half the vertical angle of the cone, $a=$ its altitude, $r=$ the distance from the point of suspension to the cone's vertex, $\theta=$ the angle between r, and a vertical line through the point of suspension and the cone's centre of gravity : then
$$r:\tfrac{3}{4}a=\sin(\theta-\varphi):\sin\theta,$$

and $$r = \frac{3a}{4} \cdot \frac{\sin(\theta - \varphi)}{\sin \theta} = \tfrac{3}{4} a \sin \varphi (\cot \varphi - \cot \theta) \quad \cdots\cdots (1),$$
which is the polar equation of the locus.

Then the area $= \frac{1}{2}\int r^2\, d\theta = \frac{9}{32} a^2 \sin^2 \varphi \int d\theta (\cot \varphi - \cot \theta)^2$

$= \frac{9}{32} a^2 \sin^2 \varphi \{(\cot^2 \varphi - 1)\theta - 2\cot\varphi . h.l. \sin\theta - \cot\theta\} + const.$

The whole area may be found by taking the limits between $\theta = \varphi$, and $\cot \theta = \cot 2\varphi - \frac{5}{8} \operatorname{cosec} 2\varphi$, or $\theta = \frac{1}{2}\pi + \tan^{-1}(\frac{5}{8} \operatorname{cosec} 2\varphi - \cot 2\varphi)$.

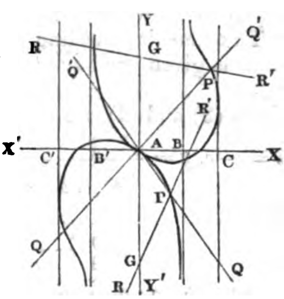
SECOND SOLUTION. *By Professor C. Avery, Hamilton College.*

(The former part of the solution is similar to that of Mr. Root's.)

The rectangular equation may now be readily found, by putting $r = \sqrt{x^2 + y^2}$, $\cot \varphi = \frac{y}{x}$, and $\frac{3}{4} a \sin \varphi = c$; then

$$x^2(x^2 + y^2) = c^2(x \cot \varphi + y)^2,$$

or $$y = \frac{c^2 x \cot \varphi \pm x^2 \sqrt{c^2 \operatorname{cosec}^2 \varphi - x^2}}{x^2 - c^2}.$$



The curve symbolized by this equation may be considered as the locus of the vertex P, of the angle φ, one of its indefinite sides QQ', sliding through the origin, while a point, G in the other side RR' at the distance $c \operatorname{cosec} \varphi$ from the vertex, slides along the axis of y. The whole curve is shown in the figure, where AB = AB' $= c$, and AC = AC' $= c \operatorname{cosec} \varphi$.

THIRD SOLUTION. *By Professor F. N. Benedict, University of Vermont.*

Let the suspended body be one of any form whose centre of gravity is given or determinable, the describing point any point given with respect to the body, and the line of suspension any line either curved or straight whose position is given in reference to the centre of gravity and the describing point, and situated in the same plane.

Let AR be the line of suspension, C the centre of gravity of the body, and P the describing point. Let x, y be the rectangular co-ordinates of the point P, referred to the origin A and axis of AC which is necessarily vertical. Let

$$F'(x', y') = 0 \quad \cdots\cdots\cdots (1),$$

be the equation of the line of suspension, in which x', y' are the rectangular co-ordinates of A referred to the origin T and axis PC of x'. putting PC $= e$, PT $= h$ which are regarded as positive on the other side of P in respect to C, we have from the triangles ACH and CIP,

$$AC^2 = y'^2 + (e + h - x')^2 = (x \pm \sqrt{e^2 - y^2})^2 \quad \cdots (2),$$

and from their similarity $$y' = \frac{y}{e}(x \pm \sqrt{e^2 - y^2}) \quad \cdots \cdots \quad (3),$$

therefore $$x' = \frac{1}{e}(eh + y^2 \mp x\sqrt{e^2 - y^2}) \quad \cdots \quad (4).$$

These values of y' and x' substituted in (1) give the equation of the locus

$$F' \left\{ \frac{1}{e}(eh + y^2 \mp x\sqrt{e^2 - y^2}), \frac{y}{e}(x \pm \sqrt{e^2 - y^2}) \right\} = 0 \quad . \quad (5).$$

In the question, if the origin of x' y' be at P the vertex of the cone, $h = 0$, and the equation of its side will be $y' = x' \tan\delta$, δ being half the vertical angle of the cone, and therefore the equation of the locus will be

$$y(x \pm \sqrt{e^2 - y^2}) = \tan\delta\,(y^2 \mp x\sqrt{e^2 - y^2}),$$

or, by reduction $$e^2(y + x\tan\delta)^2 = \sec^2\delta\, y^2(x^2 + y^2) \quad . \quad . \quad . \quad (6).$$

It is evident that (6) applies not only to the cone, but to a body of any form, provided the line of suspension be straight and pass through the describing point. If $\delta = 90°$, (6) becomes

$$e^2x^2 = y^2(x^2 + y^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

(Mr. B. then determines the area of (6) as in the first solution).

To determine the line of suspension when the locus of the describing point is given, we have (3), (4) and the equation of the locus

$$F(x, y) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

From (3) and (4), we find, by removing the origin of x' to C, or making $h = -e$, $y = \dfrac{ey'}{\sqrt{x'^2 + y'^2}}$, $x = \dfrac{x'^2 + y'^2 \pm ex'}{\sqrt{x'^2 + y'^2}}$, and the equation of the line is

$$F\left\{ \frac{x'^2 + y'^2 \pm ex'}{\sqrt{x'^2 + y'^2}}, \frac{ey'}{\sqrt{x'^2 + y'^2}} \right\} = 0 \quad \cdots \cdots \quad (9).$$

If (8) is homogeneous in respect to x and y, (9) will be

$$F\{(x'^2 + y'^2 \pm ex'), (ey')\} = 0 \quad . \quad . \quad . \quad . \quad . \quad (10).$$

Example. Let the point describe a circle, then $F(x, y) = y^2 - ax + x^2 = 0$, and (9) becomes

$$e^2y'^2 - a(x'^2 + y'^2 \pm ex')\sqrt{x'^2 + y'^2} + (x'^2 + y'^2 \pm ex')^2 = 0 \quad . \quad (11).$$

(60.) QUESTION X. *By Mr. Lenhart.*

Suppose five cards to be drawn promiscuously from a pack consisting of 52 cards, namely, 13 clubs, 13 spades, 13 hearts, and 13 diamonds; what is the chance that the five cards drawn will be all of the same suit, as clubs, or spades, &c.? What the chance that three and no more of the five cards will be aces? What the chance that three of the five cards will be alike, and also the remaining two; that is, three of them to be tens or nines, &c., and the remaining two to be fours, or fives, or knaves, &c.? What the chance that four of the five cards will be alike, say aces, kings, or queens, &c.? And, lastly, what is the chance that the five cards will compose one or other of the four foregoing hands?

SOLUTION, *principally from that of Mr. L. Abbott.*

1°. In the first place we suppose it granted that a chance depending on several chances, independent of each other, is equal to the product of all those chances. Now the chance that the second card will be of the same suit as the first is $\frac{12}{51}$, the third like these $\frac{11}{50}$, &c., the chance that all five will be of the same suit is

$$\frac{1}{1}\cdot\frac{12}{51}\cdot\frac{11}{50}\cdot\frac{10}{49}\cdot\frac{9}{48}=\frac{33}{16660}=\frac{1}{505}\ \textit{nearly.}$$

2°. The number of combinations of 3 aces out of 4 is $\frac{4.3.2}{1.2.3}=4$, and the number of combinations of the two remaining cards after the 4 aces are taken out is $\frac{48.47}{1.2}=24.47$. Then the number of combinations of 3 aces with two other cards is $4.24.47$, and this divided by the whole number of combinations of 5 cards formed from all the 52, which is $\frac{52.51.50.49.48}{1.2.3.4.5}=20.49.51.52$, gives the chance that 3 of the 5 cards and no more will be aces,

$$\frac{4.24.47}{20.49.51.52}=\frac{94}{54145}=\frac{1}{575}\ \textit{nearly.}$$

3°. There are 4 combinations of aces, 3 at a time, and 6 combinations of twos, 2 at a time; then there are $4.6=24$ cases of 3 aces combined with 2 twos, or 2 threes, &c., and 12.24 cases of 3 aces combined with 2 other cards of any like kind, and the same for 3 twos and any 2 others alike, &c.; therefore there are $12.13.24$ cases in which 3 cards which are alike may be combined with two others alike, and the chance that this will happen is

$$\frac{12.13.24}{20.49.51.52}=\frac{6}{4165}=\frac{1}{694}\ \textit{nearly.}$$

4°. The number of cases in which 4 of the 5 cards could be aces, or twos, or threes, &c., is evidently 48, so that the whole number of cases in which 4 cards can be alike is 13.48, and the chance that this will happen is

$$\frac{13.48}{20.49.51.52}=\frac{1}{4165}.$$

5°. The chance that it will be one or other of the foregoing hands, is the sum of all the preceding chances, minus the chance common to the second and third cases, or that there may be 3 aces and 2 others alike, this is

$$\frac{33}{16660}+\frac{94}{54145}+\frac{6}{4165}+\frac{1}{4165}-\frac{12.24}{20.49.51.52}=\frac{229}{43316}=\frac{1}{189}\ \textit{nearly.}$$

(61.) QUESTION XI. *By Richard Tinto, Esq., Greenville, Ohio.*

Two spheres are given in magnitude and position. It is required to

find the locus of a point at which a light being placed, the shadows of the spheres on a given plane may be of equal magnitude.

FIRST SOLUTION. *By Professor Avery.*

Let the notation be as in the second solution to (31,) Question XI. Number III. page 157; the similar quantities for the second sphere being accented at the foot of the letters, except that its radius is put $= r''$. Then we find for this question

$$\frac{\pi a^2 \tan^2 e \sec^2 \theta}{(1-\tan^2 e \tan^2 \theta)^{\frac{3}{2}}} = \frac{\pi a^2 \tan^2 e_1 \sec^2 \theta_1}{(1-\tan^2 e_1 \tan^2 \theta_1)^{\frac{3}{2}}} \quad . \quad . \quad . \quad (1),$$

and from this $$\left(\frac{\tan e}{\tan e_1}\right)^4 = \left(\frac{1-\sec^2 e \sin^2 \theta}{1-\sec^2 e_1 \sin^2 \theta_1}\right)^3 \quad . \quad . \quad . \quad . \quad (2),$$

or, since $\tan^2 e = \frac{r'^2}{r^2 - r'^2}$, and $\tan^2 e_1 = \frac{r''^2}{r_1{}^2 - r''^2}$, this becomes

$$\frac{r'^4}{r''^4} \cdot \frac{r^2 - r'^2}{r_1{}^2 - r''^2} = \left(\frac{r^2 \cos^2 \theta - r'^2}{r_1{}^2 \cos^2 \theta_1 - r''^2}\right)^3 \quad . \quad . \quad . \quad . \quad (3),$$

Now let the co-ordinates of the light be x, y, z, the plane of xy passing through the centres of the two spheres perpendicular to the shadow plane, the axis of x perpendicular to that plane, the origin being in the centre of the first sphere, and the co-ordinates of the centre of the second being b, c; then

$r^2 = x^2 + y^2 + z^2, r_1{}^2 = (x-b)^2 + (y-c)^2 + z^2, r\cos\theta = x$, and $r_1 \cos\theta_1 = x - b$;

and (3) becomes

$$\frac{r'^4}{r''^4} \cdot \frac{x^2 + y^2 + z^2 - r'^2}{(x-b)^2 + (y-c)^2 + z^2 - r''^2} = \left(\frac{x^2 - r'^2}{(x-b)^2 - r''^2}\right)^3 \quad . \quad . \quad (4),$$

which is the rectangular equation of the required surface.

Let $x = x'$ be the equation of a plane perpendicular to the axis of x, its section with the surface will be a curve, the equation of whose projection on the plane of yz will be

$$\frac{y^2 + z^2 + k^2}{(y-c)^2 + z^2 + l^2} = \frac{r''^4 k^6}{r'^4 l^6},$$

where we have put, for conveniency,

$$k^2 = x'^2 - r'^2, \quad l^2 = (x' - b)^2 - r''^2;$$

this curve is a circle whose centre is on the axis of y, its distance from the origin being

$$y' = \frac{r''^4 k^6 c}{r''^4 k^6 - r'^4 l^6} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

and its radius,

$$\text{R} = \frac{\pm kl}{r''^4 k^6 - r'^4 l^6} \cdot \sqrt{r'^4 r''^4 k^4 l^4 (k^2 + l^2 + c^2) - r'^8 l^{10} - r''^8 k^{10}} \quad . \quad (6).$$

Then the sections of the surface parallel to yz are circles, having their centres on yx, and in a curve described on that plane whose equation is found by restoring the values of k and l in (5), it is

$$\frac{y' - c}{y'} = \frac{r'^4}{r''^4} \cdot \left(\frac{(x' - b)^2 - r''^2}{x'^2 - r'^2}\right)^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

and therefore a line of the seventh order; the surface is consequently not one of revolution. When $z = o$, equation (4) becomes

$$\frac{r'^4}{r''^4}\cdot\frac{x^2+y^2-r'^2}{(x-b)^2+(y-c)^2-r''^2} = \left(\frac{x^2-r'^2}{(x-b)^2-r''^2}\right)^3 \quad . \; . \quad (8),$$

which is, in general, a line of the eighth order; it is the intersection of the plane of xy with the surface, and the line (7) is a diameter of this curve, bisecting a system of chords parallel to the axis of y, and of the variable length $2R$. All the circumstances of the surface may therefore be had by examining the nature of the line (7), and the corresponding lengths of R in (6). A full discussion of the curve would be attended with some difficulty; we may mention however that it has in general two assymptotes parallel to the axis of y, and one parallel to that of x.

Second Solution. *By Mr. O. Root.*

Take the origin at the centre of one of the given spheres, and let a, b, c be the co-ordinates of the centre of the other, x, y, z the co-ordinates of the light whose perpendicular distance from the given plane is m; then put $x^2+y^2+z^2=r^2$ and $(x-a)^2+(y-b)^2+(z-c)^2=r'^2$, also put φ and φ' for the angles between r and m, and r' and m, and θ and θ' for the angles between r and r' and the tangents to the respective spheres, then from the equality of the areas of the elliptic shadows on the plane,

$$\frac{\pi m^2 \sin^2\theta \cos\theta}{(\cos^2\varphi-\sin^2\theta)^{\frac{3}{2}}} = \frac{\pi m^2 \sin^2\theta' \cos\theta'}{(\cos^2\varphi'-\sin^2\theta')^{\frac{3}{2}}};$$

and if d, d' be put for the radii of the two spheres, then

$$\frac{d}{r} = \sin\theta,\ \frac{d'}{r'} = \sin\theta';\ \frac{x}{r} = \cos\varphi,\ \frac{x-a}{r'} = \cos\varphi',$$

and by substituting these, with the values of r and r' the equation becomes

$$\frac{d^4(x^2+y^2+z^2-d^2)}{(x^2-d^2)^3} = \frac{d'^4\{(x-a)^2+(y-b)^2+(z-c)^2-d'^2\}}{\{(x-a)^2-d'^2\}^3}$$

for the equation of the surface required.

(62.) Question XII. *By* ψ.

Let M denote the mass of the sun,
m, m' the masses of any two of the planets revolving round it,
n, n' their mean angular velocities,
a, a' their mean distances from the sun.

Show that $\dfrac{M+m}{a^3} = n^2$, and that $n^2a^3 = n'^2a'^3$.

First Solution. *By Dr. Strong.*

First Method. By the mean angular velocity of a planet is meant that which it would have were it to describe a circle round the sun, supposing the radius of the circle to equal the mean distance of the planet, and the

central force in the circle the same as that at the mean distance of the planet; whence it is manifest, by Kepler's third law, that the time of revolution in the circle will be the same as that of the planet. Hence imagine a particle of matter revolving in a circle round M at rest, at the distance a, and attracted towards it by the force $\frac{M+m}{a^2}$; its time of revolution will be the same as that of the planet describing the same circle, for the force with which the planet's centre is urged towards the sun's centre $= \frac{M+m}{a^2}$. Now since n = the angular velocity, the velocity of the particle at the distance $a = an$, and therefore $\frac{a^2n^2}{a} = an^2 =$ the centrifugal force, which must equal the centripetal force,

$$\therefore \frac{M+m}{a^2} = an^2, \text{ or } n^2 = \frac{M+m}{a^3}.$$

In the same way $n'^2 = \frac{M+m'}{a'^3}$, and if m and m' are extremely small with respect to M, so that they may be neglected,

$$n^2a^3 = n'^2a'^3 = M.$$

Second Method. (After finding the usual differential equation of the orbit,

$$\frac{d^2u}{dv^2} + u - \frac{M+m'}{c^2} = 0,$$

where $u = \frac{1}{r}$, r being the distance of the planet from the sun, and v the angle r makes with a fixed axis, which is included in the equation

$$\frac{d^2u}{dv^2} + m^2u + P = 0 \quad \cdots\cdots \quad (1),$$

m being any constant, and P *any given function of* v; Dr. S. gives the following method of integrating this last equation).

put $\sin mv = s$, $\cos mv = s'$ . . . . (2),

whence $s^2 + s'^2 = 1$, $sds + s'ds' = 0$ . . . . (3),

also, $\frac{d^2s}{dv^2} + m^2s = 0$, $\frac{d^2s'}{dv^2} + m^2s' = 0$ . . . . (4).

Multiply (1) by ds, and the first of (4) by du, and add the products,

$$\frac{d\,(ds\,du)}{dv^2} + m^2d\,(su) + P\,ds = 0,$$

$$\therefore \frac{ds\,du}{dv^2} + m^2su + \int P\,ds = const. = mA \quad \cdots \quad (5).$$

In the same way, from (1) and the second of (4),

$$\frac{ds'\,du}{dv^2} + m^2s'u + \int P\,ds' = const. = mB \quad \cdots \quad (6).$$

Multiply (5) by s and (6) by s', and add, reducing by (3), then

$$m^2u + s\int P\,ds + s'\int P\,ds' = m\,(As + Bs'), \text{ or}$$

$$mu = A\sin mv + B\cos mv - \sin mv\int P\,dv\cos mv + \cos mv\int P\,dv\sin mv \quad . (7).$$

I have found the integral of (1), in vol. 30, page 265, of Silliman's Journal, as follows: Multiply (1) by $dv \cos mv$, and integrate, then

$$\frac{du \cos mv}{dv} + mu \sin mv + \int \text{P}\, dv \cos mv = \text{const.} = \text{A},$$

also multiply (1) by $dv \,.\, \sin mv$, and take the integral, then

$$\frac{du \,.\, \sin mv}{dv} - mu \cos mv + \int \text{P}\, dv \sin mv = \text{const.} = -\text{B};$$

then multiply the first of these by $\sin mv$ and the second by $-\cos mv$, and add the products, and we get (7) by a slight reduction.

In the present case $m = 1$, $\text{P} = \text{const.} = -\dfrac{\text{M} + m'}{c^2} = -\dfrac{\text{M}'}{c^2}$, and (7) becomes

$$u = \frac{1}{r} = \text{A} \sin v + \text{B} \cos v + \frac{\text{M}'}{c^2};$$

hence, if we put $c^2 = \text{M}'p$, $\text{A}p = e \cos w$, $\text{B}p = e \sin w$, we get

$$r = \frac{p}{1 + e \cos (v - w)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

which is the polar equation of the orbit, and it will be an ellipse if $e < 1$; the method of deducing the mean angular velocity is sufficiently familiar.

SECOND SOLUTION. *By Professor Peirce.*

Considering the motions of the sun, and of one of the planets m, which arise from their mutual actions, we shall denote the co-ordinates of the sun by X and Y, and those of the planet by x and y, these being taken in the common plane of their orbits; t denoting the time, and r their distance apart. The equation is

$$\text{M}\left(\frac{dd\text{X}}{dt^2}\right)\delta\text{X} + \text{M}\left(\frac{dd\text{Y}}{dt^2}\right)\delta\text{Y} + m\left(\frac{ddx}{dt^2}\right)\delta x + m\left(\frac{ddy}{dt^2}\right)\delta y + \frac{\text{M}m}{r^2}\,\delta\, r = 0;$$

and if we take φ for the angle made by r and the axis of x, we have

$$x = \text{X} + r \cos \varphi,$$
$$y = \text{Y} + r \sin \varphi,$$

which being substituted in the above equation, give the four equations

$$\text{M}\left(\frac{dd\text{X}}{dt^2}\right) + m\left(\frac{ddx}{dt^2}\right) = 0,$$

$$\text{M}\left(\frac{dd\text{Y}}{dt^2}\right) + m\left(\frac{ddy}{dt^2}\right) = 0,$$

$$m\left(\frac{ddx}{dt^2}\right)\cos \varphi + m\left(\frac{ddy}{dt^2}\right)\sin \varphi + \frac{\text{M}m}{r^2} = 0,$$

$$-\left(\frac{ddx}{dt^2}\right)\sin \varphi + \left(\frac{ddy}{dt^2}\right)\cos \varphi = 0.$$

And the first two of these equations may again be reduced to

$$(\text{M} + m)\left(\frac{ddx}{dt^2}\right) = \text{M}\left(\frac{d^2 \,.\, r \cos \varphi}{dt^2}\right),$$

$$(M+m)\left(\frac{ddy}{dt^2}\right)=M\left(\frac{d^2 . r\sin\varphi}{dt^2}\right);$$

by which the other two are reduced to

$$\frac{d^2 r}{dt^2}-r.\frac{d\varphi^2}{dt^2}+\frac{M+m}{r^2}=0,$$

$$d\left(r^2.\frac{d\varphi}{dt}\right)=0.$$

Now, we have

$$r=a+\text{a periodic function of } t,$$
$$\varphi=nt+e+\text{a periodic function of } t;$$

which, being substituted in the preceding equations must give resulting equations in which the periodic functions cancel each other; and therefore the first equation becomes

$$-an^2+\frac{M+m}{a^2}=0, \text{ or } n^2=\frac{M+m}{a^3};$$

and neglecting the mass m in comparison with M,

$$n^2a^3=M=n'^2a'^3.$$

(63.) Question XIII. *By Professor C. Avery.*

It is required to find the time in which a rigid rod of small diameter will descend from a given, to a horizontal, position; its ends sliding along a vertical and a horizontal plane without friction.

First Solution. *By Professor Peirce.*

Let the origin of co-ordinates be in the line of intersection of the two given planes, and let this line be the axis of x; let the axis of y be in the horizontal plane, that of z being vertical. Let $2l$ be the length of the given line, s the distance of any point of this line *above* its centre, the co-ordinates of this point being x, y, z, and its magnitude Ds. If then we use S to denote the integrations relative to the whole line, and g for the force of gravity, the formula for motion becomes

$$\text{S}\left[\frac{d^2x}{dt^2}\delta x+\frac{d^2y}{dt^2}\delta y+\left(\frac{d^2z}{dt^2}+g\right)\delta z\right]\text{D}s=0.$$

If, now, we note by x' the value of x corresponding to the centre of the given line; by φ the angle which this line makes with its projection on the plane of yz, and by θ the angle which this makes with the plane of xy; we have

$$x=x'+s\sin\varphi,$$
$$y=(l-s)\cos\varphi\cos\theta,$$
$$z=(l+s)\cos\varphi\sin\theta.$$

These values being substituted in the preceding equations, the integrations are to be taken from

$$s=-l \quad\text{to}\quad s=l,$$

and it must be borne in mind that the values of φ, θ and x' are constant for any given position of the line, and that s is constant for the differen-

tiations relative to δ. The co-efficients of $\delta x'$, $\delta\varphi$ and $\delta\theta$, being put equal to zero, give

$$\frac{d^2x'}{dt^2}=0 \quad . \quad . \quad . \quad (1),$$

$$(1+3\sin^2\varphi)\frac{d^2\varphi}{dt^2}+\tfrac{3}{2}\sin 2\varphi\frac{d\varphi^2}{dt^2}+2\sin 2\varphi\frac{d\theta^2}{dt^2}-\frac{3g}{l}\sin\varphi\sin\theta=0 \quad . \quad . \quad . \quad (2),$$

$$4\cos\varphi\frac{d^2\theta}{dt^2}-8\sin\varphi\frac{d\varphi}{dt}\cdot\frac{d\theta}{dt}+\frac{3g}{l}\cos\theta=0 \quad . \quad . \quad . \quad (3).$$

These equations show that the rod is at each instant exactly *parallel to the direction which it would have had, if its upper end had been free, and its lower end had been confined to a line drawn parallel to the axis of x.*

Equation (1) shows that *if the rod started from a state of rest, its centre must remain in a plane drawn perpendicular to the axis of x.*

If equation (2) be multiplied by $2\,d\varphi$, and the product added to equation (3) multiplied by $2\cos\varphi\,d\theta$, the integral of the sum is

$$(1+3\sin^2\varphi)\frac{d\varphi^2}{dt^2}+4\cos^2\varphi\,\frac{d\theta^2}{dt^2}+\frac{6g}{l}\sin\theta\cos\varphi=\text{const.}$$

Case. If, now, the rod is originally placed nearly in the plane of yz, the angle φ, and also $\frac{d\varphi}{dt}$ must be so small that their second powers may be neglected, and this equation becomes

$$4\,.\,\frac{d\theta^2}{dt^2}+\frac{6g}{l}\sin\theta=\text{const.}=c=\frac{6g}{l}\sin\theta',$$

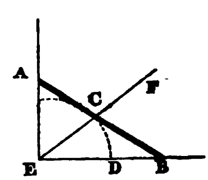
$$dt=\sqrt{\frac{2l}{3g}}\cdot\frac{d\theta}{\sqrt{\sin\theta'-\sin\theta}},$$

c being taken so that, at the beginning of motion, when $\theta=\theta'$, we may have $\frac{d\theta}{dt}=0$.

Whence *the time is the same that it would take this rod to fall to a horizontal position if its lower end were fixed and its upper end free.*

—— Dr. Strong's solution was also adapted to the case where the rod is not confined to a vertical plane.

SECOND SOLUTION. *By Professor Catlin.*



Let AEB be a vertical plane passing through the rod AB and intersecting the vertical and horizontal planes in AE, EB. Then C, the centre of the rod, will obviously describe an arc of a circle about the centre E, radius = AC = BC. Hence the centre of the rod may be considered as compelled to move in a given curve CD, while the extremity B is acted upon by a force P, causing the rod to revolve about its centre C. The angular velocity of the rod AB is the same as that of an equal rod EF whose extremity E is fixed; for the force of gravity (g) and the re-action (R) of the plane EB are the same in both cases

and since the angles CBE = CEB the force P which produces the angular velocity of either rod is the same for both. Consequently the two rods will descend in the same time. But EF may be considered a pendulum revolving about the point of suspension E; hence the time may be determined by the usual formula.

THIRD SOLUTION. *By Mr. O. Root.*

Let xy be the co-ordinates of a particle dm of the rod, the axes being the intersections of the given planes with a plane perpendicular to them. Then by the general formula of Dynamics we shall have

$$\int dm\cdot\frac{d^2x}{dt^2}\delta x+\int dm\cdot\frac{d^2y}{dt^2}\delta y+\int dm\,.\,g\delta y=0 \quad \ldots \quad (1).$$

If $2a$ = the length of the rod, φ = angle between the vertical axis and a line drawn from the origin to the middle of the rod, and m = distance from the middle of the rod to the parallel dm, then

$$x=(a-m)\sin\varphi,\quad y=(a+m)\cos\varphi;$$

and (1) will be reduced to

$$\int dm\left\{(a^2+m^2)\frac{d^2\varphi}{dt^2}+4am\sin\varphi\cos\varphi\frac{d\varphi^2}{dt^2}\right\}-\int g\,dm(a+m)\sin\varphi=0\,.\ (2)$$

which integrated from $m=-a$, to $m=+a$, gives

$$\frac{4}{3}a\,.\,\frac{d^2\varphi}{dt^2}-g\sin\varphi=0\cdot\ \ldots\ldots\ (3),$$

this multiplied by $d\varphi$, and integrated becomes

$$\frac{2}{3}a\,.\,\frac{d\varphi^2}{dt^2}+g\cos\varphi=\text{const.}=g\cos\varphi',$$

supposing the rod to start from rest, therefore

$$dt=\sqrt{\frac{2a}{3g}}\cdot\frac{d\varphi}{\sqrt{\cos\varphi'-\cos\varphi}}\ \ldots\ldots\ (4);$$

and if the initial position be nearly vertical, or $\cos\varphi'=1$,

$$dt=\sqrt{\frac{a}{3g}}\cdot\frac{d\varphi}{\sin\frac{1}{2}\varphi},$$

$$\text{and } t=\sqrt{\frac{a}{3g}}\cdot h\,.\,l\,.\tan^2\tfrac{1}{4}\varphi.$$

Hence in this case, when $\varphi=\frac{1}{2}\pi$, the whole time T of descent is

$$T=\sqrt{\frac{a}{3g}}\cdot h\,.\,l\,.(3-\sqrt{2})$$

—— The solutions of Professor Avery and Mr. Macully were also very neat.

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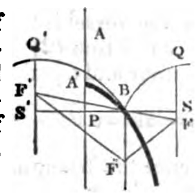
(64). QUESTION XIV. *By Professor F. N. Benedict.*

From a vessel of water, formed by the revolution of a curve about an axis perpendicular to the horizon, three jets issue at the same point; the first horizontally, the second in the direction of a normal, and the third in

the direction of a tangent, of the generating curve at the orifice. It is required to determine the form of the vessel, such that wherever the orifice may be situated, the principal vertex of the normal or of the tangent jet shall be in a given horizontal plane; and also to determine its form, such that the area of the triangle formed by connecting the foci of the three jets, shall be a given function of the depth of the orifice below the surface of the water.

First Solution. *By the Proposer.*



Let AP be the axis of the vessel A'B, A the origin of x, assumed at the surface of the water, B any small orifice, Q and Q' the principal vertices of the normal and tangent jets, F and F' their foci, BS and BS' their semi-ranges, P the intersection of AP and SS', F'' the focus of the horizontal jet, and x, y the rectangular co-ordinates of the orifice.

If θ is the angle which a jet makes with the horizon, z its semi-range, u the elevation of its vertex above the orifice, and q its principal parameter, we have (Venturoli's Mech. art. 238),

$$z = x\sin 2\theta \qquad u = x\sin^2\theta \qquad q = 4x\cos^2\theta$$
$$= \frac{2x\tan\theta}{\tan^2\theta+1}, \qquad = \frac{x\tan^2\theta}{\tan^2\theta+1}, \qquad = \frac{4x}{\tan^2\theta+1} \quad . \; . \; . \quad (1).$$

But the values of $\tan\theta$ for the normal, tangent and horizontal jets, are severally

$$\frac{dy}{dx}, -\frac{dx}{dy}, 0;$$

and if p, p', p'' represent their respective principal parameters, we have

$$\text{BS} = \frac{2x\,dx\,dy}{dx^2+dy^2}, \quad \text{QS} = \frac{x\,dy^2}{dx^2+dy^2}, \quad p = \frac{4x\,dx^2}{dx^2+dy^2} \quad . \; . \; . \quad (2),$$

$$\text{BS}' = \frac{-2x\,dx\,dy}{dx^2+dy^2}, \quad \text{Q'S'} = \frac{x\,dx^2}{dx^2+dy^2}, \quad p' = \frac{4x\,dy^2}{dx^2+dy^2} \quad . \; . \; . \quad (3),$$

$$p'' = 4x \quad . \; . \; . \; . \; . \quad (4).$$

Hence also

$$\text{PS} = \frac{y\,dx^2+y\,dy^2+2x\,dx\,dy}{dx^2+dy^2}, \quad \text{PS}' = \frac{y\,dx^2+y\,dy^2-2x\,dx\,dy}{dx^2+dy^2} \quad . \; . \quad (5);$$

$$\text{QF} = \frac{x\,dx^2}{dx^2+dy^2}, \quad \text{Q'F'} = \frac{x\,dy^2}{dx^2+dy^2}, \quad \text{BF}'' = x \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6);$$

$$\text{SF} = \frac{x(dx^2-dy^2)}{dx^2+dy^2}, \quad \text{S'F'} = -\frac{x(dx^2-dy^2)}{dx^2+dy^2} \quad . \; . \; . \; . \; . \; . \quad (7).$$

1st. The vertices of the normal jets being in a horizontal plane, at a distance h below the surface of the water, we have

$$h = x - \text{QS} = \frac{x\,dx^2}{dx^2+dy^2}.$$

Removing the denominator, &c., and integrating

$$h^{\frac{1}{2}}\,y = \tfrac{2}{3}\,(x-h)^{\frac{3}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

the vessel therefore is formed by the revolution of the semi-cubical parabola.

2d. The vertices of the tangent jets being in a horizontal plane, at a distance h' below the surface, we have

$$h' = x - \text{Q}'\text{S}' = \frac{x\,dy^2}{dx^2 + dy^2}.$$

therefore $$y^2 = 4h'\,(x - h') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

or the vessel is the common paraboloid.

3d. From (2), (3) and (7) it is evident that the triangles BSF, BS'F' are similar and equal, therefore

$$\text{BF} = \text{BF}' = \sqrt{\text{BS}^2 + \text{SF}^2} = \sqrt{\left\{\frac{4x^2dy^2dx^2}{(dx^2+dy^2)^2} + \frac{x^2(dx^2-dy^2)^2}{(dx^2+dy^2)^2}\right\}} = x = \text{BF}'';$$

hence the triangle having its angles at the foci of the three jets is right angled at F″ the focus of the horizontal one, and its area

$$= \text{BS} \times \text{BF}'' = \frac{2x^2\,dx\,dy}{dx^2 + dy^2} = \text{U} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

U being that function of the co-ordinates according to which the area varies. And if U is merely a function of x, by separating the variables,

$$dy = \frac{\text{U}\,dx}{x^2 \pm \sqrt{x^4 - \text{U}^2}} = \frac{dx}{\text{U}}\,(x^2 \mp \sqrt{x^4 - \text{U}}) \quad . \quad . \quad . \quad . \quad (11).$$

If $\text{U} = ax^2$, then we have, by integration,

$$y = bx + c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

or the vessel is a cone of revolution, and the focal triangle varies as the circular section through the orifice.

Cor. 1. The distances of the orifice from the three foci are equal, and the orifice and foci of the normal and tangents are in a straight line.

Cor. 2. The focal triangle is right angled at the focus of the horizontal jet.

Cor. 3. The ranges of the normal and tangent jets are equal.

Cor. 4. The distance of the orifice below the surface of the water $= \frac{1}{4}$ of the sum of the parameters of the normal and tangent jets, (2) and (3).

Cor. 5. The sides of the focal triangle including the right angle are respectively parallel to the normal and tangent at the orifice.

Cor. 6. The sum of the areas of the normal and tangent jets cut off by a horizontal line through the orifice $= \frac{4}{3}$ of the area of the focal triangle.

Remarks. The form of the vessel being given by the equation $f(x, y) = 0$, we have, to determine the locus of the focus F, (y', x'), the equations

$$y' = \text{FS} = y + \frac{2x\,dy\,dx}{dx^2 + dy^2},\; x' = x + \text{SF} = \frac{2x\,dx^2}{dx^2 + dy^2} \quad . \quad . \quad . \quad (13);$$

and to determine the locus of the focus F′, (y'', x''), the equations

$$y''=y-\frac{2x\,dy\,dx}{dx^2+dy^2},\quad x''=x-\text{S}'\text{F}'=\frac{2x\,dy^2}{dx^2+dy^2}\,.\quad .\quad .\quad (14).$$

The locus of the normal focus F being given by the equation $f'(x', y')=0$, the equation of the axial section of the vessel will be, by (13),

$$f'\left\{\frac{2x\,dx}{dx^2+dy^2},\; y+\frac{2x\,dy\,dx}{dx^2+dy^2}\right\}=0\quad .\quad .\quad .\quad .\quad (15).$$

And similarly for the foci F′ and F″.

For instance, if the locus of the normal or tangent jet is a cylinder of revolution, radius a, we have by (5),

$$a=y\pm\frac{2x\,dy\,dx}{dx^2+dy^2},$$

the upper sign belonging to the normal, and the lower to the tangent jet. This equation rendered homogeneous, and integrated in the usual way, gives for the equation of the generating curve of the vessel

$$2h^3=(\pm\sqrt{x^2-(a-y)^2}-2x)^2(\pm\sqrt{x^2-(a-y)^2}+x)\quad .\quad .\quad (16),$$

where $h=$ distance below the surface of the water of the common section of the surfaces of the vessel and cylinder.

SECOND SOLUTION. *By Professor Catlin:*

I. Let x and y be the co-ordinates of the required curve, x and y' the co-ordinates of the normal jet, the axis of x being vertical, that of y horizontal, and both origins in the given horizonal plane. Then since the normal of the required curve is a tangent to the parabola, we shall have

$$\frac{dx}{dy}=\frac{dy'}{dx},\text{ or } dx^2=dy\,dy',\quad .\quad .\quad .\quad .\quad .\quad (1).$$

But from the equation of the parabola $y'=p^{\frac{1}{2}}x^{\frac{1}{2}}$, where from the conditions of the problem the parameter p is evidently constant,

$$\therefore dy'=\tfrac{1}{2}p^{\frac{1}{2}}x^{-\frac{1}{2}}dx,\text{ and } p^{\frac{1}{2}}dy=2x^{\frac{1}{2}}dx,$$

$$\therefore p^{\frac{1}{2}}y+c=\tfrac{4}{3}x^{\frac{3}{2}},\text{ or } p\,(y+c)^2=\tfrac{16}{9}x^3\quad .\quad .\quad .\quad (2),$$

which is the equation of the semi-cubical parabola.

II. Since the tangent to the required curve is also a tangent to the tangent jet, we have, if x, y'' be the co-ordinates of the parabola,

$$\frac{dy}{dx}=\frac{dy''}{dx},\text{ or } dy=dy''\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad (3).$$

By integrating $y=y''+c$, or $y=y''$. Hence the required curve coincides with the jet and is a conical parabola.

III. It is well known that the three foci are in the circumference of a circle whose centre is at the orifice, and it is obvious from the nature of the parabola, that the line joining the foci of the normal and tangent jets passes through the orifice. Put $x=$ the depth, and $\theta=$ the angle which the tangent of the required curve makes with the vertical. Then we shall have the area of the triangle $=$

$$x^2\sin\theta\cos\theta=\text{X}=\text{a given function of } x\quad .\quad .\quad .\quad .\quad (4).$$

But $\tan\theta = \frac{dy}{dx}$, and $\sin\theta\cos\theta = \frac{dx\,dy}{dx^2+dy^2}$;

therefore $$\frac{x^2\,dx\,dy}{dx^2+dy^2} = \text{x}, \quad \cdots\cdots \quad (5).$$

For instance; Let $\text{x} = rx$, then putting $p = \frac{dy}{dx}$, (5) becomes

$$x = \frac{r}{p}(p^2+1) \quad \cdots\cdots\cdots \quad (6),$$

Hence, (see Ryan's Calculus, page 304)

$$y = r(p^2+1) - \int \frac{r}{p}(p^2+1)\,dp = r(\tfrac{1}{2}p^2 + 1 - h.l.p) \quad \cdots \quad (7).$$

Equation (6) gives $p = \frac{x \pm \sqrt{x^2 - 4rx}}{2r}$, therefore

$$y = r + \frac{(x \pm \sqrt{x^2-4rx})^2}{8r} - r\,h.l.\frac{x \pm \sqrt{x^2-4rx}}{2r} \quad \cdots \quad (8),$$

which is the equation of the required curve.

—— Professors Avery and Peirce, in their excellent solutions to this question, remark that since the water cannot be higher than the sides of the vessel, the first two results are absurd, unless we suppose a small pipe of water to pass through the axis to the top of the vessel from some distance above, and which is kept constantly full.

(65.) QUESTION XV. *By Professor T. S. Davies, Royal Military Academy, Woolwich.*

A prolate ellipsoid being described on the diameter of a given sphere, and cut by any meridional plane: if another given sphere be made to *roll* upon the ellipsoid, so that a given great circle of it constantly coincides with the meridional plane, the two spheres will intersect in all their positions, and it is required to find the *envelopes* of the circles of intersection made on each sphere.

SOLUTION. *By Professor B. Peirce.*

Let A = radius of first sphere,
R = that of the rolling sphere,
B = semi-conjugate axis of ellipsoid and the axis of y.

The transverse axis is the axis of x. The equation of the generating ellipse is

$$\left(\frac{x}{\text{A}}\right)^2 + \left(\frac{y}{\text{B}}\right)^2 = 1.$$

Let ψ = the angle made by the axis of x with that radius of the rolling sphere which is drawn to the point of contact, so that

$$\tan\psi = -\frac{dx}{dy} = \frac{\text{A}y}{\text{B}x},$$

Hence $$\frac{y}{\text{B}} = \frac{x}{\text{A}}\tan\psi,$$

$$x = \text{A}\cos\psi, \qquad y = \text{B}\sin\psi.$$

Let L = the distance between the centres of the two spheres,

ω = the angle which L makes with the axis of x;

and the co-ordinates of the centre of the rolling sphere are

$$L \sin \omega = y + R \sin \psi = (B + R) \sin \psi,$$
$$L \cos \omega = x + R \cos \psi = (A + R) \cos \psi.$$

Let h and α be such that

$$B + R = h \sin \alpha,$$
$$A + R = h \cos \alpha.$$

Let, also, φ = the angle which L makes with the radius of the first given sphere, drawn to one of the points of intersection of the two spheres, whence

$$2AL \cos \varphi = L^2 + A^2 - R^2 = (B + R)^2 \sin^2 \psi + (A + R)^2 \cos^2 \psi + A^2 - R^2$$
$$= h^2 \cos 2\alpha \cos^2 \psi + (B + R)^2 + A^2 - R^2.$$

First. To find the required envelope upon the first given sphere.

Using "spherical polar co-ordinates," as in page 31 of the Miscellany, let the *meridional plane* of the question be the prime meridian, and let the origin of co-ordinates be the extremity of the transverse axis of the ellipsoid. Let

r = the radius vector,

θ = the angle which r makes with the prime meridian.

The equation of the intersection of the two spheres is

$$\cos \varphi = \cos \omega \cos r + \sin \omega \sin r \cos \theta,$$

which multiplied by $2AL$, gives by substitution

$$h^2 \cos 2\alpha \cos^2 \psi + (B+R)^2 + A^2 - R^2 = 2Ah(\cos\alpha \cos\psi \cos r + \sin\alpha \sin\psi \sin r \cos\theta),$$

or $$\frac{h}{2A}\cos 2\alpha \cos^2 \psi + \frac{h}{2A} - \frac{R}{A}\cos\alpha = \cos\alpha \cos\psi \cos r + \sin\alpha \sin\psi \sin r \cos\theta.$$

Now ψ being eliminated between this equation and its differential, supposing ψ to vary, gives the equation of the required envelope. The differential of this equation, supposing ψ to vary, is

$$\frac{h}{A} \cos 2\alpha \sin \psi \cos \psi = \cos \alpha \sin \psi \cos r - \sin \alpha \cos \psi \sin r \cos \theta.$$

The sum of the products obtained by multiplying the first of these equations by $\cos \psi$ and second by $\sin \psi$ is, by reduction,

$$\frac{h}{2A} \cos 2\alpha \cos^3 \psi - \left(\frac{h}{A} \cos 2\alpha + \frac{h}{2A} - \frac{R}{A} \cos \alpha\right) \cos \psi + \cos \alpha \cos r = 0,$$

which may be used to characterize the curve instead of the equation between r and θ.

It follows from this equation that when

$$\psi = 90^\circ,$$

we have $$r = 90^\circ.$$

Which reduces the equation for the circle of intersection to

$$\frac{h}{2A} - \frac{R}{A} \cos \alpha = \sin \alpha \cos \theta,$$

or $$\cos \theta = \frac{h}{2A \sin \alpha} - \frac{R}{A} \cot \alpha = \frac{B^2 + A^2 + 2BR}{2A(B + R)}.$$

Now this value of $\cos\theta$ is >1, and therefore impossible, whenever

$$2\mathrm{R} < \mathrm{A} - \mathrm{B},$$

that is, when the diameter of the rolling sphere is less than the difference between the semi-axes of the ellipsoid; so that, in this case, the curve is composed of two parts; and we can find the extent of the branches by making

$$\theta = 0.$$

Secondly. To find the envelope upon the second sphere. Let the same plane as before be the meridional plane, and let the origin be that point upon the surface of this sphere which originally coincided with the axis of the ellipsoid; and we shall call the diameter through this origin the *axis of the sphere.*

Let, then, $\psi' =$ the angle which this axis makes with the radius drawn to the point of contact with the ellipsoid,

$\omega' =$ the angle which this axis makes with L,

$\varphi' =$ the angle which L makes with any radius of the second sphere drawn to a point of the intersection of the two spheres,

$r' =$ the radius vector,

$\theta' =$ the angle which r' makes with the prime meridian;

and we have

$$\mathrm{R}\, d\psi' = \sqrt{(dx^2 + dy^2)} = d\psi \sqrt{(\mathrm{A}^2\sin^2\psi + \mathrm{B}^2\cos^2\psi)},$$

$$\omega' = \psi' + \psi - \omega,$$

$$2\,\mathrm{RL}\cos\varphi' = \mathrm{L}^2 + \mathrm{R}^2 - \mathrm{A}^2 = h^2\cos 2\alpha\cos^2\psi + (\mathrm{B}+\mathrm{R})^2 - \mathrm{A}^2 + \mathrm{R}^2,$$

$$\cos\varphi' = \cos\omega'\cos r' + \sin\omega'\sin r'\cos\theta'.$$

And this last equation, multiplied by $2\,\mathrm{RL}$, gives by substitution

$$\begin{aligned} &h^2\cos 2\alpha\cos^2\psi + (\mathrm{B}+\mathrm{R})^2 - \mathrm{A}^2 + \mathrm{R}^2 \\ &\quad = 2h\,\mathrm{R}\cos r'[\cos\alpha\cos\psi\cos(\psi'+\psi) + \sin\alpha\sin\psi\sin(\psi'+\psi)] \\ &\quad + 2h\,\mathrm{R}\sin r'\cos\theta'[\cos\alpha\cos\psi\sin(\psi'+\psi) - \sin\alpha\sin\psi\cos(\psi'+\psi)], \end{aligned}$$

which, differentiated, becomes, supposing ψ' and ψ to vary,

$$\begin{aligned} &h\cos 2\alpha\sin\psi\cos\psi = \mathrm{R}(\cos\alpha - \sin\alpha)[\cos r'\sin(\psi'+2\psi) - \sin r'\cos\theta'\cos(\psi'+2\psi)] \\ &+ \cos r'\cos\alpha\cos\psi\cos(\psi'+\psi)\sqrt{\mathrm{A}^2\sin^2\psi + \mathrm{B}^2\cos^4\psi}\cdot[\tan(\psi'+\psi) \\ &- \tan\alpha\tan\psi - \cos\theta'\tan r'\}1 + \tan\alpha\tan\psi\tan(\psi'+\psi)\{], \end{aligned}$$

between which and the preceding equation ψ is to be eliminated by means of the preceding value of ψ'.

—— We had intended to insert here the elegant solution of Dr. Strong, but our limits would not permit us.

(66.) QUESTION XVI. *By Investigator.*

A given cylindrical surface is placed with one of its linear elements in contact with a horizontal plane, and then made to oscillate on the plane according to a given law. It is required to find the motion of a material point, placed on the smooth interior surface, and subjected to the action of gravity.

FIRST SOLUTION. *By Professor C. Avery, Hamilton College.*

I shall take it for granted that the axis of the cylinder moves parallel

to itself: hence we may abstract from the cylinder and confine our remarks to a vertical section, at right angles to the axis, moving on a straight line situated on the horizontal plane. I shall also suppose the cylinder to be right with a circular base, radius R.

Let the straight line on which the section moves be taken for the axis of x, and a perpendicular to it in the vertical plane the axis of y, the y being counted upwards.

Let, at the epoch t, x and y be the co-ordinates of the particle, and x' the abscissa of the point of contact of the section and plane. The general formula of Dynamics becomes

$$\frac{d^2 x}{dt^2}\,\delta x + \left(\frac{d^2 y}{dt^2} + g\right)\delta y = 0 \quad \ldots\ldots\ldots \quad (1).$$

But the point is confined to the circular section, therefore

$$(x - x')^2 + (R - y)^2 = R^2 \quad \ldots\ldots\ldots \quad (2),$$

$$\text{and} \qquad (x - x')\,\delta x - (R - y)\,\delta y = 0 \quad \ldots\ldots\ldots \quad (3);$$

since, in whatever manner the section moves, x' is a given function of t by the question, and therefore $\delta x' = 0$. Multiply (3) by the indeterminate co-efficient λ, adding the result to (1), and equating the co-efficients of δx and δy to zero, we have

$$\frac{d^2 x}{dt^2} + \lambda (x - x') = 0, \quad \ldots\ldots\ldots \quad (4),$$

$$\frac{d^2 y}{dt^2} + g - \lambda (R - y) = 0 \quad \ldots\ldots\ldots \quad (5).$$

And by eliminating λ we have

$$\frac{d^2 x}{dt^2}(R - y) + \frac{d^2 y}{dt^2}(x - x') + g(x - x') = 0 \quad \ldots \quad (6).$$

But if φ be the angle made by a line drawn from the particle to the centre of the moving circle with its vertical diameter we have

$$x - x' = R \sin\varphi, \qquad R - y = R\cos\varphi;$$

and these substituted in (6) give

$$R\,.\,\frac{d^2\varphi}{dt^2} + g\sin\varphi + \frac{d^2 x'}{dt^2}\cos\varphi = 0 \quad \ldots\ldots \quad (7),$$

from which the motion of the particle is determined when the nature of the function $\frac{d^2 x'}{dt^2} = F(t)$ is known.

If $x' = a + bt$, that is, if the motion of the centre be uniform, whether it arise from the rolling or sliding of the cylinder, or both, (7) becomes

$$R\,.\,\frac{d^2\varphi}{dt^2} + g\sin\varphi = 0 \quad \ldots\ldots\ldots \quad (8);$$

$$\therefore R\,.\,\frac{d\varphi^2}{dt^2} - 2g\cos\varphi = \text{const.} = k \quad \ldots\ldots \quad (9);$$

$$\therefore dt = \frac{d\varphi\sqrt{R}}{\sqrt{k + 2g\cos\varphi}} \quad \ldots\ldots\ldots \quad (10),$$

or the motion is the same as that of a simple pendulum about a horizontal axis.

If $x' = a + bt + ct^2$, that is if the motion of the centre be uniformly accelerated, $\frac{d^2x'}{dt^2} = c$, and we have

$$\text{R} \cdot \frac{d^2\varphi}{dt^2} + g \sin\varphi + c \cos\varphi = 0 \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (11),$$

$$\therefore \text{R} \cdot \frac{d\varphi^2}{dt^2} - 2g \cos\varphi + 2c \sin\varphi = \text{const.} = k \quad \cdot \cdot (12),$$

$$\therefore \quad dt = \frac{d\varphi \sqrt{\text{R}}}{\sqrt{k + 2g \cos\varphi - 2c \sin\varphi}} \cdot \cdot \cdot \cdot \cdot \cdot \quad (13);$$

which is an elliptic function.

In the case of very small oscillations (7) becomes

$$\text{R} \cdot \frac{d^2\varphi}{dt^2} + g\varphi + \text{F}(t) = 0 \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (14),$$

whose integral (La Croix, page 407,) is, if $\sqrt{\frac{g}{\text{R}}} = a$,

$$\text{R}\varphi = c \cos at + c' \sin at + \sin at \int dt \cos at\, \text{F}(t) - \cos at \int dt \sin at\, \text{F}(t) \, . \quad (15)$$

so that if $\text{F}(t)$ contain only small periodic terms, this equation will be compatible with the condition of small oscillations.

For instance, let $\text{F}(t) = p \sin mt$, p being a small constant, so that

$$x' = a + bt - \frac{p}{m^2} \sin mt,$$

which is consistent with a motion of translation of the centre, and a small oscillatory rolling motion, then (15) becomes

$$\text{R}\varphi = c \cos at + c' \sin at + \frac{ap}{a^2 - m^2} \sin mt \quad \cdot \cdot \cdot \cdot \quad (16).$$

The most interesting case of this kind is when $t = 0$, $\varphi = 0$, $\frac{d\varphi}{dt} = 0$ at the same time; that is, when the particle is originally placed at rest in the lowest point of the cylinder, and therefore owes its motion only to that of the cylinder. For this case,

$$c = 0, \, c' = \frac{-mp}{a^2 - m^2};$$

$$\text{R}\varphi = \frac{p}{a^2 - m^2}(a \sin mt - m \sin at) \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (17).$$

—— Professor Catlin's solution is also very neat. Professor Peirce, after finding an equation similar to (7) of the preceding solution, proceeds thus :

x', when arranged according to powers of t, may be written

$$x' = at + bt^2 + x'';$$

which, substituted in the preceding equation, gives

$$\text{R} \cdot \frac{d^2\varphi}{dt^2} + g \sin\varphi + \left(2b + \frac{d^2x''}{dt^2}\right) \cos\varphi = 0.$$

If now $\frac{d^2x''}{dt^2}$ is nothing, or so small as to be neglected, we may take A and B, so that

$$g = \text{A} \cos \text{B}, \quad 2b = \text{A} \sin \text{B};$$

and then this equation will be

$$\text{R} \cdot \frac{d^2\varphi}{dt^2} + \text{A} \sin (\varphi + \text{B}) = 0,$$

and making

$$\varphi' = \varphi + \text{B},$$

$$\text{R} \cdot \frac{d^2\varphi'}{dt^2} + \text{A} \sin \varphi' = 0;$$

so that *the motions are the same as those of a simple pendulum relative to an axis inclined to the vertical by the angle* B, *and whose length is*

$$\frac{g\text{R}}{\text{A}} = \frac{g\text{R}}{\sqrt{g^2 + 4b^2}}.$$

Let now $\delta\varphi$ be the perturbation arising from the small force $\frac{d^2x''}{dt^2}$, and we have to determine it, the equation

$$\text{R} \cdot \frac{d^2\delta\varphi}{dt^2} + \text{A} \cos \varphi' \cdot \delta\varphi + \frac{d^2x''}{dt^2} \cos (\varphi' - \text{B}) = 0;$$

whence

$$\delta\varphi = \frac{1}{\text{R}} \cos kt \cos (\varphi' - \text{B}) \int \frac{d^2x''}{dt} \sin kt$$

$$- \frac{1}{\text{R}} \sin kt \cos (\varphi' - \text{B}) \int \frac{d^2x''}{dt} \cos kt,$$

$$\text{where} \quad k = \sqrt{\frac{\text{A} \cos \varphi'}{\text{R}}};$$

so that if $\frac{d^2x''}{dt^2}$ is composed of terms of the form

$$\text{K} \cdot \begin{matrix}\sin.\\ \cos.\end{matrix} \left(mt + e\right)$$

each of these terms will produce in the value of $\delta\varphi$, the term

$$\frac{\text{K}\, k \cos (\varphi' - \text{B})}{m^2\text{R} - \text{A} \cos \varphi'} \cdot \begin{matrix}\sin.\\ \cos.\end{matrix} \left(mt + e\right).$$

SECOND SOLUTION. *By Dr. T. Strong.*

We shall suppose that the cylinder is right and elliptic; imagine a vertical plane to be drawn through the material point at right angles to the axis of the cylinder, then it is evident that the point will always be somewhere in the elliptic section. Let

$$\text{A}^2 y'^2 + \text{B}^2 x'^2 = \text{A}^2 \text{B}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

be the equation to this section, referred to its own axes as axes of co-ordinates. Let xy denote the rectangular co-ordinates of any point in the perimeter of the ellipse, the axis of x being the common section of the horizontal and elliptic planes, and the axis of y perpendicular to the horizontal plane through any fixed point taken in the axis of x; also let x_1

y_1 be the co-ordinates of the centre of the ellipse, referred to the same axes; put a, b for the cosines of the angles made by x', y' respectively with the axis of x, also a', b' for the cosines of the angles made by the axes with that of y; then we have

$$x = x_1 + ax' + by' \text{ and } y = y_1 + a'x' + b'y' \quad . \quad . \quad . \quad . (2).$$

$$\therefore x' = (x - x_1)\,a + (y - y_1)\,a',\ y' = (x - x_1)\,b + (y - y_1)\,b', \,. \,(3),$$

If φ = the angle made by the axis of x' or the greater axis of the ellipse with the axis of y, we shall have

$$a = \sin\varphi,\ a' = -\cos\varphi,\ b = \cos\varphi,\ b' = \sin\varphi$$

$$\therefore x' = (x - x_1)\sin\varphi - (y - y_1)\cos\varphi,\ y' = (x - x_1)\cos\varphi + (y - y_1)\sin\varphi \ . \ (4),$$

and (1) is easily changed to

$$A^2[(x - x_1)\cos\varphi + (y - y_1)\sin\varphi]^2 + B^2[(x - x_1)\sin\varphi - (y - y_1)\cos\varphi]^2 = A^2B^2 \quad . \quad . \quad (5),$$

for the equation of the ellipse. But the ellipse touches the axis of x, or when $y = 0$, we have also $dy = 0$; that is, by (5),

when $A^2[(x - x_1)\cos\varphi - y_1\sin\varphi]^2 + B^2[(x - x_1)\sin\varphi + y_1\cos\varphi]^2 = A^2B^2$ (6),

then also

$$A^2\cos\varphi[(x - x_1)\cos\varphi - y_1\sin\varphi] + B^2\sin\varphi[x - x_1)\sin\varphi + y_1\cos\varphi] = 0 \ . \ (7);$$

from which

$$y_1 = \sqrt{A^2\cos^2\varphi + B^2\sin^2\varphi},\ x - x_1 = \frac{(A^2 - B^2)\sin\varphi\cos\varphi}{\sqrt{A^2\cos^2\varphi + B^2\sin^2\varphi}} \ . \ . \ . \ . \ (8).$$

Moreover, if r be the radius of curvature at this point of contact, when $y = 0$, and $dy = 0$, we shall have

$$r = \frac{dx^2}{d \cdot y} = \frac{(B^2 - A^2)(x - x_1)\sin\varphi\cos\varphi + (B^2\cos^2\varphi + A^2\sin^2\varphi)y_1}{A^2\cos^2\varphi + B^2\sin^2\varphi}$$

$$= \frac{A^2B^2}{(A^2\cos^2\varphi + B^2\sin^2\varphi)^{\frac{3}{2}}} \ . \ . \ (9).$$

Put $g = 32, 2$ ft., R' = the normal re-action of the perimeter of the ellipse at the point xy, on the particle of matter at the time, t, from the origin of motion; then by the principles of Dynamics we get

$$\frac{d^2x}{dt^2} + \frac{R'\,dy}{\sqrt{dx^2 + dy^2}} = 0,\ \frac{d^2y}{dt^2} + g - \frac{R'dx}{\sqrt{dx^2 + dy^2}} = 0 \cdot \ . \ (10).$$

The sign $-$ has been written before the last term of the second of (10), because R' tends to increase y, and the differentials which enter into the last terms of the second members of (10) are regarded as constant with respect to x_1 and φ.

Now equation (1) is satisfied by taking

$$y' = B\sin\theta,\quad x' = A\cos\theta,$$

which, with the values of $a, a'\ b, b'$, substituted in (2), give

$$x = x_1 + A\sin\varphi\cos\theta + B\cos\varphi\sin\theta,\ y = y_1 - A\cos\varphi\cos\theta + B\sin\varphi\sin\theta \quad . \ (11);$$

$$\therefore dx = (-A\sin\varphi\sin\theta + B\cos\varphi\cos\theta)d\theta,\ dy = (A\cos\varphi\sin\theta + B\sin\varphi\cos\theta)d\theta \ . \ (12),$$

supposing θ only variable, which are the values of dx and dy to be used in the last terms of the first members of (10); and by eliminating R' from (10),

$$\frac{d^2x}{dt^2} \cdot \frac{dx}{d\theta} + \left(\frac{d^2y}{dt^2} + g\right) \cdot \frac{dy}{d\theta} = 0 \cdot \ . \ . \ . \ . \ . \ (13),$$

or, $(\text{B}\cos\varphi\cos\theta-\text{A}\sin\varphi\sin\theta)\dfrac{d^2x}{dt^2}$

$$+(\text{A}\cos\varphi\sin\theta+\text{B}\sin\varphi\cos\theta)\left(\frac{d^2y}{dt^2}+g\right)=0 \quad . \ (14).$$

Again, if we take the total differentials of x and y we shall have

$$dx=dx_1+\frac{dx}{d\varphi}d\varphi+\frac{dx}{d\theta}d\theta,\ dy=dy_1+\frac{dy}{d\varphi}d\varphi+\frac{dy}{d\theta}d\theta,$$

$$\therefore\ \frac{dx}{d\theta}d\theta=dx-dx_1-\frac{dx}{d\varphi}d\varphi,\ \frac{dy}{d\theta}d\theta=dy-dy_1-\frac{dy}{d\varphi}d\varphi \ \cdot \ \cdot \ \cdot \ (15);$$

hence, if we multiply (13) by $d\theta$, and substitute (15), we shall find

$$\frac{dxd^2x+dyd^2y}{dt^2}+g\,dy=\left(dx_1+\frac{dx}{d\varphi}d\varphi\right)\cdot\frac{d^2x}{dt^2}$$

$$+\left(dy_1+\frac{dy}{d\varphi}d\varphi\right)\left(\frac{d^2y}{dt^2}+g\right)=d\psi \ . \ . \ (16).$$

$d\psi$ being supposed an imperfect differential; and taking the integral,

$$\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}+2gy=2\int d\psi+c \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ (17),$$

c being the correction of the integral.

In order to find θ in terms of the time, t, we must substitute for y_1 its value in (8), also for x_1 and φ their values in functions of t deduced from the given law of the cylinder's motion, in (11), then x and y will be expressed in terms of θ and t, and substituted in (14) will give an equation involving $\dfrac{d^2\theta}{dt^2}$, $\dfrac{d\theta}{dt}$, and functions of t and θ; which functions may be made rational by expressing the periodic functions and radicals in series. Then by taking

$$\theta=a'+b't+c't^2+d't^3+\&c. \quad . \ . \ . \ . \ . \ (18),$$

the equation in θ and t, will be expressed in integral powers of t, and will be identical, and by putting each of the co-efficients of $t=0$, we shall have a', b', c', &c., in terms of constants, and the initial values of θ and $\dfrac{d\theta}{dt}$. By the method of La Place, vol. 1, Mec. Cel., art. 43, we can now convert θ into a periodical function of t, when it can be so expressed.

Again, if x , φ are small periodic functions of t, or if they vary so slowly with respect to θ that the second member of (16) is very small for large periods of time, we may at first neglect $2\int d\psi$ in (17), and then

$$\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}+2gy=c, \text{ very nearly} \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ (19).$$

By determining θ from this equation in terms of t as before, we shall have x and y in terms of t and φ, which substituted in $d\psi$ in the second member of (17), and the integral taken, will give a more correct value of θ, and so on, to any degree of exactness required.

If $\varphi = const.$, the ellipse will slide without rolling, and the terms in dx and dy which involve $d\varphi$ as a factor will disappear; x_1 will be a given function of t, and θ may be found as before. If also $x_1 = const.$, then $dx_1 = 0$, and the ellipse is at rest, the motion of the particle being then exactly defined by equation (19). Again, by (8) we know

$$x_1 = x - \frac{(\mathrm{A}^2 - \mathrm{B}^2)\sin\varphi\cos\varphi}{\sqrt{\mathrm{A}^2\cos^2\varphi + \mathrm{B}^2\sin^2\varphi}} = x + \frac{d\sqrt{\mathrm{A}^2\cos^2\varphi + \mathrm{B}^2\sin^2\varphi}}{d\varphi},$$

or

$$dx_1 = dx + d\varphi \,.\, \frac{d^2.\sqrt{\mathrm{A}^2\cos^2\varphi + \mathrm{B}^2\sin^2\varphi}}{d\varphi^2} \quad . \; . \; . \; (20),$$

and if $ds = dx$ = the element of the ellipse in contact with the plane, we have, by (9), since $ds = r d\varphi$,

$$dx = ds = r d\varphi = \frac{\mathrm{A}^2\mathrm{B}^2 d\varphi}{(\mathrm{A}^2\cos^2\varphi + \mathrm{B}^2\sin^2\varphi)^{\frac{3}{2}}},$$

$$\therefore dx_1 = \frac{\mathrm{A}^2\mathrm{B}^2\, d\varphi}{(\mathrm{A}^2\cos^2\varphi + \mathrm{B}^2\sin^2\varphi)^{\frac{3}{2}}} + d\varphi \,.\, \frac{d^2\sqrt{\mathrm{A}^2\cos^2\varphi + \mathrm{B}^2\sin^2\varphi}}{d\varphi^2} \quad . \; (21),$$

which is the value to be used for dx_1 in (16), when the ellipse rolls without sliding.

In the case of small oscillations, supposing that quantities of the orders φ^2, θ^2, &c., are neglected, we shall have

$$dx = dx_1 + \mathrm{A}\, d\varphi + \mathrm{B}\, d\theta, \; dy = 0;$$

and when x_1 and φ vary very slowly compared with θ, we shall have by (17),

$$\mathrm{B}^2 \,.\, \frac{d\theta^2}{dt^2} + \frac{g}{\mathrm{A}}(\mathrm{A}\theta + \mathrm{B}\varphi)^2 = c \text{ very nearly} \quad . \; . \; . \; . \; (22),$$

which will give θ in functions of t and φ, therefore by repeating the process we shall find θ in terms of t to any degree of exactness. But if x_1 and φ do not vary very slowly with respect to the variation of θ, we may use the first method above given.

If $\mathrm{A} = \mathrm{B} = \mathrm{R}$, the ellipse becomes a circle whose radius is R, and the cylinder will be the common circular cylinder; the motion of the point under these circumstances may be deduced from the preceding equations. In the case of small oscillations, since $dx = dx_1 + \mathrm{R}d(\varphi + \theta)$, by putting $\varphi + \theta = \beta$, (19) becomes

$$\frac{dx_1 + \mathrm{R}\, d\beta}{dt} = \sqrt{g\mathrm{R}} \,.\, \sqrt{\beta'^2 - \beta^2} \quad . \; . \; . \; . \; . \; . \; (23),$$

where β' is the initial value of β, and if the cylinder rolls without sliding, $dx_1 = \mathrm{R}\, d\varphi$, and then

$$\frac{d\varphi}{dt} + \frac{d\beta}{dt} = \sqrt{\frac{g}{\mathrm{R}}} \,.\, \sqrt{\beta'^2 - \beta^2} \quad . \; . \; . \; . \; . \; . \; . \; (24),$$

where since $\frac{d\varphi}{dt}$ is given by the conditions in terms of t, β will be had in terms of t, and will determine the motion of the particle relatively to the point of the cylinder in contact with the axis of x, subject however to the necessary corrections on account of the imperfect differential $d\varphi$, by successive substitutions in (17).

From what has been done, the method of proceeding for a cylinder of any other shape will be sufficiently evident.

List of Contributors and of Questions answered by each. The figures refer to the number of the Question, as marked in Number III., Art. XII.

LYMAN ABBOTT, Niles, N. Y., ans. 1 to 14.
PROFESSOR C. AVERY, Hamilton College, N. Y., ans. all the questions.
A., ans. 5.
P. BARTON, jun., Athol, Mass., ans. 1, 2, 3, 4, 7.
PROFESSOR F. N. BENEDICT, University of Vermont, ans. 6, 9, 11, 12, 13, 14.
W. R. BIDDLECOM, Clinton Liberal Institute, N. Y., ans. 1.
B. BIRDSALL, New Hartford, Oneida Co. N. Y., ans. 1, 2, 3, 7, 12.
PROFESSOR M. CATLIN, Hamilton College, N. Y., ans. all the questions.
PROFESSOR JNO. CHAMBERLAIN, Oakland College, Miss., ans. 3.
J. B. H., of the Sophomore Class, Harvard University, ans. 1, 2, 3, 4, 6.
R. S. HOWLAND, of the Sophomore Class, St. Paul's College, L. I., ans. 2.
INVESTIGATOR, ans. 16.
W. LENHART, York, Pa., ans. 2, 3, 4, 7.
J. F. MACULLY, Teacher of Mathematics, New York, ans. 1 to 13.
PROFESSOR B. PIERCE, Harvard University, ans. all the questions.
ψ, ans. 12.
O. ROOT, Principal of Syracuse Academy, N. Y., ans. all the questions.
PROFESSOR T. STRONG, LLD., New Brunswick, N. J., ans. all the questions.
RICHARD TINTO, Greenville, Ohio, ans. 11.
N. VERNON, Frederick, Md., ans. 1, 2, 3, 4, 7, 9.

*** All communications for Number VI, which will be published on the first day of November, 1838, must be post paid, addressed to the Editor, St. Paul's College, Flushing, L. I.; and must arrive before the first of August, 1838. New questions must be accompanied with their solutions.

The Editor would thank the Contributors for their kindness in sending *all* their communications for this number, at the specified time; which has materially lightened the labor of preparing it for the press.

Mr. Lenhart's Tables will appear with Number VI, which will complete the first volume of the Mathematical Miscellany.

Professor Catlin informs us of the death of our correspondent, Mr. J. Ketchum. He is spoken of as a young man of high promise.

We would apologize to *Professor Harnay, Hanover College, South Hanover, Ind.*, for an error in the printing of his name in the last Number; see Article XVII.

ARTICLE XX.

NEW QUESTIONS TO BE ANSWERED IN NUMBER VII.

Their Solutions must arrive before Feb. 1st, 1839.

(82.) QUESTION I. *By an Engineer.*

The following is an extract from my Note-Book:

No.	Bearing.	Distance.	Elevation.
1	N. 10° 15′ E.	27,54 ch.	+ 17° 54′
2	N. 28° 40′ W.	100,00	+ 20° 19′
3	N. 20° 00′ W.	15,00	+ 7° 43′
4	N. 20° 00′ W.	37,26	— 5° 26′
5	N. 36° 17′ E.	68,75	— 11° 13′

It is required to find the Bearing, Distance and Elevation of a line drawn from the beginning of the first to the end of fifth line, by a method applicable to all practical cases of the kind. The Bearing is considered as the inclination of a vertical plane through the two places with the plane of the meridian.

(83.) QUESTION II. *By J. F. Macully, Esq., New York.*

It is required to draw a chord through the focus of a given ellipse, which shall divide the area in a given ratio.

(84.) QUESTION III. *By Investigator.*

Find the *polar* equation of a straight line on a plane; and bring it to the form best adapted to general use. Apply it to finding the equation of a tangent to the ellipse at any point, the pole being at the focus and the angular axis the line of the foci.

(85.) QUESTION IV. *By Mr. P. Barton, Jun.*

The co-ordinates of the vertex of a cone of revolution are

$$x = -4\ ,\quad y = 3\ ,\quad z = -2\ ;$$

the equations of its axis are

$$x = \tfrac{1}{2}z - 3\ ,\quad y = -\tfrac{1}{3}z + 2\tfrac{1}{3}\ ;$$

and its vertical angle is 90°. It is required to find where its surface is intersected by the line whose equations are

$$x = z + 6\ ,\quad y = -z - 5.$$

(86.) QUESTION V. *By ψ.*

The circumference of a circle is divided into n equal parts, and from the points of division perpendiculars are drawn upon a given diameter of the circle. If lines be drawn from any given point in the plane of the circle to the points where these perpendiculars intersect the diameter, it is required to find the sum of the squares of these lines.

(87.) QUESTION VI. *By* ——.

Def. In the ellipse or hyperbola, the parameter of any diameter is that chord of the system it bisects which is a third proportional to that diameter and its conjugate.

It is required

1°. To find what diameters may properly be said to have parameters.

2°. To find the locus of the middle points of all the parameters of the same curve.

3°. Having given a parameter, to find, if possible, another one perpendicular to it.

(88.) QUESTION VII. *By* ——.

The theorem of M. Sturm, published in the "Memoirs présêntes par des Savans Estrangers," for 1835, may be stated thus:

Let $X = 0$, be any algebraical equation whose co-efficients are real, and whose roots are unequal, and let $X_1 = \frac{dX}{dx}$. Apply to the two polynomials X, X_1 the process for finding their greatest common measure, the several remainders having all their signs changed from $+$ to $-$, and from $-$ to $+$, before they are used as new divisors, and in that state let them be represented by $X_2, X_3, X_4 \ldots\ldots X_m$. In the series of polynomials

$$X, X_1, X_2, X_3, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad X_m,$$

which are of continually decreasing dimensions in x, X_m being independent of x, let any two numbers p and q be successively substituted for x, noting the signs of the two series of results. *Then the difference between the number of variations of the first series of signs, and that of the second, expresses exactly the number of real roots of the given equation, which are comprised between the two numbers p and q.*

It is required to apply this theorem to the general equation

$$x^4 + ax^2 + bx + c = 0,$$

in order to determine the number and nature of its real roots.

(89.) QUESTION VIII. *By Professor B. Peirce, Harvard University.*

Prove that if all the roots of the equation

$$x^n - \mathrm{A}x^{n-1} + \mathrm{B}x^{n-2} - \&c., = 0,$$

are real, that we shall have

$$n(n-1)(3\mathrm{B})^2 < (n-2)(2\mathrm{A})^3.$$

(90.) QUESTION IX. *By Professor F. N. Benedict, University of Vermont.*

To determine the locus of the intersection of two tangents or normals to the common parabola which include an angle whose tangent varies as a given function of the co-ordinates of the point of intersection.

(91.) QUESTION X. *By Wm. Lenhart, Esq., York, Pa.*

Having given a series of whole numbers whose third order of differences are constant, and of which a given term is divisible by a given prime number m; it is required to find that term in the series which is divisible by m^n, n being a given whole number.

(92.) QUESTION XI. *By J. F. Macully, Esq.*

Required the value of n terms of the continued product

$$(\tfrac{2}{3}+3\cos\theta)(\tfrac{2}{3}+3\cos\tfrac{1}{4}\theta)(\tfrac{2}{3}+3\cos\tfrac{1}{16}\theta)\ \ldots\ldots\ldots$$

(93.) QUESTION XII. *By Professor Peirce.*

To find a curve whose radius of curvature is a given function of its arc.

(94.) QUESTION XIII. *By the same Gentleman.*

Find a curve which is its own involute, subject, if possible, to the restriction that when it is expressed by an equation between its arc and radius of curvature, this equation is algebraic.

(95.) QUESTION XIV. *By Professor C. Avery, Hamilton College.*

Suppose a rod to descend as in Question (63), Miscellany, and that a particle, whose weight is inconsiderable with respect to that of the rod, is placed on it and begins to descend by gravity, without friction, at the instant the rod commences its motion. Required the point on the rod where the particle must be placed, in order that it may arrive at the lowest extremity of the rod at the time the rod becomes horizontal.

(96.) QUESTION XV. *By Mr. W. S. B. Woolhouse, Actuary of the National Loan Fund Life Assurance Society, London.*

(*From the Gentleman's Diary, for* 1836.)

A crown piece being twirled any how on a perfectly smooth horizontal plane, it is required to investigate the circumstances of the motion and the velocities of its points when it acquires any given position, disregarding the thickness of the metal.

(97.) QUESTION XVI. *By the same Gentleman.*

(*From the Gentleman's Diary, for* 1837.)

It is required to solve the preceding Question, when, instead of the circular disc, any solid of revolution is substituted, as for instance, a *spheroid*, the semi-axes of which are a and b.

ARTICLE XXI.

DEMONSTRATION OF A THEOREM IN GEOMETRY.

By Dr. Strong.

" *Theorem.* Let S be the number of solid angles in a polyedron, H the number of its faces, A the number of its edges; then, in all cases, we shall have

$$S + H = A + 2."$$

(*See Appendix to Books* 6 *and* 7 *Le Gendre's Geometry*)

DEMONSTRATION.

Imagine the polyedron to stand on one of its faces, supposed horizontal; then since the number of angles in any rectilinear figure is equal to that of its sides, the number of sides of the face on which the solid stands equals the number of solid angles through which the face passes. Also the number of linear edges drawn from the angles of the base, or face on which the solid stands, to the next angles above, equals the number of faces which terminate in the base, and the remaining edges of these faces is the same in number as that of the solid angles through which they pass; then the linear edges drawn from the last solid angles to the next in succession above them, equals the number of faces added, and the number of solid angles added is the same as the number of edges added, which pass through the angles, and so on, until we arrive at the last face or solid angle. It is hence evident that if we do not consider the first and last faces (or the last solid angle) the number of edges is the same as that of the angles and faces, therefore, by including the first and last faces (or solid angle) we shall have

$$S + H = A + 2.$$

Should one or more of the linear edges which have been supposed to be drawn from the angles of the base to the next angles above them $= 0$, then the number of linear edges added will equal the number of solid angles added together with the number of the linear edges which are $= 0$, and the same remarks will apply in every case when one or more of the linear edges which have been supposed to be drawn from any angles to the next in succession are $= 0$; hence as before we shall have

$$S + H = A + 2.$$

ARTICLE XXII.

DIOPHANTINE SPECULATIONS.

By Wm. Lenhart, Esq., York, Penn.

NUMBER TWO.

1. In preparing the general investigation published in Number II. of the Miscellany, it was deemed most advisable to omit the following case of Art. 10, not only on account of its extensive bearing, and its not interrupting the arrangement of that paper, but because it was thought better suited, as an entire speculation, for some future number; as such, therefore, it is now communicated.

2. Suppose $s'x \pm r'm^3$ and $s'y \mp r'm^3$ to represent any two numbers; their sum will be

$$s'(x+y) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the sum of their cubes will be

$$s'^3(x^3+y^3) \pm 3\,s'^2r'm^3(x^2-y^2)+3\,s'r'^2m^6(x+y) \quad . \quad . \quad (2)$$

Or, dividing (2) by (1) and substituting $m'm^3$ for x^2-xy+y^2 and dividing by m^3,

$$\left(\frac{s'x\pm r'm^3}{m}\right)^3+\left(\frac{s'y\mp r'm^3}{m}\right)^3=s'(x+y).(3r'^2m^3\pm3s'r'(x-y)+s'^2m') \quad (3)$$

Or, if x and y were such as to make $x+y=a^3$ at the same time that $x^2-xy+y^2=m'm^3$, then

$$\left(\frac{s'x\pm r'm^3}{am}\right)^3+\left(\frac{s'y\pm r'm^3}{am}\right)^3=s'(3r'^2m^3\pm3s'r'(x-y)+s'^2m') \quad . \quad . \quad (4)$$

In which, if $r'=1$ and $s'=1, 2, 3$ &c., the first difference of the right hand factors will be $3\,(m'\pm(x-y))$ and the difference of the differences $2m'$. Or, if $s'=1$ and $r'=1, 2, 3$, &c., the first difference will be $3\,(3m^3\pm(x-y))$ and the difference of the differences $6m^3$. Or, if r' and s' be assumed any numbers prime to each other, as the system will not be changed, the differences may be easily ascertained and the calculations made accordingly.

3. But here the question arises how are we to find x and y such that $x+y$ may be a cube at the same time that x^2-xy+y^2 is a multiple of a cube? or, that we may have

$$\left(\frac{x}{am}\right)^3+\left(\frac{y}{am}\right)^3=m' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Be that our task, and preparatory thereto, let us propose and resolve the following

Problem.

Suppose A, B, C, D, &c. be a series of numbers whose first order of differences forms an arithmetical progression. Now, if any term in the

given series be divisible by a prime number m, it is required to find a term in that series that shall be divisible by m^n.

Solution.

4. Let the arithmetical progression which forms the first order of differences be denoted by

$$a,\ a+b,\ a+2b\ \ .\ \ .\ \ .\ \ a+n'b$$

the series itself may then be thus expressed, viz:

$$(\text{A}).\left\{\begin{array}{lll} a'_0 & & \\ & a & \\ a'_1 & & b \\ & a+b & \\ a'_2 & & '' \\ & a+2b & \\ a'_{n'} & & '' \\ & a+n'b & \end{array}\right.$$

Let d, d' represent the term in the first and second order of differences respectively of any term a'_n in (A), then the m^{th} term from $a'_{n'}$ or $a'_{n'+m}$ in (A) will be expressed, as is well known, by

$$a'_{n'}+md+\frac{m(m-1)}{2}d' \quad .\ .\ .\ .\ (6).$$

5. Now, if $a'_{n'}$ in (A) be divisible by m, so also will $a'_{n'+m}$. Let $a'_{n'}\div m=a''_0$ and $a'_{n'+m}\div m=a''_1$; and since $a+n'b$ and b are the terms of the first and second order of differences corresponding to $a'_{n'}$ in (A), by substituting them respectively for d, d' in (6) and reducing we shall find

$$a''_1-a''_0=(2a+(2n'-1)b+mb)\div 2 \quad .\ .\ .\ .\ (7)$$

which is the term of the first order of differences corresponding to a''_0 the first term of a new or second series, and mb the term of the second order or common difference. Hence,

$$(\text{B}).\left\{\begin{array}{lll} a''_0 & & \\ . & (2a+(2n'-1)b+mb\div 2 & .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ (7) \\ . & & mb \\ . & & . \\ . & & . \\ . & & . \\ a''_{n''} & & . \\ & (2a+(2n'-1)b+(2n''+1)mb)\div 2 & .\ .\ \ .\ .\ (8). \end{array}\right.$$

6. Now, in (B) suppose $a''_{n''}\div m=a'''_0$. Then, as (8) and mb are the terms of the first and second order of differences corresponding to $a''_{n''}$ in (B), by substituting them for $a+n'b$ and b in (7), we shall find

$$(2a+(2n'-1)b+2n''mb+m^2b)\div 2 \quad .\ .\ .\ (9),$$

the term of the first order of differences corresponding to a'''_0 and m^2b the common difference. Hence the third series

$$(\text{C}).\left\{\begin{array}{ll} a'''_0 & \\ . & (2a+(2n'-1)b+2n''mb+m^2b)\div 2 \quad .\ .\ .\ .\ (9) \\ . & . \\ . & . \\ . & . \\ . & . \\ a'''_{n'''} & \\ & (2a+(2n'-1)b+2n''mb+(2n'''+1)m^2b)\div 2 \quad (10). \end{array}\right.$$

7. And in general we shall have $a_0^{(n)}$ = first term of the n^{th} series, and

$$(2a+(2n'-1)b+2n''mb+2n'''m^2b \ldots 2n^{(n-1)} \cdot m^{n-2}b+m^{n-1}b)+2 \quad (11)$$

the term of the first order of differences corresponding to $a_0^{(n)}$ and $m^{n-1}b$ the term of the second order or common difference.

8. We shall here notice a few cases, which not unfrequently occur in practice, of which a knowledge may be useful. 1st. Suppose $a'_{n'}$ in (A) to be divisible by m^2, or which is the same thing, suppose a''_0 to divide by m, then $a''_0 \div m = a'''_0$, $n''=0$ and $2n''mb$ in (9) vanishes, therefore instead of (9) we shall have

$$(2a+(2n'-1)b+m^2b)+2 \quad \ldots\ldots \quad (12)$$

for the term of the first order of differences in (C).

2d. Suppose $a''_{n''}$ in (B) to divide by m^2, then $a'''_0 \div m = a^{\text{iv}}_0$, $n'''=0$, and $2n'''m^2b$ in the term of the first order of differences corresponding to a^{iv}_0 in the fourth series would vanish, and instead of $(2a+(2n'-1)b + 2n''mb+2n'''m^2b+m^3b)+2$, we should have

$$(2a+(2n'-1)b+2n''mb+m^3b)+2 \quad \ldots\ldots \quad (13)$$

for the term of the first order of differences in (D) or fourth series. From these two cases the reader will perceive their nature, and hence be enabled, when either occurs, to use them to advantage.

9. An impossible case of the problem is plainly indicated in the term of the first order of differences in (B), for, if that, or (7), be divisible by m, we shall have (7) equal to $m'm$ and the common difference being mb, each term in the first order of differences will be a multiple of m; but no multiple of m added to a''_0 can divide by m, because, by hypothesis, a''_0 will not. Hence, if (7) be not prime to m, we shall fail to find a term in (B) divisible by m; and, as the same reasoning holds in regard to the first order of differences in each successive series, the problem in such cases becomes impossible.

10. Since then (7), (9), &c. must be prime to m, and it is plain that each term in the first order of differences in the series (B), (C), &c. divided by m, will leave the same remainder, we shall therefore have this extremely simple method for finding n'' n''' &c., and consequently general expressions for $a''_{n''}$, $a'''_{n'''}$ .. $a^{(n)}_{n^{(n)}}$ that will divide by m. Thus, suppose $a''_0 \div m$ leaves the remainder R, and (7) $\div m$ leaves the remainder R',* we shall then have R + R', R + 2R', R + 3R' proceeding to R + n''R' which will divide by m (n'' being always less than m) an operation as simple as it is curious and beautiful, then $a''_{n''}$ will be the term of that series divisible by m; n'' will denote the number of terms that $a''_{n''}$ is from a''_0 and $a''_{n''}$ will be expressed by

$$a''_0+n''d+\frac{n''(n''-1)}{2}d' \quad \ldots\ldots \quad (14)$$

* But as R' is the same in each successive operation, after the first we need only calculate R, and if they be not prime to each other they may be made so.

in which $d = (7)$ and $d' = mb$. Hence, by substitution, the general expression for $a''_{n''}$ that will divide by m, will be

$$a''_{n''} = a''_0 + \frac{n''}{2}(2a + (2n' - 1)b + n''mb) \cdot \cdot \cdot \cdot (15).$$

In the same manner precisely, we get

$$a'''_{n'''} = a'''_0 + \frac{n'''}{2}(2a + (2n' - 1)b + 2n''mb + n''m^2b) \cdot \cdot \cdot (16),$$

and, in general,

$$a^{(n)}_{n^{(n)}} = a^{(n)}_0 + \frac{n^{(n)}}{2}\{2a + (2n' - 1)b + 2n''mb + 2n'''m^2b + \ldots + 2n^{(n-1)}m^{n-2}b$$
$$+ n^{(n)}m^{n-1}b\} \ldots (17),$$

which will divide by m.

11. Now, as (17), or $a^{(n)}_{n^{(n)}}$ in the n^{th} series, is divisible by m, let the quotient be q, then will

$$qm^n \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (18),$$

be the term sought of the given series: the number of terms from $a'_{n'}$ in (A) divisible by m, to qm^n in (A) divisible by m^n will, as the notation and the nature of the process, plainly indicate, be

$$n''m + n'''m^2 + n'''m^3 \; . \; . \; . \; n^{(n)}m^{n-1} \; . \; . \; . \; . \; (19).$$

12. Let us now take a column of equations (see Misc., page 116) for, a, an odd number, namely:

$$(D) \begin{cases} \left(\frac{s+1}{a}\right)^3 + \left(\frac{s}{a}\right)^3 = s^2 + s + 1 \\ \qquad\qquad\qquad\qquad 6 \\ \left(\frac{s+2}{a}\right)^3 + \left(\frac{s-1}{a}\right)^3 = s^2 + s + 7 \\ \qquad\qquad\qquad\qquad 12 \\ \left(\frac{s+3}{a}\right)^3 + \left(\frac{s-2}{a}\right)^3 = s^2 + s + 19 \\ \qquad\qquad\qquad\qquad 18 \\ \quad \&c. \qquad\qquad \&c. \end{cases}$$

and suppose the right hand side of each equation to correspond with the terms in the series a'_0, a'_1, a'_2 . . $a'_{n'}$ respectively, then, if the equation $\left(\frac{s+3}{a}\right)^3 + \left(\frac{s-2}{a}\right)^3 = s^2 + s + 19$, for example, be divisible by m, and we can find the equation

$$\left(\frac{s+(n+1)}{a}\right)^3 + \left(\frac{s-n}{a}\right)^3 = s^2 + s + (n+1).(2n+1) + n^2 \; *$$

* See Misc., Note, page 117.

that will divide by m^3, which may be done in the manner described in the solution of the foregoing problem, we shall evidently have

$$\left(\frac{s+(n+1)}{am}\right)^3+\left(\frac{s-n}{am}\right)^3=\frac{s^2+s+(n+1)(2n+1)+n^2}{m^3}=m' \quad . \ (20),$$

which is identical with q in (18) when $n=3$; and also with (5), and thence $x=s+(n+1)$ and $y=s-n$; which values substituted in (4), give

$$\left(\frac{s'(s+\delta n+1)\pm r'm^3}{am}\right)^3+\left(\frac{s'(s-n)\mp r'm^3}{am}\right)^3$$
$$=s'(3r'^2m^3\pm 3s'r'(2n+1)+s'^2m') \quad . \quad . \ (21).$$

13. The same process holds in using the column of equations for a, an even number. In this case we have

$$\left(\frac{s+n}{am}\right)^3+\left(\frac{s-n}{am}\right)^3=\frac{s^2+3n^2}{m^3}\text{*}=m' \quad . \quad . \quad . \quad . \ (22),$$

$x=s+n$, $y=s-n$, and from (4)

$$\left(\frac{s'(s+n)\pm r'm^3}{am}\right)^3+\left(\frac{s'(s-n)\mp r'm^3}{am}\right)^3=s'(3r'^2m^3\pm 6r's'n+s'^2m') . \ (23),$$

in all of which s', r' may be positive or negative and assumed at will.

14. *Remark*. As s, m and m' are known, n, in the case of a, an odd number, may be found by an adfected, and in the case of a, an even number, by a pure quadratic, as will plainly appear to the reader, and thence the roots to correspond with m' in (20) and (22.) They may however be more easily ascertained by finding the number of terms from the first term that divides by m, to that which divides by m^3, in the column of equations, as directed in Art. (11) form (19), which added to the numerator of the one and deducted from the numerator of the other of the roots corresponding to the term that is divisible by m, will be the numerators of the roots to correspond with the number m'; and will furthermore be such as to make $x+y=a^3$ and $x^2-xy+y^2=m'm^3$.

Application.

1st. Suppose a, in the column of equations D (see Misc., page 116) to be an odd number, say 11, then $s=665$, and we shall have

$$\begin{array}{lll}
(\tfrac{666}{11})^3+(\tfrac{665}{11})^3=442891 & \ldots\ldots\ a'_0 & \\
\qquad\qquad\qquad\qquad 6 & & a \\
(\tfrac{667}{11})^3+(\tfrac{664}{11})^3=442897 & \ldots\ldots\ a'_{n'} & \\
\qquad\qquad\qquad\qquad 12 & & a+n'b \\
\quad\&c. \qquad\qquad \&c. & &
\end{array}$$

Now, assume $m=7$ and $n=3$; then $a'_{n'}\div m=a''_0=63271$, $a=6$, $b=6$, $n'=1$ and (7) becomes 30. $a''_0\div m$ leaves a remainder $5=\text{R}$ and $(7)\div m$ leaves a remainder $2=\text{R}'$, consequently $n''=1$ and (15) or $a''_{n''}=63301$. Hence $a''_{n''}\div m=a'''_0=9043$. Again: (9) becomes 198, $a'''_0\div m$ leaves a remainder $6=\text{R}$, $(9)\div m$ leaves $2=\text{R}'$, then $n'''=4$ and (16)

* See Note, page 116.

or $a'''_{n'''} = 11599$. Consequently $a'''_{n'''} + m = q = 1657 = m'$. Now, from the quadratic in (20), viz:

$$(n+1)(2n+1) + n^2 = m'm^3 - s^2 - s = 125461$$

we find $n = 204$. Consequently (18) or (5) becomes

$$(\tfrac{870}{77})^3 + (\tfrac{461}{77})^3 = 1657 = m' \quad . \quad . \quad . \quad . \quad . \quad (24).$$

Or, in the briefer and more simple way, by (19) we have

$$n''m + n'''m^2 = 1 \times 7 + 4 \times 49 = 203$$

which added to 667 and deducted from 664 the numerators of the roots opposite to $a'_{n'}$ above, as directed in the remark, Art. (14), gives the same roots as in (24); that is $x = 870$ and $y = 461$. Now, assume $r' = 1$ and $s' = 1 . 2 . 3$, &c., and substitute the values of x, y, m', m and a, in equation (4), and we shall have, by taking the upper signs,

$$\begin{array}{lll}
(\tfrac{1213}{77})^3 + (\tfrac{118}{77})^3 = 1 \times 3913 & & \\
 & 6198 & \\
(\tfrac{2083}{77})^3 + (\tfrac{579}{77})^3 = 2 \times 10111 & & 3314. \\
 & 9512 & \\
(\tfrac{2953}{77})^3 + (\tfrac{1040}{77})^3 = 3 \times 19623 & & '' \\
 & 12826 & \\
(\tfrac{3923}{77})^3 + (\tfrac{1501}{77})^3 = 4 \times 32449 & & '' \\
\quad \&c. \qquad \&c. & &
\end{array}$$

and using the lower signs

$$\begin{array}{lll}
(\tfrac{527}{77})^3 + (\tfrac{804}{77})^3 = 1 \times 1459 & & \\
 & 3744 & \\
(\tfrac{1397}{77})^3 + (\tfrac{1265}{77})^3 = 2 \times 5203 & & 3314 \\
 & 7058 & \\
(\tfrac{2267}{77})^3 + (\tfrac{1726}{77})^3 = 3 \times 12261 & & '' \\
 & 10372 & \\
(\tfrac{3137}{77})^3 + (\tfrac{2187}{77})^3 = 4 \times 22633 & & '' \\
\quad \&c. \qquad \&c. & &
\end{array}$$

2d. Suppose a, an even number of the form 2^n, say 16, then s in the column of equations B (see Misc., page 115) will be 2048 and we shall have

$$\begin{array}{lll}
(\tfrac{2049}{16})^3 + (\tfrac{2047}{16})^3 = 4194307 & . \; . \; . \; . & a'_0 \\
 & 24 & \quad a \\
(\tfrac{2051}{16})^3 + (\tfrac{2045}{16})^3 = 4194331 & . \; . \; . \; . & a'_1 \\
 & 48 & \quad a + b \\
(\tfrac{2053}{16})^3 + (\tfrac{2043}{16})^3 = 4194379 & . \; . \; . \; . & a'_{n'} \\
 & 72 & \quad a + n'b \\
\quad \&c. \qquad \&c. & &
\end{array}$$

Assume $m = 7$ and $n = 3$; then $a'_{n'} + m = a''_0 = 599197$. $a = 24$, $b = 24$, $n' = 2$ and (7) becomes 144. Now, $a''_0 + m$ leaves a remainder $4 = \text{R}$ and $(7) + m$ leaves a remainder $4 = \text{R}'$, consequently $n'' = 6$, (15) or $a''_{n''} = 602581$, and $a''_{n''} + m = a'''_0 = 86083$. Again: (9) becomes 1656, $a'''_0 + m$ leaves a remainder $4 = \text{R}$ and $\text{R}' = 4$; then $n''' = 6$, (16) or $a'''_{n'''} = 113659$ and $a'''_{n'''} + m = q = 16237 = m'$. Then, by the quadratic in (22) we find $n = 677$ and thence $x = s + n = 2725$, $y = s - n = 1371$. Or, by (19), the briefer method

$$n''m + n'''m^2 = 6 \times 7 + 6 \times (7)^2 = 336$$

the double of which, however, as the roots in the column of equations above increase and decrease by 2, must be added to and subtracted from the numerators of the roots opposite to $a'_{x'}$; That is $2053 + 672 = 2725$ and $2043 - 672 = 1371$ the same as above. Consequently (18) or (5) becomes

$$\left(\tfrac{2725}{112}\right)^3 + \left(\tfrac{1371}{112}\right)^3 = 16237 = m' \quad . \quad . \quad . \quad . \quad . \quad (25),$$

And with these values of x and y, and others found in the same way, making $x+y=a^3$ and $x^2 - xy + y^2 = m'm^3$, by assuming for r' and s' in this paper, and for a and a' in art. 10, page 119 Misc., any numbers positive or negative, and substituting them in the various general formulas in art. 10, and in this paper, we shall find many numbers and the two cubes to whose sum or difference they are equal. If we take $s' = 1$ and $r' = 2$, $x = 2725$ and $y = 1371$, we shall get from (4), taking the lower signs,

$$\left(\tfrac{2032}{112}\right)^3 + \left(\tfrac{2047}{112}\right)^3 = 12229$$

which shows that had we taken two terms more in the above column, we should at once have had $n'_4 + m^3 = 12229$.

WM. LENHART.

York, Penn., January, 1838.

ARTICLE XXIII.

ON THE THEORY OF EXPONENTIAL AND IMAGINARY QUANTITIES.

BY W. S. B. WOOLHOUSE, F.R.A.S.

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(From the Gentleman's Diary, for 1837.)

The various expressions that can be deduced from the well known formulæ,

$$(\cos x + \sin x\sqrt{-1})(\cos y + \sin y\sqrt{-1})$$
$$= \cos(x+y) + \sin(x+y)\sqrt{-1} \quad . \quad . \quad . \quad (1)$$
$$(\cos x + \sin x\sqrt{-1})^n = \cos nx + \sin nx\sqrt{-1} \quad . \quad . \quad . \quad (2)$$
$$\cos x + \sin x\sqrt{-1} = \varepsilon^{x\sqrt{-1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

are very remarkable when viewed in all their generality. The second of these equations is directly inferred from the first, and can only be received as general so far that n may be any whole number, positive or negative; for if n be a fraction, the left hand member will comprise twice* as many different values as the denominator of that fraction when expressed in its lowest terms; and if n cannot be expressed by either a whole number or fraction, it will comprise an indefinite number of values, while the right hand member contains only two values. A similar remark applies to the third equation, in reference to the exponent

* Considering both values of the factor $\sqrt{-1}$.

$x\sqrt{-1}$. The equation (2) will, however, be rendered perfectly general, on the substitution of $x+2r\pi$ instead of x; it then assumes the form

$$(\cos x+\sin x\sqrt{-1})^n=\cos(nx+2rn\pi)+\sin(nx+2rn\pi)\sqrt{-1},$$

in which r designates any whole number whatever, positive or negative; and here both sides possess the same number of corresponding values, those of the right hand member depending on the successive values of the number r. Each value of r appertains to a particular value of the form on the left. It will be convenient to indicate that particular value of the form by placing the symbol r at the foot of the quantity involved, thus:

$$(\cos x+\sin x\sqrt{-1})^n_r=\cos(nx+2rn\pi)+\sin(nx+2rn\pi)\sqrt{-1} \quad . \ . \quad (\text{A}).$$

By writing down the equations in this manner, they will be found to be much more distinct and intelligible, since, instead of the sometimes perplexing idea of multiple forms, they present to the mind a clear association of their corresponding and identical values; and this plan will likewise possess the advantage of affording greater precision in all our reasonings on exponential and imaginary equivalents. For instance, if $\varepsilon^x=p$ and $\varepsilon^x=q$, it does not necessarily follow that $p=q$, since these quantities may express different values of their common generating form ε^x. Let the specific equations be $\varepsilon^x_r=p$ and $\varepsilon^x_{r'}=q$; then, in order that we may satisfactorily make the deduction $p=q$, it will be requisite that we should also be in possession of the condition $r=r'$. It is the want of attention to this distinction, and others of a similar nature, so absolutely necessary in all our reasonings on multiple forms, that has involved mathematicians, from time to time, in so many controversies connected with this peculiar department of analysis.

In the general equation (A) let the arc x be made equal to *zero*, and we get

$$1^n_r=\cos(2rn\pi)+\sin(2rn\pi)\sqrt{-1}$$

or, changing n into x,

$$1^x_r=\cos(2rx\pi)+\sin(2rx\pi)\sqrt{-1} \quad . \ . \ . \ . \quad (\text{B}),$$

which represents all the values of 1^x if r be supposed to assume successively the values 0, 1, 2, 3, &c.

We proceed to determine equally general expressions for ε^x, $\varepsilon^{x\sqrt{-1}}$, &c. Let ε^x_m denote any particular value of ε^x; then if 1^x be supposed to pass through all its successive values, it may be shown that the expression $\varepsilon^x_m . 1^x$ will pass through all the values of the form ε^x, though not, perhaps, in any direct order of progression. For if the fraction which expresses the value of x be $\frac{h}{k}$, when reduced to its lowest terms, it is evident that the expression $\varepsilon_m^{\frac{h}{k}} . 1^{\frac{h}{k}}$ will pass through k different values, or just *as many* values as are contained in $\varepsilon^{\frac{h}{k}}$; and since, for every value

$(\varepsilon_m^{\frac{h}{k}} \cdot 1^{\frac{h}{k}})^k = (\varepsilon^{\frac{h}{k}})^k (1^{\frac{h}{k}})^k = \varepsilon^h \,.\, 1^h = \varepsilon^h$, it follows that each of them must represent a value of $(\varepsilon^h)^{\frac{1}{k}}$ or $\varepsilon^{\frac{h}{k}}$, and that all of them must constitute the k values comprised in this form.

The only real value of the expression (B) is $1_o^x = 1$. Let the ordinary value of ε^x be similarly denoted by ε_o^x, and the general value will be $\varepsilon_r^x = \varepsilon_o^x \,.\, 1_r^x$, or by (B),

$$\varepsilon_r^x = \varepsilon_o^x \{\cos(2rx\pi) + \sin(2rx\pi)\sqrt{-1}\} \quad . \quad . \quad . \quad . \quad (\text{C}).$$

But the ordinary value of ε^x is well known to be,

$$\varepsilon_o^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2\,.\,3} + \frac{x^4}{2\,.\,3\,.\,4} + \&\text{c}.$$

which by the series that represent $\cos x$ and $\sin x$ is readily transformed into

$$\begin{aligned} \varepsilon_o^x &= \cos(x\sqrt{-1}) - \sin(x\sqrt{-1})\sqrt{-1} \\ &= \cos(-x\sqrt{-1}) + \sin(-x\sqrt{-1})\sqrt{-1} \quad . \quad . \quad . \quad (\text{D.}) \end{aligned}$$

Hence the equation (C) becomes

$$\begin{aligned} \varepsilon_r^x = \{&\cos(-x\sqrt{-1}) + \sin(-x\sqrt{-1})\sqrt{-1}\} \\ \{&\cos(2rx\pi) + \sin(2rx\pi)\sqrt{-1}\} \end{aligned}$$

That is, according to the equation (1),

$$\varepsilon_r^x = \cos(-x\sqrt{-1} + 2rx\pi) + \sin(-x\sqrt{-1} + 2rx\pi)\sqrt{-1} \quad . \quad . \quad (\text{E}).$$

In (D) put $x\sqrt{-1}$ for x, and it gives

$$\varepsilon_o^{x\sqrt{-1}} = \cos x + \sin x\sqrt{-1} \quad . \quad . \quad . \quad . \quad . \quad (\text{F}),$$

which specifically expresses what is meant by the particular equation (3).

By comparing this with (E) we obtain the remarkable formula,

$$\begin{aligned} \varepsilon_r^x &= \varepsilon_o^{(-x\sqrt{-1} + 2rx\pi)\sqrt{-1}} \\ \text{or } \varepsilon_r^x &= \varepsilon_o^{x + 2rx\pi\sqrt{-1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{G}). \end{aligned}$$

The formula (D) gives

$$\begin{aligned} \varepsilon_o^x &= \cos(-x\sqrt{-1}) + \sin(-x\sqrt{-1})\sqrt{-1} \\ \varepsilon_o^y &= \cos(-y\sqrt{-1}) + \sin(-y\sqrt{-1})\sqrt{-1} \end{aligned}$$

which, multiplied by means of (1), we get

$$\varepsilon_o^x \,.\, \varepsilon_o^y = \cos\{-(x+y)\sqrt{-1}\} + \sin\{-(x+y)\sqrt{-1}\}\sqrt{-1}.$$

That is, by (D),

$$\varepsilon_o^x \,.\, \varepsilon_o^y = \varepsilon_o^{x+y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{H}).$$

Let $x = (2r')\pi$ and $(2r'+1)\pi$ in (F), and

$$\varepsilon_o^{(2r')\pi\sqrt{-1}} = +1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad , \quad (\text{I}),$$

$$\varepsilon_o^{(2r'+1)\pi\sqrt{-1}} = -1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{K}).$$

To find the general values of these forms substitute $(2r')\,\pi\,\sqrt{-1}$ and $(2r'+1)\,\pi\,\sqrt{-1}$ for x in (G), and we derive

$$\varepsilon_r^{(2r')\pi\sqrt{-1}} = \varepsilon_0^{(2r')\pi\sqrt{-1}-(4rr')\pi^2}$$

$$\varepsilon_r^{(2r'+1)\pi\sqrt{-1}} = \varepsilon_0^{(2r'+1)\pi\sqrt{-1}-2r(2r'+1)\pi^2}$$

or, by eliminating the factors (I) and (K), in virtue of the property (H),

$$\varepsilon_r^{(2r')\pi\sqrt{-1}} = +\,\varepsilon_0^{-(3rr')\pi^2} \quad \ldots\ldots\ldots\ldots \quad (\text{L}),$$

$$\varepsilon_r^{(2r'+1)\pi\sqrt{-1}} = -\,\varepsilon_0^{-2r(2r'+1)\pi^2} \quad \ldots\ldots\ldots \quad (\text{M}).$$

Applying (D) to these expressions, they become

$$\varepsilon_r^{(2r')\pi\sqrt{-1}} = \cos\{(4rr')\pi^2\} + \sin\{4rr')\pi^2\}\sqrt{-1} \quad \ldots\ldots \quad (\text{N}),$$

$$\varepsilon_r^{(2r'+1)\pi\sqrt{-1}} = \cos\{2r(2r'+1)\pi^2\} + \sin\{2r(2r'+1)\pi^2\}\sqrt{-1}\,.\,(\text{P}),$$

which may be directly deduced from (E). By (E) we have

$$\varepsilon_r^x = \cos(-x\sqrt{-1}+2rx\pi) + \sin(-x\sqrt{-1}+2rx\pi)\sqrt{-1}$$

$$\varepsilon_{r'}^y = \cos(-y\sqrt{-1}+2r'y\pi) + \sin(-y\sqrt{-1}+2r'y\pi)\sqrt{-1},$$

which combined by means of (1), give

$$\varepsilon_r^x\,\varepsilon_{r'}^y = \cos\{-(x+y)\sqrt{-1}+2(rx+r'y)\pi\}$$
$$+\sin\{-(x+y)\sqrt{-1}+2(rx+r'y)\pi\}\sqrt{-1} \quad \ldots \quad (\text{Q}).$$

The equation (F) applied to this formula transforms it into

$$\varepsilon_r^x\,\varepsilon_{r'}^y = \varepsilon_0^{x+y+2(rx+r'y)\pi\sqrt{-1}} \quad \ldots \quad (\text{R}).$$

On comparing (Q) and (R) with (E) and (G), it appears that $\varepsilon_r^x\,\varepsilon_{r'}^y$ is not contained amongst the values of ε^{x+y} unless $r=r'$, in which case we have

$$\varepsilon_r^x\;\varepsilon_r^y = \varepsilon_0^{x+y+2r(x+y)\pi\sqrt{-1}}$$

$$\text{or}\quad \varepsilon_r^x\;\varepsilon_r^y = \varepsilon_r^{x+y} \quad \ldots\ldots\ldots\ldots \quad (\text{S}).$$

The ordinary solution of the equation $\varepsilon^x = a$, is $x = \log a$. It is evident, however, that each value of the form ε^x will offer a distinct solution. Let $x = \alpha$ be a solution of the particular equation $\varepsilon_0^x = a$, and which, therefore, fulfils the relation $\varepsilon_0^\alpha = a$. Then, since by (H), and (I) this value is not altered when put in the form $\varepsilon_0^{\alpha+2r'\pi\sqrt{-1}}$, it is plain that a more general solution to $\varepsilon_0^x = a$ is $x = \alpha + 2r'\pi\sqrt{-1}$. To particularize this solution denote it by $\log_0^{r'}(a)$; thus we shall have

$$\log_0^{r'}(a) = \alpha + 2r'\pi\sqrt{-1} \quad \ldots\ldots \quad (a).$$

Also, since by (I) the equation $\varepsilon_0^x = -a$, or $-\varepsilon_0^x = a$, is equivalent to $\varepsilon_0^{x+(2r'+1)\pi\sqrt{-1}}$ we similarly get, by pursuing the same notation,

$$\log_0^{r'}(-a) = \alpha + (2r'+1)\pi\sqrt{-1} \quad \ldots \quad (b),$$

in which α, as before, denotes a solution of $\varepsilon_0^x = a$.

It thus appears that even the particular equations $\varepsilon_o^x = a$, $\varepsilon_o^x = -a$, possess an indefinite number of imaginary solutions.

Again, according to the equation (σ) the general equation $\varepsilon_r^x = a$ is equivalent to $\varepsilon_o^{x+2rx\pi\sqrt{-1}} = a$. The general solution is therefore $x + 2rx\pi\sqrt{-1} = \log_o^{r'}(a)$, or

$$\log_r^{r'}(a) = \frac{\log_o^{r'}(a)}{1+2r\pi\sqrt{-1}} \quad . \quad . \quad . \quad . \quad (c),$$

In the same manner we have

$$\log_r^{r'}(-a) = \frac{\log_o^{r'}(-a)}{1+2r\pi\sqrt{-1}} \quad . \quad . \quad . \quad . \quad (d),$$

which is exactly the same in principle.

By (a) and (b) the two last equations become

$$\log_r^{r'}(a) = \frac{\alpha + 2r'\pi\sqrt{-1}}{1+2r\pi\sqrt{-1}} \quad . \quad . \quad . \quad . \quad . \quad (e),$$

$$\log_r^{r'}(-a) = \frac{\alpha + (2r'+1)\pi\sqrt{-1}}{1+2r\pi\sqrt{-1}} \quad . \quad . \quad . \quad (f).$$

We have denoted the value of $\log_o^o(a)$ by α; if we similarly denote the value of $\log_o^o(b)$ by β, and suppose, according to the ordinary property of logarithms, that

$$\log_o^o(a) + \log_o^o(b) = \log_o^o(ab),$$

the formula (a) will give

$$\log_o^{r'}(a) + \log_o^{r''}(b) = \log_o^{r'+r''}(ab) \quad . \quad . \quad . \quad . \quad (g),$$

and (c) will hence give the general relation

$$\log_r^{r'}(a) - \log_r^{r''}(b) = \log_r^{r'+r''}(ab) \quad . \quad . \quad . \quad . \quad (h).$$

If $r' = o$, $r'' = o$, it becomes

$$\log_r^o(a) + \log_r^o(b) = \log_r^o(ab) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (i),$$

and this is the only way in which the logarithms can each assume the same identical form so as to sustain the principle on which they are usually founded.

Consider now the equation $\varepsilon_o^x = a^m$. A particular solution is $x = m\alpha$, α being any one of the solutions of $\varepsilon_o^x = a$. By putting the more general value of $\alpha + 2r'\pi\sqrt{-1}$ for α, the solution becomes $x = m\alpha + 2r'm\pi\sqrt{-1}$, and as the equation is not altered in its values when written $\varepsilon^{x+2r''\pi\sqrt{-1}}$ we have, for the most general form of solution,

$$\log_o^{r'}(a^m) = m\alpha + 2r'm\pi\sqrt{-1} + 2r''\pi\sqrt{-1} \quad . \quad . \quad . \quad (k).$$

By the formula (c) the most general solution of $\varepsilon_r^x = a^m$ is therefore

$$\log_r^{r'}(a^m) = \frac{m\alpha + 2r'm\pi\sqrt{-1} + 2r''\pi\sqrt{-1}}{1+2r\pi\sqrt{-1}} \quad . \quad . \quad . \quad (l).$$

We have similarly,

$$\log_e^{r'}(-a^m) = m\alpha + 2r'm\pi\sqrt{-1} + (2r''+1)\pi\sqrt{-1} \quad . \quad . \quad . \quad . \quad (m),$$

$$\log_r^{r'}(-a^m) = \frac{m\alpha + 2r'm\pi\sqrt{-1} + (2r''+1)\pi\sqrt{-1}}{1 + 2r\pi\sqrt{-1}} \quad . \quad . \quad . \quad . \quad (n).$$

If the symbols α, m, represent positive arithmetical values, the quantity α will denote the common hyperbolic logarithm of a, and the preceding expressions will be equally general as solutions to the equations proposed. It ought to be observed, however, that the expressions (k), (l), (m), (n), though strictly true, as far as ultimate values are concerned, refer to the most general derivation of the quantity a^m, which would be just the same in value if any of the values of $(a^m)^{\frac{1}{m}}$ were employed as the root; these expressions may be considered as referring to each value of a^m simply as a *result*, without any regard to the form from which it has been derived. If a^m is to be strictly regarded as derived from $+a$ as a root, the equation (l) will become

$$\log_r^{r'}(+a)^m = \frac{m\alpha + 2r'm\pi\sqrt{-1}}{1 + 2r\pi\sqrt{-1}} \quad . \quad . \quad . \quad (p).$$

To explain these limitations more distinctly, suppose the index m to be expressed by the fraction $\frac{h}{k}$. The formula (l) expresses the values of the logarithm when $a^{\frac{h}{k}}$ is derived from any one of the h values $a = (a^h)^{\frac{1}{h}}$ as a root, and when any one of the k values of $a^{\frac{h}{k}}$ is taken into consideration; it therefore comprises as many different modes of derivation as are expressed by the product hk. But the formula (p) relates only to the k values deduced from the individual root a, and it will be found to comprise the whole of these k values.

It appears to me that Professor Young, (Treatise on Logarithms, pages 107-8,) from not attending to this last consideration, has committed an oversight in his criticisms on what Mr. Peacock has said concerning the logarithms of negative quantities in his "Report on certain branches of Analysis." Mr. Peacock makes it out that the logarithm of a negative number will be identical with the logarithm of the same number with a positive sign. That this is not true is very obvious from a comparison of the most general values (e), (f), that we have obtained, which cannot in any case be reduced to the same identical form. But Professor Young, instead of pointing out the real source of the error, has objected to an equation that seems to be in every respect justifiable. This equation is

$$\log(+a)^2 = 4r\pi\sqrt{-1} + 2\alpha \quad . \quad . \quad . \quad . \quad (\alpha);$$

instead of which Professor Young substitutes

$$\log(+a)^2 = 2r\pi\sqrt{-1} + 2\alpha \quad . \quad . \quad . \quad . \quad (\beta).$$

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He deduces the value of a^2 by multiplying together the two forms $a = a\varepsilon^{2r\pi\sqrt{-1}}$, $a = a\varepsilon^{2r'\pi\sqrt{-1}}$, and so gets $a^2 = a^2\varepsilon^{2(r+r')\pi\sqrt{-1}}$ which is of the form $a^2 = a^2\varepsilon^{2r''\pi\sqrt{-1}}$; but the combination of *different* forms of the value a, though true with respect to resulting values, is not admissible as an operation of *involution*, when regard is had to the particular derivation. In such a case the operation of involution requires that the values combined shall be identical in form or derivation as well as in result. It is true that the form $a^2\varepsilon^{2r''\pi\sqrt{-1}}$ strictly represents the general value of a^2, but it is without any reference to the form of the root, since it equally represents $(-a)^2$ and $(+a)^2$. It indicates $(+a)^2$ when r'' is an even number, and $(-a)^2$ when it is an odd number; and hence when the value a^2 is predicated to be derived from $+a$ as the root, the only admissible form is $(+a)^2 = a^2\varepsilon^{4r\pi\sqrt{-1}}$, and this is quite consistent with the particular equation (α).

Mr. Peacock puts $m = \frac{1}{2}$, and $r'' = o$ in the formula (m), which gives

$$\log_o^{r'}(-\sqrt{a}) = \tfrac{1}{2}\alpha + (r'+1)\pi\sqrt{-1} \quad . \quad . \quad (\gamma).$$

The expression thus deduced from the general formula must necessarily comprise the results of the operation, indicated by $\log_o^{r'}$, on *both* values of the form $-\sqrt{a}$. Thus, if we make $m = \frac{1}{2}$, $r'' = o$ in (k), we get

$$\log_o^{r'}(\sqrt{a}) = \tfrac{1}{2}\alpha + r'\pi\sqrt{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\delta),$$

which is of the same form as (γ), and contains precisely the same values. The equations (a) and (b) show that in each case the positive value is predicated to be under $\log_o^{r'}$ when the factor of $\pi\sqrt{-1}$ is an even number, and *vice versa*. It is hence clear that when Mr. Peacock makes also $r' = -1$, or $r' + 1 = 0$ in the equation (γ), that the quantity $-\sqrt{a}$ takes the positive arithmetical value, that $\sqrt{a}$ takes the negative value, and that, supposing $\sqrt{a}$ to designate the positive value, the equation, instead of being $\log(-\sqrt{a}) = \frac{1}{2}\alpha$, as Mr. Peacock has it, is no more than the ordinary form $\log\sqrt{a} = \frac{1}{2}\alpha$.

The formulæ contained in this paper are not offered as being principally new; but it was thought that a more simple deduction than had been given of some of them, and the addition of some others, might be acceptable to the readers of the Diary. The extension of the expression of a logarithm to the introduction of *two* arbitrary integers, was first made public by Mr. Graves, in the Philosophical Transactions for the year 1829. The numbers r, r', which appear respectively in the denominator and numerator of the formula (e), he designates the *order* and *rank* of the logarithm. The equation (h) shows that in the usual operations of the addition and subtraction of logarithms, the ranks will undergo the same operations while the order will remain the same.

By the equation (e) the general form of the hyperbolic logarithm of *unity* is $\dfrac{2r'\pi\sqrt{-1}}{1+2r\pi\sqrt{-1}}$. This expression has been the subject of some recent discussion which has contributed, in some degree, towards a more general elucidation of the extended theory.

METEOROLOGICAL OBSERVATIONS,

MADE AT THE INSTITUTE, FLUSHING, L. I., FOR THIRTY-SEVEN SUCCESSIVE HOURS, COMMENCING AT SIX A. M., OF THE TWENTY-FIRST OF DECEMBER, EIGHTEEN HUNDRED AND THIRTY-SEVEN, AND ENDING AT SIX P. M., OF THE FOLLOWING DAY.

(*Lat.* 40° 44′ 58″ N., *Long.* 73° 44′ 20″ W. *Height of Barometer above low water mark of Flushing Bay,* 54 *feet.*)

Hour.	Barometer Corrected.	Attached Therm'ter.	External Therm'ter.	Wet Bulb Therm'ter.	Winds —from—	Clouds —to—	Strength of wind.	REMARKS.
6	29.977	43	15		NW		Gentle.	Clear.
7	.977	40	15½		"		"	"
8	.965	42	17		"		"	"
9	.957	44	19	18	"		"	"
10	.953	44	20	19	"		"	"
11	.949	43	22	21	"		"	"
12	.917	44	26	24	"		"	"
1	.896	45	25¼	23	"		"	"
2	.895	46	25¼	23	"		"	"
3	.894	46	26½	24	"	E	"	Cirrus Clouds.
4	.896	48	24¾	22½	NNW	"	"	"
5	.898	48	23¾	21½	"	"	"	"
6	.916	45	23¾	21½	"	"	"	Cumulus Clouds.
7	.923	44	23½	21½	"	"	"	"
8	.928	48	23	21½	"	"	"	Stratus Clouds.
9	.932	44	22½	21	"	"	"	"
10	.936	44	22¾	21¼	"	"	"	"
11	.938	44	23½	21¾	N	"	"	"
12	.938	46	23	21	"	"	"	Clearing.
1	.937	43	19	18	"		"	Clear.
2	.971	41	17	16	NE		"	"
3	.973	41	17	16	"		"	"
4	.969	41	17½	16½	"	E	"	Cirrus Clouds.
5	.968	41	18	17	"	"	"	"
6	.987	40	16½	16	"		"	Clear, except on the horizon.
7	32.023	42	15½	15	"		"	"
8	.048	40	15¾	15¼	"		"	Cirro-Cumulus Clouds.
9	.072	40	17	16½	"	E	"	"
10	.089	40	19	18	"	"	"	"
11	.090	40	20½	19½	"	"	"	"
12	.087	41	26¼	25¼	"	"	"	"
1	.087	41	26¾	25¾	"	"	"	"
2	.085	41	26½	25½	"	"	"	"
3	.091	41	25½	25	"	"	"	"
4	.110	40	21¼	20¼	"	"	"	Clear.
5	.122	40	20	19¼	"		"	"
6	.159	41	17	16½	"		"	"
	29.988	42¾	21 $\frac{1}{12}$	20½		Means.		

METEOROLOGICAL OBSERVATIONS,

MADE AT THE INSTITUTE, FLUSHING, L. I., FOR THIRTY-SEVEN SUCCESSIVE HOURS, COMMENCING AT SIX A. M., OF THE TWENTY-FIRST OF MARCH, EIGHTEEN HUNDRED AND THIRTY-EIGHT, AND ENDING AT SIX P. M., OF THE FOLLOWING DAY.

(*Lat.* 40° 44′ 58″ N., *Long.* 73° 44′ 20″ W. *Height of Barometer above low water mark of Flushing Bay*, 54 *feet*.)

Hour.	Barometer Corrected.	Attached Therm'ter.	External Therm'ter.	Wet Bulb Therm'ter.	Winds —from—	Clouds —to—	Strength of wind.	REMARKS.
6	29.806	55	38	36	NE	S	Gentle.	Thin stratus Clouds.
7	.850	50	40½	38½	"	"	"	"
8	.859	50	42¼	39½	"	"	"	"
9	.919	51	46	42	"	"	Fresh.	"
10	.930	50	46	42½	"	"	"	"
11	.949	51	47	44	"	"	"	"
12	.971	50	48¼	44	"	"	"	"
1	.994	53	47¾	43	"	"	"	"
2	30.029	57	47¾	42¼	"	"	Brisk.	" thinn er.
3	.036	59	47¾	42½	"	"	"	" "
4	.071	57	46¾	41¼	"	SE	Fresh.	Cirrus Clouds.
5	.080	56	46	41½	"	"	"	"
6	.103	54	42½	39½	"	"	Gentle.	"
7	.136	55	40	37½	"		"	Clear.
8	.171	55	38	36	"		"	"
9	.186	55	37½	35½	"		"	"
10	.206	54	36¾	35	"		"	"
11	.236	55	35½	34½	"		"	"
12	.234	58	35	34	"		"	"
1	.242	58	33¼	32½	"		Light.	"
2	.239	57	34	33	"	SE	"	Cirrus Clouds.
3	.236	55	34¼	33	"	"	"	"
4	.239	54	33	32	"	"	"	" spreading.
5	.243	54	33	32	"	S	"	Thin stratus Clouds.
6	.251	53	33	32	"	"	"	"
7	.273	54	34	33¼	"	"	"	" sun shining
8	.301	56	37	36½	"	"	Gentle.	" [through them.
9	.308	54	39	37¾	"	"	"	" "
10	.320	44	42¾	41	E	SW	"	" "
11	.319	43	42	40	SE	W	"	Stratus Clouds.
12	.304	43	41	39	"	"	"	"
1	.273	46	40	38¼	E	"	"	" fine misty rain.
2	.260	47	38¾	38	ENE	"	"	" "
3	.252	48	39	38½	"	"	"	" "
4	.243	48	36¾	36½	"	"	"	" "
5	.243	48	35¾	35½	"	"	"	" "
6	.239	48	35½	35	"	"	"	" "
	30.150	52½	39¾	37½	Means.			

THE

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JUNIOR DEPARTMENT.

ARTICLE XII.

HINTS TO YOUNG STUDENTS. (*Continued from page* 276.)

25. THE definitions and principles in the last seven paragraphs, are the result of a more intimate knowledge of Mathematical Analysis in all its relations than is perhaps possessed by any other living Mathematician; they have probably cost their illustrious author more labor in their composition than many of his exquisite specimens of the higher analysis that have enriched the periodicals of Europe for the last twenty years. Do not throw them aside, then, because they may appear at first sight abstract or difficult to comprehend; but return to them again and again, and make your knowledge of them, as you may with safety, a measure of your progress in Algebra.

26. The next obstacle to the advance of the student, is the difficulty he finds in comprehending or even crediting the full value of his results. The symbols he uses may, in general, represent any arithmetical number, or any quantity whatever, as this term is defined in the preceding paragraphs; their meaning may even be much more comprehensive, but to this extent at least he must interpret them until he is farther advanced. Thus every operation which he performs on these symbols comprehends a general property of numbers or of quantities, which may be interpreted and used according to the established principles of his notation; then, this property or theorem will give birth to innumerable others, and it is skill in deducing, modifying, and interpreting these results, that should be the first and principal object of the student.

To give an example of these deductions: the first of equations (2), page 210, contains the theorem:

The product of any number or quantity by the sum of any number of numbers or quantities, is equal to the sum of the products obtained by multiplying the first severally by each of the others.

From this results immediately the rule of arithmetical multiplication: for the product of any number, k, by the number 3584, for instance, is

$$k(3000+500+80+4)=3000k+500k+80k+4k,$$

which is precisely the arrangement used in common arithmetic. Again, since k can be any quantity, if we write for it the quantity $c+d$, we get

$$(6). \quad (c+d)(a+b)=a(c+d)+b(c+d)$$
$$=ac+ad+bc+bd,$$

which may be similarly enunciated in ordinary language. If in (6) we write a and b for c and d, we find

$$(a+b)(a+b)=aa+ab+ba+bb$$

or (7), $$(a+b)^2=a^2+2ab+b^2;$$

that is, *the square of the sum of any two numbers or quantities is equal to the sum of their squares plus twice their product.* If in this formula we write $-b$, for b, since $a.-b=-ab$, and $(-b)^2=-b.-b=b^2$, it becomes

(8), $$(a-b)^2=a^2-2ab+b^2;$$

that is, *the square of the difference of any two numbers or quantities is equal to the sum of their squares minus twice their product.* And from these might be deduced the square of any polynomial.

Again, if in (6,) we write a and $-b$ for c and d, we find

(9), $$(a-b)(a+b)=aa-ab+ab-bb$$
$$=a^2-b^2;$$

that is, *the product of the sum and difference of any two numbers or quantities is equal to the difference of their squares, &c. &c.*

27. If the members of these equalities be interchanged, they give rise to converse properties, equally general and important, thus the first of (2) becomes

$$ka+kb+kc+\ldots=k(a+b+c+\ldots);$$

that is, *the sum of any numbers or quantities which have a common factor is equal to the product of the common factor by the sum of the unequal factors:* thus,

$$99.28+28=28(99+1)=28.100=2800,$$
$$x(c+d)+x(c-d)=x(c+d+c-d)=2cx,$$
$$x(c+d)-x(c-d)=x(c+d-c+d)=2dx,$$
$$x(a^2+ab+b^2)-x(a^2-ab+b^2)=2abx,$$
$$\&c. \qquad \&c.$$

In this manner equations (7) and (8) become

$$a^2\pm 2ab+b^2=(a\pm b)^2,$$

which shows that, if any trinominal be arranged according to the powers of any letter in it, and if, in that form, the second term is equal to twice the product of the square roots of the first and third terms, then *that trinominal is the square of a binominal formed by connecting the roots of the first and third terms by the sign of the second term, of the trinomial:* thus,

$$a^2+4ab+4b^2=(a+2b)^2,$$
$$a+2\sqrt{ab}+b=(\sqrt{a}+\sqrt{b})^2,$$
$$(x-y)z^4-2z^2\sqrt{x^2-y^2}+(x+y)=(z^2\sqrt{x-y})-\sqrt{z+y})^2,$$
$$\&c. \qquad \&c.$$

Similarly, equation (9) gives

$$a^2-b^2=(a+b)(a-b);$$

that is, *a quantity which is the difference of two squares, has for its factors the sum and the difference of the roots of these squares:* thus

$(m+n)^2-(m-n)^2=\{(m+n)+(m-n)\}\{(m+n)-(m-n)\}=2m.2n=4mn,$
$m^2-k^2+2kn-n^2=m^2-(k^2-2kn+n^2)=m^2-(k-n)^2$
$=\{m+(k-n)\}\{m-(k-n)\}=(m+k-n)(m-k+n),$

(10), $m^4+m^2n^2+n^4=m^4+2m^2n^2+n^4-m^2n^2=(m^2+n^2)^2-(mn)^2$
$=(m^2+n^2+mn)(m^2+n^2-mn),$

$1=2-1=(\sqrt{2})^2-(1)^2=(\sqrt{2}+1)(\sqrt{2}-1),$
&c. &c.

In this way the result of every operation in multiplication furnishes the general form of a product whose factors are the two numbers multiplied, and it is important that many of these results, such as (6), (7,) . . . . . (10), as well as

$$(11), \quad \begin{cases}(a+b)(a^2-ab+b^2)=a^3+b^3,\\(a-b)(a^2+ab+b^2)=a^3-b^3,\end{cases}$$

which, from their symmetry and brevity, can be extensively applied, should be imprinted on the mind; so that without any further mechanical means the factors of polynomials can often be detected.

28. To give an example of this application; let it be required to sum the fractions

$$\frac{a+7b}{a^3+b^3},\quad \frac{4b-10a}{a^3-b^3},\quad \frac{5a^2-8ab+b^2}{a^4+a^2b^2+b^4},\quad \frac{4}{a^2-b^2}.$$

By dividing their denominators into their simple factors, we have

$a^3+b^3=(a+b)(a^2-ab+b^2)$, by eq. (11),
$a^3-b^3=(a-b)(a^2+ab+b^2)$, "
$a^4+a^2b^2+b^4=(a^2-ab+b^2)(a^2+ab+b^2)$, by eq. (10,)
$a^2-b^2=(a+b)(a-b)$, by eq. (9).

Then the least common multiple of these denominators, and the least common denominator of the fractions is

$$(a+b)(a-b)(a^2-ab+b^2)(a^2+ab+b^2)=(a^2-b^2)(a^4+a^2b^2+b^4)$$
$$=a^6-b^6.$$

Now to find the new fractions, we shall want the quotients obtained by dividing this multiple by the several denominators. Then

$$\frac{(a+b)(a^2-b)(a-ab+b^2)(a^2+ab+b^2)}{a^3+b^3}=\frac{(a+b)(a-b)(a^2-ab+b^2)(a+ab+b^2)}{(a+b)(a^2-ab+b^2)}.$$

But since we have

$$\frac{klmn}{km}=ln,$$

and k, l, m, n may be any quantities whatever, they may represent the several factors in the above fraction, and

$$\frac{(a+b)(a-b)(a^2-ab+b^2)(a^2+ab+b^2)}{(a+b)(a^2-ab+b^2)}=(a-b)(a^2+ab+b^2)=a^3-b^3;$$

in other words, the polynominal factors are to be regarded in the operation as simple monomial factors. Or, in this case, since by eq. (9),

$$\frac{a^2-b^2}{a+b}=\frac{(a-b)(a+b)}{a+b}=a-b,$$

so,
$$\frac{a^6-b^6}{a^3+b^3}=\frac{(a^3-b^3)(a^3+b^3)}{a^3+b^3}=a^3-b^3.$$

In a similar manner the others are divided, and the multiples may be ex-

hibited in a form which, after a little practice, the student will find himself able to write down in most cases by inspection, thus:

$$
\begin{aligned}
a^6 - b^6 &= (a^3 + b^3) \times (a^3 - b^3) \\
&= (a^3 - b^3) \times (a^3 + b^3) \\
&= (a^4 + a^2b^2 + b^4) \times (a^2 - b^2) \\
&= (a^2 - b^2) \times (a^4 + a^2b^2 + b^4),
\end{aligned}
$$

the second factors being the multipliers to be used for the transformation. It follows that

$$
\begin{aligned}
\frac{a+7b}{a^3+b^3} &= \frac{a+7b}{a^3+b^3}\cdot\frac{a^3-b^3}{a^3-b^3} = \frac{a^4+7ba^3-b^3a-7b^4}{a^6-b^6}, \\
\frac{4b-10a}{a^3-b^3} &= \frac{4b-10a}{a^3-b^3}\cdot\frac{a^3+b^3}{a^3+b^3} = \frac{-10a^4+4ba^3-10b^3a+4b^4}{a^6-b^6}, \\
\frac{5a^2-8ab+b^2}{a^4+a^2b^2+b^4} &= \frac{5a^2-8ab+b^2}{a^4+a^2b^2+b^4}\cdot\frac{a^2-b^2}{a^2+b^2} = \frac{5a^4-8a^3b-4a^2b^2+8b^3a-b^4}{a^6-b^6}, \\
\frac{4}{a^2-b^2} &= \frac{4}{a^2-b^2}\cdot\frac{a^4+a^2b^2+b^4}{a^4+a^2b^2+b^4} = \frac{4a^4+4a^2b^2+4b^4}{a^6-b^6};
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\frac{a+7b}{a^3+b^3} + \frac{4b-10a}{a^3-b^3} + \frac{5a^2-8ab+b^2}{a^4+a^2b^2+b^4} + \frac{4}{a^2-b^2} &= \frac{3ba^3-3b^3a}{a^6-b^6} \\
&= \frac{3ab\,(a^2-b^2)}{(a^2-b^2)(a^4+a^2b^2+b^4)} \\
&= \frac{3ab}{a^4+a^2b^2+b^4}.
\end{aligned}
$$

For a second example, let there be given the equations

$$
ax^2 + by^2 = c,
$$
$$
xy = d;
$$

to find the values of x and y.

By multiplying the second equation by $2\sqrt{ab}$, and then adding it to, and subtracting it from the first equation, we obtain

$$
ax^2 + 2xy\sqrt{ab} + by^2 = c + 2d\sqrt{ab},
$$
$$
ax^2 - 2xy\sqrt{ab} + by^2 = c - 2d\sqrt{ab};
$$

but the trinomials forming the first members of these two equations are the squares of binomials, by the last paragraph, and extracting the roots of both members, we find

$$
x\sqrt{a} + y\sqrt{b} = \pm\sqrt{c + 2d\sqrt{ab}},
$$
$$
x\sqrt{a} - y\sqrt{b} = \pm\sqrt{c - 2d\sqrt{ab}};
$$

hence, by addition and subtraction,

$$
x = \pm\tfrac{1}{2}\sqrt{\frac{c}{a} + 2d\sqrt{\frac{b}{a}}} \pm \tfrac{1}{2}\sqrt{\frac{c}{a} - 2d\sqrt{\frac{b}{a}}},
$$
$$
y = \pm\tfrac{1}{2}\sqrt{\frac{c}{b} + 2d\sqrt{\frac{a}{b}}} \pm \tfrac{1}{2}\sqrt{\frac{c}{b} - 2d\sqrt{\frac{a}{b}}}.
$$

For a third example, let there be given the two equations

(A), $$x^3 + x^{\frac{3}{2}}y^{\frac{3}{2}} + y^3 = a,$$

(B), $$x^{\frac{3}{2}} - x^{\frac{3}{4}}y^{\frac{3}{4}} + y^{\frac{3}{2}} = b;$$

to find x and y.

If, in (10), we make $m = x^{\frac{3}{4}}$, $n = y^{\frac{3}{4}}$, it will become

$$x^3 + x^{\frac{3}{2}}y^{\frac{3}{2}} + y^3 = (x^{\frac{3}{2}} + x^{\frac{3}{4}}y^{\frac{3}{4}} + y^{\frac{3}{2}})\,(x^{\frac{3}{2}} - x^{\frac{3}{4}}y^{\frac{3}{4}} + y^{\frac{3}{2}});$$

and therefore, dividing the first equation by the second, member by member, we obtain

$$(\text{C}), \qquad x^{\frac{3}{2}} + x^{\frac{3}{4}}y^{\frac{3}{4}} + y^{\frac{3}{2}} = \frac{a}{b}.$$

Take (B) from (C), then

$$2x^{\frac{3}{4}}y^{\frac{3}{4}} = \frac{a}{b} - b = \frac{a-b^2}{b}, \text{ or}$$

$$(\text{D}), \qquad x^{\frac{3}{4}}y^{\frac{3}{4}} = \frac{a-b^2}{2b}.$$

Add (D) to (C), and then subtract it from (B), then

$$x^{\frac{3}{2}} + 2x^{\frac{3}{4}}y^{\frac{3}{4}} + y^{\frac{3}{2}} = \frac{3a-b^2}{2b},$$

$$x^{\frac{3}{2}} - 2x^{\frac{3}{4}}y^{\frac{3}{4}} + y^{\frac{3}{2}} = \frac{3b^2-a}{2b};$$

and extracting the roots,

$$x^{\frac{3}{4}} + y^{\frac{3}{4}} = \pm\sqrt{\frac{3a-b^2}{2b}},$$

$$x^{\frac{3}{4}} - y^{\frac{3}{4}} = \pm\sqrt{\frac{3b^2-a}{2b}}.$$

Then, by addition and subtraction,

$$x^{\frac{3}{4}} = \frac{\pm\sqrt{3a-b^2} \pm \sqrt{3b^2-a}}{2\sqrt{2b}}, \text{ or } x = \frac{\{\pm\sqrt{3a-b^2} \pm \sqrt{3b^2-a}\}^{\frac{4}{3}}}{4b^{\frac{2}{3}}}$$

$$y^{\frac{3}{4}} = \frac{\pm\sqrt{3a-b^2} \mp \sqrt{3b^2-a}}{2\sqrt{2b}}, \text{ or } y = \frac{\{\pm\sqrt{3a-b^2} \mp \sqrt{3b^2-a}\}^{\frac{4}{3}}}{4b^{\frac{2}{3}}}.$$

29. The remarks in paragraphs 4 to 10, respecting arithmetical fractions, apply with slight modifications to algebraic fractions.

Example. From *Peirce's Algebra.*

Divide $x^2 + \dfrac{x^4}{a^2-x^2}$, by $\dfrac{ax}{a-x} - x$.

Here $\quad x^2 + \dfrac{x^4}{a^2-x^2} = \dfrac{x^2(a^2-x^2)}{a^2-x^2} + \dfrac{x^4}{a^2-x^2} = \dfrac{a^2x^2}{a^2-x^2}$,

and $\quad \dfrac{ax}{a-x} - x = \dfrac{ax}{a-x} - \dfrac{x(a-x)}{a-x} = \dfrac{x^2}{a-x}$.

Then

$$\frac{a^2x^2}{a^2-x^2} \div \frac{x^2}{a-x} = \frac{a^2x^2}{a^2-x^2}\cdot\frac{a-x}{x^2}$$

$$= \frac{a^2x^2(a-x)}{x^2(a-x)(a+x)}$$

$$= \frac{a^2}{a+x}.$$

Or, the division may be indicated thus:

$$\frac{x^2 + \dfrac{x^4}{a^2 - x^2}}{\dfrac{ax}{a-x} - x},$$

and if this fraction be reduced, as in § 7, by multiplying its two terms by $a^2 - x^2$, recollecting that

$$\frac{x^4}{a^2 - x^2}.(a^2 - x^2) = x^4, \text{ and } \frac{ax}{a-x}.(a^2 - x^2) = ax(a+x),$$

it becomes

$$\frac{a^2x^2 - x^4 + x^4}{a^2x + ax^2 - a^2x + x^3} = \frac{a^2x^2}{x^2(a+x)} = \frac{a^2}{a+x}.$$

The usual rule for freeing an equation of fractions is to reduce all the terms to fractions having a common denominator, and then suppress the denominator. It depends upon the principle that all the terms of the equation may be multiplied by the same quantity without destroying the equality. But the following rule depends more immediately on this principle, will be found less inconvenient in practice, and is more in accordance with the spirit of the modern analysis.

Multiply the several terms of the equation by the least common multiple of the denominators of its fractional terms.

The process will render all the terms entire, since the product of a fraction by any multiple of its denominator, is the same multiple of its numerator, and therefore entire.

Example. Take the equation, *(from Peirce's Algebra,)*

$$\frac{x^2+1}{x^2-1} - \frac{x-1}{(x+1)^2} = \frac{x+1}{x-1}.$$

Since $x^2 - 1 = (x-1)(x+1)$, the least common multiple of the denominator is

$$(x-1)(x+1)^2.$$

Now
$$\begin{aligned}(x-1)(x+1)^2 &= (x^2-1) \times (x+1)\\ &= (x+1)^2 \times (x-1)\\ &= (x-1) \times (x+1)^2;\end{aligned}$$

the second factors show what multiples of the several denominators the multiplier is, and therefore if the terms of the equation be multiplied by $(x-1)(x+1)^2$, there results

$$(x^2+1)(x+1) - (x-1)^2 = (x+1)^3,$$

and developing the terms

$$x^3 + x^2 + x + 1 - x^2 + 2x - 1 = x^3 + 3x^2 + 3x + 1$$

or
$$0 = 3x^2 + 1.$$

In practice, however, it will often be found more convenient to multiply first by a multiple of one or two of the denominators, and the equation may then be susceptible of great reduction before any further multiplication is necessary.

Example 1. Take the equation

$$\frac{2x + 8\frac{1}{2}}{9} - \frac{13x-2}{17x-32} = \frac{x}{4} - \frac{x+16}{36}.$$

Multiply first by 36, then

$$8x + 34 - \frac{36(13x - 2)}{17x - 32} = 9x - x - 36,$$

$$\therefore\ 70 - \frac{36(13x - 2)}{17x - 32} = 0,$$

or

$$35 - \frac{18(13x - 2)}{17x - 32} = 0;$$

so that we have now only two terms to multiply by $17x - 32$ instead of four; then

$$35(17x - 32) - 18(13x - 2) = 0,$$

or $$595x - 1120 - 234x + 36 = 0,$$

or $$361x - 1084 = 0,$$

or $$x - 4 = 0,$$

Example 2. Take the equation

$$\frac{7}{15 + 2x} + \frac{8x - 17}{2 + 4x} = \frac{4x + 3}{2x + 12}.$$

Multiply by $2(1 + 2x)(x + 6)$, the least common multiple of $2 + 4x$ and $2x + 12$, then

$$\frac{14(2x + 1)(x + 6)}{15 + 2x} + (8x - 17)(x + 6) = (4x + 3)(2x + 1),$$

or $$\frac{14(2x + 1)(x + 6)}{15 + 2x} + 8x^2 + 31x - 102 = 8x^2 + 10x + 3,$$

and $$\frac{14(2x + 1)(x + 6)}{15 + 2x} + 21x - 105 = 0,$$

or $$\frac{2(2x + 1)(x + 6)}{15 + 2x} + 3(x - 5) = 0;$$

therefore $$2(2x + 1)(x + 6) + 3(x - 5)(2x + 15) = 0,$$

and $$10x^2 + 41x - 213 = 0.$$

ARTICLE XIII.

SOLUTIONS TO QUESTIONS PROPOSED IN NUMBER V.

---

(19). QUESTION I. *By* ——.

1°. Reduce 19° 43′ 27″ to time, at the rate of 15° to the hour.
2°. Reduce 19 hr. 43 m. 27 s. to degrees, at the same rate.

SOLUTION. *By Alfred.*

Let a be the number of degrees, b the minutes, c the seconds, &c.; then the number expressed in *degrees* will be

$$a + \frac{b}{60} + \frac{c}{60^2} + \&c.$$

these are reduced to *hours* by dividing by 15, and the result will be

$$\frac{a}{15}+\frac{b}{15\,.\,60}+\frac{c}{15.60^2}+\&c.$$

$$=\frac{4a}{60}+\frac{4b}{60^2}+\frac{4c}{60^3}+\&c.\text{ hours.}$$

Hence to reduce degrees, &c., to time, ***multiply by 4, and in the product regard each denomination as one lower than before,*** that is, degrees as minutes of time, minutes as seconds of time, &c., thus:

$$\begin{array}{r}19^\circ \quad 43' \quad 27'' \\ 4 \\ \hline \text{1 hr. 18 m. 53 sec. 48 th.} \\ \hline \end{array}$$

2°. Time may be expressed in *hours*, by

$$a+\frac{b}{60}+\frac{c}{60^2}+\&c.,$$

and this, reduced to *degrees*, is

$$15a+\frac{15b}{60}+\frac{15c}{60^2}+\&c.$$

$$=\frac{60a+b}{4}+\frac{c}{4.60}+\&c.$$

Hence to reduce time to angular magnitude, *reduce the degrees to minutes and then divide by* 4, calling, in the quotient, minutes of time degrees, seconds of time minutes, &c.; thus:

19 hr. 43 m. 27 s. = 1183 m. 27 s.,

and dividing by 4, gives 295° 51′ 45″.

---

(20). Question II. *By* ——.

Find the vulgar fraction equivalent to the circulating decimal, 3,81231231 23 . . . . .

First Solution. *By Mr. G. W. Coaklay, Peekskill Academy, N. Y.*

The given number s may be written thus:

$$s=3+\frac{8}{10}+\frac{123}{10^4}+\frac{123}{10^7}+\frac{123}{10^{10}}+\&c.$$

But since $\frac{123}{10^4}\,\frac{123}{10^7}+\frac{123}{10^{10}}+\&c.$ is an infinite geometrical progression whose first term is $\frac{123}{10^4}$, and ratio $\frac{1}{10^3}$; its sum is

$$\frac{123}{10^4}\cdot\frac{1}{1-10^{-3}}=\frac{123}{10^4}\cdot\frac{10^3}{10^3-1}=\frac{123}{9990}=\frac{41}{3330},$$

and $$s=3+\frac{8}{10}+\frac{41}{3330}=3+\frac{541}{666}=\frac{2539}{666}.$$

SECOND SOLUTION. *By Mr. B. Birdsall, New Hartford, Oneida Co., N. Y.*

Let	$s =$	$3,8123123\ldots,$
then	$10s =$	$38,123123\ldots,$
and	$10000s =$	$38123,123123\ldots;$
By subtraction,	$9990s =$	$58085,$
and	$s =$	$\frac{58085}{9990} = \frac{2539}{666}.$

GENERAL SOLUTION. *By Mr. P. Barton, Jr., Duanesburgh, N. Y.*

Put $b =$ the non-repeating number,
$m =$ the number of *decimal* figures in b,
$a =$ repeating period,
$n =$ the number of figures in a period.

Then the number s will be expressed thus:

$$s = \frac{b}{10^m} + \frac{a}{10^{m+n}} + \frac{a}{10^{m+2n}} + \frac{a}{10^{m+3n}} + \&c.$$

$$= \frac{b}{10^m} + \frac{a}{10^{m+n}} \times \left(1 + \frac{1}{10^n} + \frac{1}{10^{2n}} + \&c.\right)$$

$$= \frac{b}{10^m} + \frac{a}{10^{m+n}} \cdot \frac{10^n}{10^n - 1}$$

$$= \frac{b}{10^m} + \frac{a}{10^m(10^n - 1)}$$

$$= \frac{b(10^n - 1) + a}{10^m(10^n - 1)}.$$

In the given example, $b = 38, m = 1, a = 123, n = 3$; therefore

$$s = \frac{38}{10} + \frac{123}{10.999} = \frac{38}{10} + \frac{41}{3330} = 3\frac{541}{666}.$$

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(21). QUESTION III. *By* ——.

A banker borrows a sum of money at 4 per cent. per annum, and pays the interest at the end of the year. He lends it out at the rate of 5 per cent. per annum, and receives the interest half yearly. By this means he gains \$100 a year; how much does he borrow?

SOLUTION. *By Mr. Merries Hurlburt, Clinton Liberal Institute, N. Y.*

Let $10000x =$ the principal, in dollars; then
$400x =$ the interest to *pay*, at 4 per cent.,
$500x =$ the interest to *receive*, at 5 per cent.,
$\frac{250x}{2.20} = \frac{25}{4}x =$ the interest of the first half year's interest during the second half year at 5 per cent.;

therefore, $500x + \frac{25}{4}x - 400x = 100,$
or $4x + \frac{1}{4}x = 4,$
and $x = \frac{16}{17} = ,94117647.$
Hence $10000x = 9411,76478.$

(22.) QUESTION IV. *By* ——.

Prove that

$$(m+n)\sqrt{\frac{m}{n}}-(m-n)\sqrt{\frac{n}{m}}=(m+n)\sqrt{\frac{n}{m}}+(m-n)\sqrt{\frac{m}{n}}$$
$$=m\sqrt{\frac{m}{n}}+n\sqrt{\frac{n}{m}}.$$

FIRST SOLUTION. *By a member of the Freshman Class, University of N. C.*

$$(m+n)\sqrt{\frac{m}{n}}-(m-n)\sqrt{\frac{n}{m}}=m\sqrt{\frac{m}{n}}+\sqrt{mn}-\sqrt{mn}+n\sqrt{\frac{n}{m}}$$
$$=m\sqrt{\frac{m}{n}}+n\sqrt{\frac{n}{m}},$$
$$(m+n)\sqrt{\frac{n}{m}}+(m-n)\sqrt{\frac{m}{n}}=\sqrt{mn}+n\sqrt{\frac{n}{m}}+m\sqrt{\frac{m}{n}}-\sqrt{mn}$$
$$=m\sqrt{\frac{m}{n}}+n\sqrt{\frac{n}{m}},$$

and therefore the property is true.

SECOND SOLUTION. *By Mr. J. Campbell, St. Paul's College, L. I.*

First, $m\sqrt{\frac{n}{m}}=n\sqrt{\frac{m}{n}}=\sqrt{mn}$; and therefore

$$(m+n)\sqrt{\frac{m}{n}}-(m-n)\sqrt{\frac{n}{m}}$$
$$=(m+n)\sqrt{\frac{m}{n}}-(m-n)\sqrt{\frac{n}{m}}+2m\sqrt{\frac{n}{m}}-2n\sqrt{\frac{m}{n}}$$
$$=(m+n-2n)\sqrt{\frac{m}{n}}+(2m-m+n)\sqrt{\frac{n}{m}}$$
$$=(m-n)\sqrt{\frac{m}{n}}+(m+n)\sqrt{\frac{n}{m}}$$
$$=m\sqrt{\frac{m}{n}}-n\sqrt{\frac{m}{n}}+m\sqrt{\frac{n}{m}}+n\sqrt{\frac{n}{m}}$$
$$=m\sqrt{\frac{m}{n}}+n\sqrt{\frac{n}{m}}.$$

THIRD SOLUTION. *By Mr. B. Birdsall.*

If we multiply each of the three equalities by $\sqrt{\frac{m}{n}}$, they reduce to

$$\frac{m^2}{n}+n;$$

therefore they are equal.

(23). QUESTION V. *By* ——.

Determine A and B, so that

$$\frac{cx+d}{(x+a)(x+b)}=\frac{A}{x+a}+\frac{B}{x+b},$$

whatever x may be.

Example. Let $a=2$, $b=-3$, $c=7$, $d=-6$.

FIRST SOLUTION. *By Mr. Warren Colburn, St. Paul's College, L. I.*

By freeing the given equation of fractions, we have

$$cx + d = \text{A}x + \text{A}b + \text{B}x + \text{B}a,$$

or $$0 = (\text{A} + \text{B} - c)x + \text{A}b + \text{B}a - d,$$

independently of x, and therefore by the principle of Indeterminate Coefficients,

$$\text{A} + \text{B} - c = 0,$$
$$\text{A}b + \text{B}a + d = 0;$$

which, solved for A and B, give

$$\text{A} = \frac{ac - d}{a - b}, \quad \text{B} = \frac{d - bc}{a - b}.$$

In the given example, $\text{A} = 4$, $\text{B} = 3$, and therefore

$$\frac{7x - 6}{(x + 2)(x - 3)} = \frac{7x - 6}{x^2 - x - 6} = \frac{4}{x + 2} + \frac{3}{x - 3}.$$

SECOND SOLUTION. *By a member of the Freshman Class, University of N. C.*

Clear the equation of fractions,

$$cx + d = \text{A}x + \text{A}b + \text{B}x + \text{B}a, \quad \ldots\ldots \quad (1).$$

Take $x = 0$, then

$$d = \text{A}b + \text{B}a, \text{ or } \text{A}b + \text{B}a - d = 0, \quad \ldots\ldots \quad (2),$$

add (1) and (2), and divide by x, then

$$c = \text{A} + \text{B}, \text{ or } \text{A} + \text{B} - c = 0, \quad \ldots\ldots \quad (3).$$

Multiply by b and subtract from (2), then

$$(a - b)\text{B} - d + bc = 0, \text{ or } \text{B} = \frac{d - bc}{a - b},$$

and $$\text{A} = c - \frac{d - bc}{a - b} = \frac{ac - d}{a - b}.$$

Then $$\frac{cx + d}{(x + a)(x + b)} = \frac{ac - d}{a + b} \cdot \frac{1}{x + a} + \frac{d - bc}{a - b} \cdot \frac{1}{x + b},$$

and $$\frac{7x - 6}{(x + 2)(x - 3)} = \frac{4}{x + 2} + \frac{3}{x - 3}.$$

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(24). QUESTION VI. *By a Lady.*

Given the equation

$$x^4 - 3x^3 - \tfrac{35}{4}x^2 + \tfrac{33}{2}x - 2075\tfrac{3}{4} = 0,$$

to find x by quadratics.

FIRST SOLUTION. *By Mr. De Roy Luther, Syracuse Academy.*

Multiply by 4, and the equation is

$$4x^4 - 12x^3 - 35x^2 + 66x - 8303 = 0.$$

If we attempt to take the square root of the first member, the root will be found to be $2x^2 - 3x - 11$, with a remainder of -8424; hence, by adding 8424 to both members, it becomes

$$(2x^2 - 3x - 11)^2 = 8424,$$

and $$2x^2 - 3x - 11 = \pm\sqrt{8424}.$$

By multiplying by 8, and adding 97 to both members, we have

$$16x^2-24x+9=97\pm\sqrt{8424},$$

and $$4x-3=\pm\sqrt{97\pm\sqrt{8424}},$$

or $$x=\tfrac{3}{4}\pm\tfrac{1}{4}\sqrt{97\pm\sqrt{8424}}.$$

SECOND SOLUTION. *By Mr. E. H. Delafield, St. Paul's College.*

In order to take away the second term of this equation, take

$$x=u+\tfrac{3}{4},$$

and it becomes

$$u^4-\frac{97}{8}u^2=\frac{529727}{256};$$

therefore, $u^2=\dfrac{97\pm\sqrt{539136}}{16}=51{,}9536756$, or $-39{,}8286756$,

and $u=\pm 7{,}2078898$ or $\pm 6{,}3109964\sqrt{-1}$.

Hence $x=u+\frac{3}{4}=7{,}9578898$ or $-6{,}4578898$ or $\frac{3}{4}\pm 6{,}3109964\sqrt{-1}$.

THIRD SOLUTION. *By Mr. Merriss Hurlburt.*

Clearing of fractions it becomes

$$4x^4-12x^3-35x^2+66x=8303.$$

By adding and subtracting $9x^2$, it may be put under the form

$$(2x^2-3x)^2-22(2x^2-3x)=8303.$$

By quadratics, $2x^2-3x=11\pm\sqrt{18324}=11\pm18\sqrt{26}$.

Again, quadratics, $x=\dfrac{3\pm\sqrt{97\pm144\sqrt{26}}}{4}$.

(25.) QUESTION VII. *By Mr. Geo. W. Coaklay.*

If a and b be two sides of a triangle including the angle C, and l the line bisecting the angle C and terminating in the third side, prove that

$$\cos\tfrac{1}{2}C=\frac{l(a+b)}{2ab}.$$

SOLUTION. *By Mr. B. Franklin Chapman, Hamilton College, N. Y.*

Let A = area of the triangle, and B, B′ the two parts into which this area is divided by the beseeting line l, then we have

$$2A=ab\sin C=2ab\sin\tfrac{1}{2}C\cos\tfrac{1}{2}C$$

$$2B=al\sin\tfrac{1}{2}C \text{ and } 2B'=bl\sin\tfrac{1}{2}C.$$

But $$A=B+B',$$

and $$2ab\sin\tfrac{1}{2}C\cos\tfrac{1}{2}C=(a+b)l\sin\tfrac{1}{2}C,$$

$$\therefore\ \cos\tfrac{1}{2}C=\frac{l(a+b)}{2ab}.$$

—— Mr. Blickensderfer's solution was like this.

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(26). QUESTION VIII. *By ——.*

Within a given sphere two equal ones are inscribed, their radii being each half that of the given one. It is required to prove that there can

be six other equal spheres inscribed within the first, each touching the three former ones, and each also touching two of the others.

See Solution to Question (50), equation (27), page 248, where k must be taken $= 1$.

FIRST SOLUTION. *By Mr. R. S. Howland, St. Paul's College.*

Let two circles, radii $= \text{R}$, be inscribed within a circle, radius $= 2\text{R}$; their centres are in the same straight line. Now if a circle whose radius is r be inscribed to touch these three circles, by the properties of circular contact, we shall have

$$(\text{R} + r)^2 = \text{R}^2 + (2\text{R} - r)^2,$$

or $$r = \tfrac{2}{3}\text{R}.$$

Let now, the four circles revolve round the common diameter of the three first circles, these circles will generate the spheres described in the question, and the fourth will generate an annulus or ring, in constant contact with the three spheres, and therefore any sphere inscribed within this ring will be also in contact with the other three spheres. A section of this ring, made by a plane through the centre of 2R, perpendicular to the axis, would bisect the ring, and would make great circle sections of all the spheres so inscribed within it, the circles touching the two concentric circumferences of the ring. Hence, when the circles and consequently the spheres are tangent to each other, we shall have, by writing in the equation referred to, $\frac{2}{3}\text{R}$ and 2R for r_s and R,

$$\tfrac{2}{3}\text{R} = \frac{2\text{R} \sin \dfrac{\pi}{n}}{1 + \sin \dfrac{\pi}{n}},$$

or $$1 + \sin \frac{\pi}{n} = 3 \sin \frac{\pi}{n},$$

$$\sin \frac{\pi}{n} = \tfrac{1}{2},$$

$$\frac{\pi}{n} = 30^\circ = \frac{\pi}{6},$$

and $$n = 6,$$

the number of circles inscribed in the ring, and therefore the number of spheres inscribed in the annulus, tangent to each other.

SECOND SOLUTION. *By Mr. B. Birdsall.*

Let $2r =$ radius of the sphere, then $r =$ radius of each of the inscribed ones. Let $x =$ radius of each of the other equal ones. If we join the centres of the spheres $2r$, r and x, a triangle will be formed, right angled at the centre of $2r$, and we shall have

$$\sqrt{2rx + x^2} + x = 2x,$$

and $$x = \tfrac{2}{3}r.$$

If a circumference, whose radius is $= 2r - \frac{2}{3}r = \frac{4}{3}r$, be made to pass through the centres of the spheres x, the lines joining the centres of the spheres, will be chords of this circle. But since the spheres are in con-

tact, these chords will $= 2x = \frac{4}{5}r =$ the radius of the circle; and therefore there can be six such chords inscribed in the circle, and therefore six spheres in contact with each other and the three first.

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(27). QUESTION IX. *By* ——.

The diagonals of a given regular pentagon form, by their intersection, another regular pentagon. It required to find its side and area.

FIRST SOLUTION. *By Mr. Geo. W. Coakley.*

Let a be the side of the given pentagon, x that of the required one, and d one of the diagonals. It is evident that any two sides, together with the two diagonals drawn from their extremities to the fourth and fifth angles, form a parallelogram, and hence

$$x = 2a - d.$$

But d is the unequal side of an isosceles triangle, whose equal sides are a, and their included angle the angle of the pentagon $= 108°$, therefore

$$d = 2a \cos \tfrac{1}{2}(180° - 108°) = 2a \cos 36°,$$

and
$$x = 2a(1 - \cos 36°);$$

hence the area is easily found.

SECOND SOLUTION. *By Mr. B. F. Chapman.*

Let AB, BC be two adjacent sides of the pentagon, including an angle of 108°, (the figure can be easily sketched); and let the diagonals from B, intersect the diagonal AC in a and b. Then ABC is an isosceles triangle, CAB = ABa $= \frac{1}{2}(180° - 108°) = 36°$ and the triangles ABa, Bbc are also isosceles, and Ba, Bb trisect the angle B. Also ABb $= 72°$ = AbB, and the triangle ABb is isosceles, having AB = Ab $= a$, the side of the polygon, and if we put Bb $= c$,

$$c = \sqrt{2a^2 - 2a^2 \cos 36°} = \sqrt{4a^2 \sin^2 18°} = 2a \sin 18°;$$

and the required side is
$$ab = a - c = a(1 - 2 \sin 18°).$$

If $d =$ the area of a pentagon whose side is 1, the required area is

$$a^2 d(1 - 2 \sin 18°)^2.$$

THIRD SOLUTION. *By Mr. F. Gardner, St. Paul's College.*

Let s be the side of the given pentagon, and s' that of the required one. Let d be a diagonal of the polygon, and d' the part of d intercepted between the angles of the two polygons, then

$$s' = d - 2d'.$$

But d is the base of an isosceles triangle whose equal sides, s, include the angle of the pentagon $= \frac{3}{5}\pi$, therefore

$$d = 2s \sin \tfrac{3}{10}\pi = 2s \cos \tfrac{1}{5}\pi;$$

and d' is one of the equal sides of an isosceles triangle, whose base is s, and vertical angle $= \frac{3}{5}\pi$, therefore

$$s = 2d' \cos \tfrac{1}{5}\pi, \text{ and } d' = \tfrac{1}{2}s \sec \tfrac{1}{5}\pi;$$

hence
$$s' = 2s \cos \tfrac{1}{5}\pi - s \sec \tfrac{1}{5}\pi = s \cdot \frac{2 \cos^2 \frac{1}{5}\pi - 1}{\cos \frac{1}{5}\pi} = s \cdot \frac{\cos \frac{2}{5}\pi}{\cos \frac{1}{5}\pi}$$

$$= - s \cdot \frac{\cos \frac{3}{5}\pi}{\cos \frac{1}{5}\pi} = s(1 - 2 \cos \tfrac{2}{5}\pi) = \tfrac{1}{2}s(3 - \sqrt{5}).$$

and its area $= \frac{5}{4}s'^2 \cot\frac{1}{5}\pi = \frac{5}{4}s^2 . \frac{\cos^2\frac{2}{5}\pi}{\cos^2\frac{1}{5}\pi} . \cot\frac{1}{5}\pi = \frac{5}{8}s^2 . \frac{\cos^2\frac{2}{5}\pi}{\sin\frac{2}{5}\pi}$

$$= s^2 . \frac{5^{\frac{3}{4}}(\sqrt{5}-1)^{\frac{5}{2}}}{16\sqrt{2}}.$$

Scholium. If S be the area of the given pentagon, S_1 the area of the pentagon formed by the intersection of the diagonals, S_2, that of a pentagon formed by the diagonals of S_1, and so on; we have

$$S + S_1 + S_2 + \&c. = S\{1 + (1 - 2\cos\tfrac{2}{5}\pi)^2 + (1 - 2\cos\tfrac{2}{5}\pi)^4 + \&c.\}$$

$$= \frac{S}{1-(1-\cos\frac{2}{5}\pi)^2} = \frac{S}{8\cos\frac{2}{5}\pi\sin^2\frac{1}{5}\pi}.$$

(28). QUESTION X. *By* ——.

In any right angled spherical triangle, prove that the ratio of the cosines of the two sides including the right angle is equal to the ratio of the sines of twice their opposite angles.

SOLUTION. *By Mr. P. Barton, Jun.*

Let the two sides be b and c, and their opposite angles B and C; then

$$\cos b = \frac{\cos B}{\sin C}, \text{ and } \cos c = \frac{\cos C}{\sin B};$$

and dividing these equations, member by member,

$$\frac{\cos b}{\cos c} = \frac{\sin B \cos B}{\sin C \cos C} = \frac{2 \sin B \cos B}{2 \sin C \cos C} = \frac{\sin 2B}{\sin 2C}.$$

(29). QUESTION XI. *By* ——.

Through a point, given by its rectangular co-ordinates, to draw two straight lines, including a given angle, and intercepting a segment on the axis of y, of a given length.

SOLUTION. *By Alfred.*

Let $(x'y')$ denote the given point, v = the given angle, and b = the given intercept. Let the equations of the two lines be

$$Y = a(X - x') + y' \text{ and } y = a'(x - x') + y',$$

$\therefore$ when $X = x$, $\quad Y - y = (a - a')(x - x')$,

the intercept between the two lines, of any parallel to the axis of y, at the distance x from that axis.

And, when $x = 0$, $\quad Y - y = (a' - a)x' = b$,

$$\text{But} \quad \frac{a' - a}{aa' + 1} = \tan v,$$

$$\text{Hence} \quad a' - a = \frac{b}{x'}, \text{ and } aa' = \frac{b}{x}\cot v - 1,$$

$$\therefore \quad a' + a = \sqrt{(a' - a)^2 + 4aa'} = \sqrt{\frac{b^2}{x'^2} + 4\frac{b}{x'}\cot v - 4},$$

$$\therefore a' = \frac{1}{2x'}(\sqrt{b^2 + 4bx'\cot v - 4x'^2} + b), a = \frac{1}{2x'}(\sqrt{b^2 + 4bx'\cot v - 4'^2} - b),$$

which determine the position of the two lines.

(30). QUESTION XII. *By Mr. P. Barton, Jun.*

In a given semicircle, it is required to inscribe the greatest isosceles triangle, having its vertex in the extremity of the diameter, and one of its equal sides coinciding with the diameter.

SOLUTION. *By Mr. J. Blickensderfer, Jun., Roscoe, Ohio.*

Let the radius $= 1$, and denote the arc subtended by that side of the triangle which is a chord of the semicircle, by $2x$. Then will that side $= 2 \sin x$, and the angle contained by the equal sides of the triangle $= 90^\circ - x$. Hence, if $y =$ the area of the triangle, we have

$$y = 2 \sin^2 x \sin (90^\circ - x) = 2 \sin^2 x \cos x;$$

and $$\frac{dy}{dx} = 4 \sin x \cos^2 x - 2 \sin^3 x = 2 \sin x (2 \cos^2 x - \sin^2 x) = 0.$$

Hence, either $\sin x = 0$, or $x = 0^\circ$ or 180°,

or $2 \cos^2 x - \sin \quad x = 2 - 3 \sin^2 x = 0$,

and $\sin x = \sqrt{\tfrac{2}{3}} = \tfrac{1}{3}\sqrt{6} = ,8164965816$,

$x = 54^\circ \ 44' \ 9''$,

the former solutions giving *minima*, the later a *maximum*.

List of Contributors to the Junior Department, and of Questions answered by each. The figures refer to the number of the Questions, as marked in Number V., Article XI., page 283.

A LADY, ans. 6.
ALFRED, ans. all the Questions.
P. BARTON, Jun., Duanesburgh, N. Y., ans. all the Questions.
B. BIRDSALL, New-Hartford, Oneida Co., N. Y., ans. all the Questions.
J. BLICKENSDERFER, Jun., Roscoe, Ohio, ans. 7, 11, 12.
J. V. CAMPBELL, Freshman Class, St. Paul's College, ans. 1, 2, 3, 4, 5.
B. F. CHAPMAN, Hamilton College, N. Y., ans. 1, 2, 5, 7, 9, 12.
GEO. W. COAKLAY, Peekskill Academy, N. Y., ans. all the Questions.
WARREN COLBURN, Freshman Class, St. Paul's College, ans. 5, 6, 9.
E. H. DELAFIELD, First Preparatory Class, St. Paul's College, ans. 1, 6, 7.
F. GARDNER, Freshman Class, St. Paul's College, ans. 9.
R. S. HOWLAND, Sophomore Class, St. Paul's College, ans. 7, 8, 9, 10, 11, 12.
MERRISS HURLBURT, Clinton Liberal Institute, N. Y., ans. 1, 2, 3, 4, 5, 6, 7.
DE ROY LUTHER, Syracuse Academy, N. Y., ans. 1, 2, 3, 4, 5, 6.
A MEMBER of the Freshman Class, University of N. C., ans. 1, 2, 3, 4, 5, 6.

ARTICLE XIV.

QUESTIONS TO BE ANSWERED IN NUMBER VII.

Their Solutions must arrive before February 1st, 1839.

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(31). QUESTION I. *By* ——.

Given $a = ,280796$, $b = 1,528307$, $c = 3$, $d = ,087648$, $e = ,002879$; to calculate the numerical value of the expression

$$x = \sqrt{(ab + c)}\frac{d}{e},$$

true to five places of decimals; and exhibit the work, without using logarithms.

(32). QUESTION II. *By* ——.

Express the number 1006005 in a system of notation whose scale of relation is 6.

(33). QUESTION III. *By* ——.

Given, to find x and y, the two equations

$$x + \frac{x^2}{y} + y = a,$$

$$x^2 + \frac{x^4}{y^2} + y^2 = b.$$

(34). QUESTION IV. *By the Editor.*

Given, that

$$2(a + b)^2 + ab = (2a + b)(a + 2b).$$

It is required to divide the number

$$2x + y$$

into two factors.

(35). QUESTION V. *From Peirce's Algebra.*

A, B, C, D, E play together on this condition, that he who loses shall give to all the rest as much as they already have. First A loses, then B, then C, then D, and at last also E. All lose in turn, and yet at the end of the fifth game they all have the same sum, viz., each \$32. How much had each when they began to play?

(36). QUESTION VI. *By Mr. Geo. W. Coakley.*

Find what relation must exist among the co-efficients of the equation

$$x^4 + Ax^3 + Bx^2 + Cx + D = 0,$$

so that it may be put in either of the forms

$$(x^2 + ax)^2 + b(x^2 + ax) + c = 0,$$

or $$(x^2 + a'x + b')^2 + c' = 0.$$

(37). QUESTION VII. *By β.*

Prove that, if θ be any angle,

$$\tan^2\theta - \tan^2\tfrac{1}{3}\theta = \frac{8\sin^2\frac{1}{3}\theta\cos\frac{2}{3}\theta}{\cos^4\theta}.$$

(38). QUESTION VIII. *By the Editor.*

In a plane triangle, given that

$$b = a\sin C, \quad c = a\cos B.$$

Find its angles.

(39). QUESTION IX. *By ——.*

Let $a_1, a_2, a_3, \ldots\ldots$, be the sides of any plane polygon, and $\varphi_1, \varphi_2, \varphi_3, \ldots\ldots$, the angles they severally make with any straight line in the same plane, all counted in the same direction; prove that

$$a_1\sin\varphi_1 + a_2\sin\varphi_2 + a_3\sin\varphi_3 + \ldots\ldots = 0,$$
$$a_1\cos\varphi_1 + a_2\cos\varphi_2 + a_3\cos\varphi_3 + \ldots\ldots = 0.$$

(40). QUESTION X. *By ——.*

Having given the sum of the sides that include the right angle of a spherical triangle, and the difference of their opposite angles; to determine the sides and angles of the triangle.

(41). QUESTION XI. *By θ.**

Given the equation

$$y^2 - yx + 1 = 0,$$

to express y, by the method of Indeterminate Coefficients, in a series of monomials arranged 1°. according to the ascending powers of x, 2°. according to the descending powers of x.

(42). QUESTION XII. *By ——.*

The equation of a plane is

$$Ax + By + Cz + D = 0;$$

prove that the area of the triangle intercepted on the plane by the three rectangular co-ordinate planes, is

$$\frac{D^2}{2ABC}\cdot\sqrt{A^2 + B^2 + C^2}.$$

* We hope this gentleman, as well as others of our correspondents, will continue to assist us in selecting questions for this department.

SENIOR DEPARTMENT.

ARTICLE XXIV.

SOLUTIONS TO THE QUESTIONS PROPOSED IN ARTICLE XV, NUMBER IV.

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(76). QUESTION I. *By Petrarch, New-York.*

A boat moving uniformly in a current, performs a mile in t seconds when going with the current, and a mile in T seconds when going against the current. To find the velocity of the current.

FIRST SOLUTION. *By Mr. O. Root, Syracuse, N. Y.*

Let V = the velocity of the boat per second; v = that of the current; then

$$V+v=\frac{1}{t},\text{ and } V-v=\frac{1}{T}:$$

$$\text{therefore}\quad 2V=\frac{1}{t}+\frac{1}{T},\text{ and }\quad 2v=\frac{1}{t}-\frac{1}{T},$$

$$\text{or}\quad V=\frac{T+t}{2Tt},\text{ and }\quad v=\frac{T-t}{2Tt}.$$

SECOND SOLUTION. *By Alfred.*

Let V = velocity of the boat in still water, v = that of the current.

$$\text{Then}\qquad (V+v)t=1,\text{ or } V=\frac{1-tv}{t}.$$

$$\text{Again,}\quad (V-v)T=\frac{1-2tv}{t}\,.\,T=1,\text{ or } v=\frac{T-t}{2Tt}\text{ of a mile.}$$

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(68). QUESTION II. *By ——.*

To find the relation between the parts into which any system of conjugate diameters divides the surface of an ellipse.

FIRST SOLUTION. *By Professor M. Catlin, Hamilton College.*

Every diameter bisects all lines drawn parallel to its conjugates; consequently the two conjugate diamaters divide the ellipse into four equal parts.

Otherwise. If a circle be divided into quadrants by two perpendicular diameters, the orthographic projections of these radii will be conjugate diameters of the elliptic projection, and will include the projections of the quadrants, which will therefore be equal.

—— See also Dr. Strong's solution which will be found in a subsequent article of this number on Orthographic Projection.

SECOND SOLUTION. *By Mr. B. Birdsall.*

We have for the equation of the ellipse referred to any system of conjugate diameters ($2A'$, $2B'$) as axes,

$$A'^2y^2+B'^2x^2=A'^2B'^2, \text{or } y=\frac{B'}{A'}\sqrt{A'^2-x^2}.$$

Hence, if these axes include the angle φ, the area between them and the ellipse

$$=\sin\varphi\int ydx=\frac{B'}{A'}\sin\varphi\int_0^{A'}dx\sqrt{A'^2-x^2}$$
$$=\tfrac{1}{4}\pi'A'B\sin\varphi=\tfrac{1}{4}\pi AB$$
$$=\tfrac{1}{4}\text{ of the area of the whole ellipse,}$$

and therefore the conjugates divide the ellipse into four equal parts.

THIRD SOLUTION. *By Mr. J. F. Macully, New-York.*

Taking the centre of the ellipse as pole, its polar equation is

$$z^2=\frac{A^2B^2}{A^2\sin^2\varphi+B^2\cos^2\varphi}=\frac{2A^2B^2}{A^2+B^2-(A^2-B^2)\cos^2\varphi}.$$

Hence the area

$$=\tfrac{1}{2}\int z^2d\varphi=\tfrac{1}{2}\int\frac{A^2B^2d.2\varphi}{A^2+B^2-(A^2-B^2)\cos^2\varphi}=\tfrac{1}{2}AB\tan^{-1}\left(\frac{A}{B}\tan\varphi\right)+C.$$

Hence, if φ' and $\varphi'+\beta$ be the angles that any pair of conjugate diameters make with the axis, so that

$$\tan\varphi'\tan(\varphi'+\beta)=-\frac{B^2}{A^2},$$

the area between the conjugate semidiameters and the ellipse is

$$\tfrac{1}{2}AB\left\{\tan^{-1}\left(\frac{A}{B}\tan(\varphi'+\beta)\right)-\tan^{-1}\left(\frac{A}{B}\tan\varphi'\right)\right\}$$

$$=\tfrac{1}{2}AB\tan^{-1}\left(\frac{\frac{A}{B}\tan(\varphi'+\beta)-\frac{A}{B}\tan\varphi'}{1+\frac{A^2}{B^2}\tan\varphi'\tan(\varphi'+\beta)}\right)$$

$$=\tfrac{1}{2}AB\tan^{-1}\infty=\tfrac{1}{4}AB\pi,$$

which is the same for all parts contained between a pair of conjugate semi-diameters and the ellipse, and therefore for the four parts formed by the same system of conjugates.

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(69). QUESTION III. *By Mr. P. Barton, Jun., Duanesburgh, N. Y.*

Find x so that

$$376x^2+114x+34=\text{a square number.}$$

FIRST SOLUTION. *By Professor Peirce.*

I shall adopt the method and notation of Gauss in this solution.

Let, then
$$y^2=376x^2+114x+34,$$
$$x'=376x+57;$$
and we have
$$376y^2=(376x+57)^2+34.376-57^2=x'^2+9535,$$
or
$$376y^2-x'^2=9535.$$

We are, therefore, to find all the representations of the number 9535 by the form $(376,0,-1)$ of which the determining quantity is 376. Now, since 9535 is not divisible by a square number, y and x' must be prime to each other. We have, then, to find $\sqrt{376}$ (mod. 9535); or since $9535 = 5.1907$, we have to find

$$\sqrt{376}\ (\text{mod. } 5) = \sqrt{1}\ (\text{mod. } 5) = \pm 1, \text{ and } \sqrt{376}\ (\text{mod. } 1907);$$

which last is easily solved by Gauss' method of exclusions, and gives

$$\sqrt{376}\ (\text{mod. } 1907) = \pm 911 \text{ or } = \pm 996.$$

Now the forms

$(376,0,-1),(9535,911,87),(9535,-911,87),(9535,996,104),(9535,-996,104)$,

are properly equivalent, for their reduced forms are respectively

$(-1,19,15),(15,16,-8),(15,19,-1),(-12,14,15),(-12,10,23)$,

which all belong to the period of forms

$(-1,19,15),(15,11,-17),(-17,6,20),(20,14,-9),(-9,13,23),(23,10,-12)$,
$(-12,14,15),(15,16,-8),(-8,16,15),(15,14,-12),(-12,10,23),(23,13,-9)$,
$(-9,14,20),(20,6,-17),(-17,11,15),(15,19,-1)$.

The form $(376,0,-1)$ is then transformed into the form $(9535,911,87)$ through the series of forms

$(376,0,-1),(-1,19,15),(15,41,87),(87,-911,9535),(9535,911,87)$;

and into the form $(9535,996,87)$ through the forms

$(376,0-1),(-1,19,15),(15,11,-17),(-17,6,20),(20,14,-9),(-9,13,23)$,
$(23,10,-12),(-12,2,31),(31,60,104),(104,-996,9535),(9535,996,87)$,

which give the representations

$$x' = 789t + 15416u, \text{ or } x' = 19119t + 370736u,$$
$$y = 41t + 789u, \text{ or } y = 986t + 19119u,$$

in which t and u are the roots of the equation

$$t^2 - 376u^2 = 1,$$

which can be obtained by the method of Gauss from the above period, and are

$$t = \tfrac{1}{2}(2143295 + 110532\sqrt{376})^e + \tfrac{1}{2}(2143295 - 110532\sqrt{376})^e,$$
$$u = \frac{(2143295 + 110532\sqrt{376})^e - (2143295 - 110532\sqrt{376})^e}{2\sqrt{376}}.$$

Now to obtain *integral and positive values* of x, we are to find such values of x' that $x' - 57$ may be divisible by 376; that is

$$57t \equiv \pm 57 \ (\text{mod. } 376),$$
$$\text{or} \qquad t \equiv \pm 1 \ (\text{mod. } 376).$$

But since we have for all positive values of t,

$$t \equiv 1 \text{ or } \equiv 131 \ (\text{mod. } 376),$$

the question is obviously impossible for integral values of x', contained in the first formula. But if e' is a positive integer, we have for positive integral values of x,

$$e = 2e',\ t' = -t,\ x = 986u + 51t' - \frac{37(t'+1)}{376}.$$

If we make $e' = 1$, we have $x = 6092965917$.

The least negative value of x is $x = -51$, obtained by making $t = 1$, $u = 9$.

SECOND SOLUTION. *By Dr. T. Strong.*

If for x we put $\frac{x}{y}$, we shall have $376x^2 + 114xy + 34y^2 = z^2$, or if

we put $\quad 2z = z'$, and $376x + 57y = x'$ . . . . (1),
we shall have $\quad x'^2 + 9535x^2 = 94z'^2$ . . . . (2),
where $9535 = 5 \,.\, 1907$, 1907 being a prime number of the form $4n + 3$.

Put $\quad x' = vz' - 9535z''$ . . . . (3), and (2) will be changed to

$$\frac{v^2 - 94}{9535} \cdot z'^2 - 2vz'z'' + 9535z''^2 = -y^2 \quad . \ . \ . \ (4).$$

Therefore, when the question is possible we must have,

$$\frac{v^2 - 94}{9535} = p, \ . \ . \ . \ (5), \text{ or } v^2 = 9535p + 94 \ . \ . \ . \ (6).$$

After a few trials we find that (6) is satisfied by putting $p = 26$, which gives $v = 498$, and (4) will be changed to $-26z'^2 + 996z'z'' - 9535z'' = y^2$, or if $26z' - 498z'' = w$, . . . (7), we get $w^2 = 94z''^2 - 26y^2$ . . . (8), which is satisfied by putting $z'' = \pm 5$ and $y = \pm 3$; then $w = \pm 46$, and by (7), we have $26z' = \pm 2490 \pm 46$, or taking the sign $+$ before 2490 and $-$ before 46, we get $z' = 94$, hence by (2), $x' = 863$, and by (1), $x = \frac{172}{94}$; but since we have used $\frac{x}{y}$ instead of x, we shall have the required value of $x = \frac{173}{282}$, and we may now find other values of x, by the usual methods.

Remark. The value of v which satisfies the equation (5), may be found after the following ingenious method given by Legendre at p. 211, of his *Théorie des Nombres*. Since $9595 = 5.1907$, we must have

$$\frac{v^2 - 94}{1907} = \text{an integer, and } \frac{v^2 - 94}{5} = \text{an integer.}$$

Put $1907 = 4n + 3 = p'$, and we must have v^2 congruous to 94, or using the characteristic of Gauss, $v^2 \equiv 94$; therefore

$$(v^2)^{p'-1} \equiv 94^{p'-1}, \text{ or } (v^{p'-1})^2 \equiv 94^{p'-1} \equiv 94^{4n+2}, \text{ or } v^{p'-1} \equiv 94^{2n+1}.$$

But since p' is a prime number, we have by the well known theorem of Fermat, $\frac{v^{p'-1} - 1}{p'} =$ an integer, provided v is an integer prime to p'. But since $v^{p'-1} \equiv 94^{2n+1}$, therefore $v^{p'-1} - 1 \equiv 94^{2n+1} - 1$, and $\frac{v^{p'-1} - 1}{p'} \equiv \frac{94^{2n+1} - 1}{p'}$.

Therefore $\frac{94^{2n+1} - 1}{p'} =$ an integer, and $\frac{94^{2n+2} - 94}{p'} = \frac{(94^{n+1})^2 - 94}{p'} =$ an integer also; hence we may put $v = 94^{n+1} = 94^{477}$, or we may reject the multiples of p' contained in 94^{477}, and use the remainder, that is take $v = -498$, and $\frac{v^2 - 94}{9535} = 26$ as before.

THIRD SOLUTION. *By the Proposer.*

Put $\quad 376x^2 + 114x + 34 = (11x + 7)^2 + (17x + 3)(15x - 5)$
$\qquad = \{(11x + 7) + a(15x - 5)\}^2$,

and we get $\quad x = \frac{5a^2 - 14a + 3}{15a^2 + 22a - 17}$.

If $\overset{a}{x} = 3$, $\overset{x}{a} = \frac{3}{\underset{92}{34}}$.

FOURTH SOLUTION. *By Mr. B. Birdsall.*

The value $x = \frac{1}{3}$ will satisfy this equation; hence, if $x = y + \frac{1}{3}$,

$$376x^2 + 114x + 34 = 376y^2 + \tfrac{1024}{3}y + (\tfrac{32}{3})^2 = (px + \tfrac{32}{3})^2,$$

then $$y = \frac{64p + 1094}{3p^2 - 1128}.$$

If we take $p = 20$, then $y = \frac{4442}{12}$, and $x = y + \frac{1}{3} = \frac{4452}{12}$.

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(70). QUESTION IV. *By P.*

Find the diameter of the sphere, which placed in a given conical glass full of water, shall cause the greatest quantity of water to overflow.

FIRST SOLUTION. *By Mr. N. Vernon.*

Let $2a$ = the diameter of the cone's base, b = its depth, c = the slant height, and x = the radius of the sphere. Then $\dfrac{cx}{a}$ = the distance of the sphere's centre from the vertex of the cone, $b - \dfrac{cx}{a}$ = its distance from the top of the glass, and $h = b + x - \dfrac{cx}{a}$ = the height of the segment immersed in water. Now, by mensuration, the solidity of this segment

$$= (6x - 2h)h^2 . \tfrac{1}{6}\pi = \tfrac{1}{3}\pi\left(2x - b + \frac{cx}{a}\right)\left(b + x - \frac{cx}{a}\right)^2 = \text{a } max.$$

Taking the differential and reducing

$$x = \frac{abc}{(2a + c)(c - a)}, \text{ as required.}$$

SECOND SOLUTION. *By Mr. L. Abbott, Niles, N. Y.*

Put a = altitude of the glass, 2θ = angle at its vertex, r = radius of the sphere, x = the immersed part of the diameter, and B = the immersed segment; then $r = \dfrac{(a - x)\sin\theta}{1 - \sin\theta}$, and

$$\text{B} = \tfrac{1}{3}\pi x^2(3r - x) = \tfrac{1}{3}\pi . \frac{3ax^2\sin\theta - (1 + 2\sin\theta)x^3}{1 - \sin\theta}.$$

This differentiated and put = zero, gives

$$x = \frac{2a\sin\theta}{1 + 2\sin\theta}, \text{ and } r = \frac{(a - x)\sin\theta}{1 - \sin\theta} = \frac{a\sin\theta}{(1 - \sin\theta)(1 + 2\sin\theta)}.$$

—— We notice that the value of the radius reduces to

$$r = \frac{a\sin\theta}{2\sin\frac{1}{2}(\frac{1}{2}\pi - \theta)\sin\frac{3}{2}(\frac{1}{2}\pi - \theta)}.$$

Professor Peirce, after a solution similar to the last, says

"since $\quad x = 2(1 - \sin\theta)r$,

the centre is not immersed when $x < r$, or $\sin\theta > \frac{1}{2}$, or $\theta > 30°$; but when $\theta = 30°$, then $x = r$, and the sphere is half immersed, having $r = \frac{1}{3}a$."

(71). QUESTION V. *By Mr. O. Root.*

If through the extremity of the diameter of a semicircle, chords be drawn, and semicircles be described upon them as diameters, their vertices will be in the semicircle described on the chord which passes through the vertex of the given semicircle.

FIRST SOLUTION. *By Mr. L. Abbott, Jun.*

Let v, av be any two radii vectors having their origin in a common point P, and a a constant quantity. Then if v and av, be made to revolve about P, making always the same angle with each other, the two curves described by the extremities of v and av, whatever be their form, which depends upon the variation of v, must be similar to each other, since the radii have a constant ratio, and the same angular velocities. Hence if v be made to describe a semicircle, then $\frac{v}{\sqrt{2}}$, the chord through the vertex of the semicircle on v, will also describe a semicircle, the two diameters making an angle of 45°.

SECOND SOLUTION. *By Professor C. Avery, Hamilton College.*

If through the extremity of the chord of an arc of θ degrees, chords be drawn and segments be described on them containing θ degrees, their vertices will be in an arc of θ described on the chord passing through the vertex of the given arc. For the angle made by the two radii drawn to the extremity of the given arc, and the point where the line joining the extremity of any chord and the vertex of the arc described on it meets the given arc will be $\pi - \theta$; this subtracted from π gives the angle between these radii when the one drawn to the point on the given arc is produced, equal to θ. This shows that the line joining the extremity of any chord and the vertex of the arc described on it, always bisects the given arc; hence the general proposition is manifest, and consequently the particular case in the question.

THIRD SOLUTION. *By Mr. J. Blickensderfer, Jun., Roscoe, Ohio.*

Let the axes of co-ordinates be the diameter of the given semicircle, radius r, and a perpendicular to it through the centre. The co-ordinates of one extremity of the variable chord are $r, 0$; those of the other, x', y'; and those of their middle point, or the centre of the semicircle on the chord $\frac{1}{2}(r + x'), \frac{1}{2}y'$; a line through the origin and the last point, will necessarily pass through the vertex, its equation is

$$y = \frac{y'}{r + x'} \cdot x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

But a line drawn from $r, 0$, making an angle of 45° with (1), will also pass through this vertex, its equation is

$$y = \frac{r + x' + y'}{r + x' - y'}(x - r) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Then the vertex is at the point of intersection x, y of these lines, and eliminating x' and y', we get the equation of the vertex

$$y^2 + x^2 - ry - rx = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

which is that of a circle whose radius is $\frac{1}{2}r\sqrt{2}$, and centre at the point $\frac{1}{2}r, \frac{1}{2}r$, the property ennunçiated.

—— Equation (3) is reduced from (1) and (2), without any relation between x', y', that is, without any reference to the curve in which the point x', y' is compelled to move, and therefore it is true whatever that curve may be.

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(72). QUESTION VI. *By*——.

Def. In the parabola, the parameter of any diameter is that chord of the system it bisects, which is equal to four times the distance of its middle point from the vertex of the diameter.

It is required to show that all parameters of the parabola pass through a given point, and to find the locus of their middle points.

FIRST SOLUTION. *By Alfred.*

Let $y^2 = 2px$ be the equation of the parabola, referred to its axis and tangent through its vertex, as axes of co-ordinates. Then by equations (2) and (13) of the solution to Question (56), the equation of any diameter is $y = \frac{p}{a}$, and that of the system of chords it bisects $y = ax + b$. For the intersections (y', x', and y'', x'') of this line with the parabola we have

$$a^2x^2 + (2ab - 2p)x + b^2 = 0,$$

so that

$$x' + x'' = \frac{2}{a^2}(p - ab), \quad x'x'' = \frac{b^2}{a^2},$$

and

$$x' - x'' = \sqrt{(x' + x'')^2 - 4x'x''} = \frac{2}{a^2}\sqrt{p^2 - 2abp}.$$

Hence, if l be the length of the chord

$$l^2 = (y' - y'')^2 + (x' - x'')^2 = (a^2 + 1)(x' - x'')^2 = \frac{4(a^2+1)}{a^4}(p^2 - 2abp).$$

For the vertex of the diameter we have $\frac{p^2}{a^2} = 2px$, or $x = \frac{p}{2a}$, and for its intersection with the chord, $\frac{p}{a} = ax + b$, or $x = \frac{p}{a^2} - \frac{b}{a}$; then the distance from the vertex to the middle of the chord is $\frac{p}{2a^2} - \frac{b}{a}$; and, by the definition,

$$l^2 = \frac{4(a^2+1)}{a^4}(p^2 - 2abp) = 16\left(\frac{p}{2a^2} - \frac{b}{a}\right)^2 = \frac{4}{a^4}(p - 2ab,$$

or

$$(a^2 + 1)p = p - 2ab, \text{ and } b = -\tfrac{1}{2}ap)^2,$$

therefore the equation of any *parameter* is

$$y = ax - \tfrac{1}{2}ap.$$

The equation of a second parameter is, then, $y = a'(x - \frac{1}{2}p)$, and these intersect at the point

$$y = 0, \quad x = \tfrac{1}{2}p;$$

these co-ordinates, being independent of the position of the parameters, shows that *all* parameters pass through the same point, which is evidently the focus

The co-ordinates of the middle point of any parameter or its intersection with its diameter are, then,

$$y=\frac{p}{a},\quad x=\frac{p}{a^2}+\tfrac{1}{2}p,$$

and eliminating a between these equations we have

$$y^2=p(x-\tfrac{1}{2}p),$$

which is a parabola, whose vertex is in the focus, and parameter half that of the given one, their axes being on the same line.

SECOND SOLUTION. *By Mr. P. Barton, Jun.*

Let (x_1, y_1), (x_2, y_2), (x_3, y_3) denote, respectively, the extremities and middle point of any chord of the parabola, (a, y_3) the vertex of the diameter passing through (x_3, y_3), and $4m$ the principal parameter; then $y_1^2=4mx_1$, $y_2^2=4mx_2$, $y_3^2=4ma$, $x_3=\frac{1}{2}(x_1+x_2)$, $y_3=\frac{1}{2}(y_1+y_2)$ (1), and by the definition of the parameter,

$$(x_1-x_2)^2+(y_1-y_2)^2=4(a-x_3)^2 \quad . \quad . \quad . \quad . \quad (2).$$

By eliminating x_1, x_2, a and x_3 from these equations, we find

$$y_1y_2+4m^2=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

a singular property of the parameter chord.

The equation of a chord through (x_1, y_1) and (x_2, y_2), is

$$(y-y_1)(x_1-x_2)=(y_1-y_2)(x-x_1),$$

or, eliminating x_1 and x_2, by means of (1)

$$y(y_1+y_2)=4mx+y_1y_2=4m(x-m),$$

that is, $$yy_3=2m(x-m) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

which is the equation of a parameter, and shows that it passes through the point $(0, m)$ or the focus, whatever y_3 may be.

From (4), we have for the middle point of any parameter where $y=y_3$,

$$y^2=2m(x-m) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5);$$

and the locus is a parabola, parameter $2m$, and vertex in the focus of the original one.

THIRD SOLUTION. *By Professor Catlin.*

I. If a tangent be drawn at the vertex of any diameter to intersect the axis, the parts of the diameter and axis, intercepted between the tangent and parameter chord will be equal; but, by definition, the intercepted part of the diameter $=\frac{1}{4}p$, and it is well known that the part of the axis intercepted between the focus and tangent $=\frac{1}{4}p$; hence the parameter chord passes through the focus.

II. The equation of the parabola is $y^2=px$. Let $p'=$ the parameter of any diameter, and x', y' the co-ordinates of the required curve. Then we obviously have $y=y'$, and $x=x'-\frac{1}{4}p'=x'-(x+\frac{1}{4}p)$, or $x=\frac{1}{2}x'-\frac{1}{8}p$, therefore

$$y'^2=p(\tfrac{1}{2}x'-\tfrac{1}{8}p)=\tfrac{1}{2}p(x'-\tfrac{1}{4}p),$$

which shows that the required locus is a parabola whose vertex is the focus of the given parabola, and parameter half that of given one.

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(73). QUESTION VII. (*From the Phil. Mag. and Jour., Aug.*, 1836.)

Theorem. The circumference drawn through the point of intersection of any three tangents of a parabola, passes through the focus of that parabola.

FIRST SOLUTION. *By Dr. Strong.*

Let p denote the parameter of the axis of the given parabola, its equation is $y^2 = px$, also the equation of a tangent drawn from any point Y,X to the point of contact y',x' is

$$\text{Y} = \frac{p}{2y'}(\text{X} + x') \quad \ldots \ldots \quad (1),$$

and, for another tangent from the same point, to the point y'',x'',

$$\text{Y} = \frac{p}{2y''}(\text{X} + x'') \quad \ldots \ldots \quad (2)$$

By (1) and (2), $\text{X} = \dfrac{y'x'' - y''x'}{y'' - y'} = \dfrac{y'y''}{p}$, $\text{Y} = \dfrac{p}{2} \cdot \dfrac{x'' - x'}{y'' - y'} = \frac{1}{2}(y' + y'')$ (3)
which are interesting properties of the curve, and evidently retain the same form when the curve is referred to any diameter and its ordinates. Also if X', Y'; X'', Y'' are the co-ordinates of the points of intersection of a third tangent with the first and second tangents, x''', y''' being its point of contact, we have similarly,

$$\text{X}' = \frac{y'y'''}{p}, \ \text{Y}' = \frac{y' + y'''}{2}, \ \text{X}'' = \frac{y''y'''}{p}, \ \text{Y}'' = \frac{y'' + y'''}{2} \quad \ldots \quad (4).$$

Again, if we make a circle pass through the three points of intersection of the three tangents, the co-ordinates of its centre being a, b, and r its radius, its equation will be

$$(x - a)^2 + (y - b)^2 = r^2 \quad \ldots \ldots \quad (5).$$

so that

$$(\text{X} - a)^2 + (\text{Y} - b)^2 = r^2 = (\text{X}' - a)^2 + (\text{Y}' - b)^2 = (\text{X}'' - a)^2 + (\text{Y}'' - b)^2 \ . \ (6).$$

Hence

$$(\text{X} + \text{X}')(\text{X} - \text{X}') + (\text{Y} + \text{Y}')(\text{Y} - \text{Y}') = 2a(\text{X} - \text{X}') + 2b(\text{Y} - \text{Y}'),$$

and

$$(\text{X} + \text{X}'')(\text{X} - \text{X}'') + (\text{Y} + \text{Y}'')(\text{Y} - \text{Y}'') = 2a(\text{X} - \text{X}'') + 2b(\text{Y} - \text{Y}''),$$

which give, by substituting the values of X, X', &c.,

$$a = \frac{y'y'' + y'y''' + y''y'''}{2p} + \frac{p}{8}, \ b = \frac{y' + {}'' + y'''}{4} - \frac{y'y''y'''}{p^2},$$

hence $r^2 = \left(\dfrac{y' + y'' - y'''}{4} + \dfrac{y'y''y'''}{p^2}\right)^2 + \left(\dfrac{y'y'' - y'y''' + y''y'''}{2p} + \dfrac{p}{8}\right)^2$,
which will enable us to describe the circle when the co-ordinates of the three points of contact are known. Again (5) and (6) give

$$x^2 + y^2 - 2ax - 2by = \text{X}^2 + \text{Y}^2 - 2a\text{X} - 2b\text{Y} = -\tfrac{1}{4}(y'y'' + y'y''' + y''y'''),$$

and to find the point where the circle cuts the axis of x, we must put $y = 0$, then

$$x^2 - 2ax = -\tfrac{1}{4}(y'y'' + y'y''' + y''y'''),$$

or $\quad (x - a)^2 = a^2 - \frac{1}{4}(y'y'' + y'y''' + y''y''')$

$$= a^2 - \tfrac{1}{2}ap + \tfrac{1}{16}p^2 = (a - \tfrac{1}{4}p)^2,$$

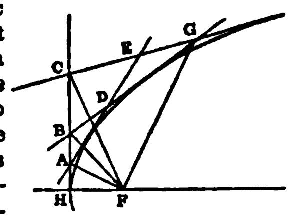
and $\quad x - a = \pm(a - \frac{1}{4}p)$,

or $\quad x = 2a - \frac{1}{4}p$, and $x = \frac{1}{4}p$,

which show that the circumference passes through the focus.

SECOND SOLUTION. *By Mr. O. Root.*

Let D, E, G be the points of intersection of the three tangents drawn from the focus F, the perpendiculars FA, FB, FC intersecting these tangents in the tangent through the vertex H. It is manifest that a circle will pass through F, B, C, G, hence CGB = CFB, being in the same segment; also a circle will pass through F, A, B, D, hence FBA = FDA. But FBA = FCB + CFB = FGB + CGB = FGC = FDA, and FDA + FDE = two right angles, therefore FGC + FDE = two right angles; hence the points FDEG are situated on the circumference of a circle.



(74). QUESTION VIII. *By Wm. Lenhart, Esq., York, Pa.*

It is required to find three consecutive natural numbers, that are divisible by cube numbers greater than unity.

FIRST SOLUTION. *By Professor C. Avery.*

Let x, $x+1$, $x+2$ be the required numbers, and a, b, c the roots of the cubes. We are to have, t, t', t'' being integers,

$$x = a^3t,\ x+1 = b^3t',\ x+2 = c^3t'' \quad . \quad . \quad . \quad . \quad . \quad (1),$$

therefore $\quad b^3t' - a^3t = 1$, and $c^3t'' - b^3t' = 1 \quad . \quad . \quad . \quad . \quad . \quad (2).$

For any assumed values of a, b, c, which must be prime to each other, let T, T', be the least values of t, t', that satisfy the first of equations (2), found by continued fractions or otherwise, and T'_1, T'' the least values of t', t'' that satisfy the second equation; we may take

$$t' = a^3s + T' \quad = c^3s' + T_1',$$
$$t = b^3s + T,\ t'' = b^3s' + T'',$$

and therefore $\quad a^3s - c^3s' = T_1' - T' = a^3k + l \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$

k being the greatest integer contained in $\dfrac{T_1' - T'}{a^3}$, and l' the remainder.

Let s, s' be the least values that satisfy the equation

$$a^3s - c^3s' = 1,$$

and we may take $\quad s = c^3u + ls + k,\ s' = a^3u + ls',$

where u may be taken at pleasure, and we shall have

$$t = b^3c^3u + lb^3s + T + kb^3,$$

and $\quad x = a^3b^3c^3u + a^3b^3ls + a^3T + ka^3b^3.$

Example. Take $a=2$, $b=3$, $c=5$. Then $T=10$, $T'=3$; $T_1'=37$, $T''=8$, $s=47$, $s'=3$, $k=4$, $l=2$ and $x = 27000u + 21248$. If $u=0$, $x=21248$, and the numbers are 21248, 21249, 21250. If $u = -1$, the numbers are 5752, 5751, 5750.

—— By taking $a=3$, $b=2$, $c=5$, Mr. Root finds the numbers 1375, 1376, 1377, which are the least number furnished us. It is evident that by the same method we could find any number of consecutive numbers, divisible by given numbers prime to each other.

SECOND SOLUTION. *By Mr. N. Vernon.*

When, in the formula $aq - bp = \pm c$ . . (1), we have found one set of numbers to answer, we can obtain others by using the formula

$\left(\frac{p+ma}{q+mb}\right)$ . . (2). And when $c=1$, we may find values for p and q to answer any other value of c, by using the formula $\left(\frac{cp \pm ma}{cq \pm mb}\right)$ . . (3). This premised, let us make the question general, by finding n numbers rx^3, $r'x'^3$, $r''x''^3$, &c., such, that

$$rx^3 - r'x'^3 = 1, r'x'^3 - r''x''^3 = 1, r''x''^3 - r'''x'''^3 = 1, \text{\&c.}$$

Let us now substitute for x, x', x'', &c., the prime numbers 2, 3, 5, 7, &c., and find for each pair of these, cubed, and reduced to a continued fraction, a series of converging fractions, which may be tabulated, and by the help of which, with formulas (2) and (3), a general solution may always be obtained.

To find two numbers, $8r - 27r' = \pm 1$, and by continued fractions we get

$$8.10 - 27.3 = -1, \text{ and } 8.17 - 27.5 = 1.$$

Hence $r=10$, $r'=3$, and the numbers are 80 and 81, or $r=17$, $r'=5$, and the numbers are 136 and 135, and as many numbers as we please may be obtained from formula (2).

Let us find three numbers. Then, by continued fractions, we find $8.47 - 125.3 = 1$, $27.51 - 125.11 = 2$, hence by (2),

$$r = 47 + 125m,\ r' = 51 + 125m',\ r'' = 3 + 8m = 11 + 27m',$$
$$\text{or} \qquad 8(m-1) = 27m'.$$

Then if $m=1$; $m'=0$, and $r=172$, $r'=51$, $r''=11$, the numbers being $8.172 = 1376$, $27.51 = 1377$ and $125.11 = 1375$.

—— Mr. Vernon then proceeds to find four consecutive numbers in the same ingenious manner; but the question is not of sufficient interest to allow us to insert the whole of his solution.

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(75). Question IX. *By P.*

Given the area and vertical angle of a plane triangle, its base being on a straight line given in position, and one extremity of it at a given point of that line. To find the locus of the intersection of perpendiculars from the angles on the opposite sides; to trace the curve, and find its form under every relation of the constants.

Solution. *By the Proposer.*

Let A, the fixed extremity of the base, be the origin of rectangular coordinates, the given line being the axis of x. Represent by $0, x_1$ the coordinates of B the other extremity, and by y_2, x_2 those of the third vertex C, of the triangle. Then

the equation of AC is $\qquad y = \frac{y_2}{x_2} \cdot x,$

the equation of BC is $\qquad y = \frac{y_2}{x_2 - x_1} \cdot (x - x_1).$

Then if C be the angle between these two lines, we shall have

$$\tan C = \frac{\frac{y_2}{x_2 - x_1} - \frac{y_2}{x_2}}{1 + \frac{y_2^2}{x_2^2 - x_1 x_2}} = \frac{y_2 x_1}{y_2^2 + x_2^2 - x_1 x_2};$$

or, since if s be the given area, we have

$$y_2 x_1 = 2s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

the above equation becomes

$$y_2^2 + x_2^2 - x_1 x_2 = 2s \cot c \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Moreover, the equation of the perpendicular from C upon AB is

$$x = x_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

and of that from A upon BC, is

$$y y_2 = (x_1 - x_2) x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

By eliminating y_2, x_1, x_2 between these equations, we find

$$x(y + x \cot c)(y^2 + x^2) = 2s(x - y \cot c)^2 \quad . \quad . \quad . \quad . \quad (5)$$

for the equation of the curve.

If we change these into polar co-ordinates v, φ, the angular axis being the positive axis of x, it becomes

$$v^2 = \frac{a^2 \sin^2(c - \varphi)}{\cos\varphi \cos(c - \varphi)}, \text{ where } a^2 = \frac{2s}{\sin c}.$$

By writing $c + \varphi$ for φ in this equation, we shall change it into a system, better adapted to our purpose, in which the angular axis is inclined to the former one in an angle c, the polar equation then is

$$v^2 = \frac{a^2 \sin^2\varphi}{\cos\varphi \cos(c + \varphi)},$$

and the co-ordinates of any point, in a rectangular system of which the new angular axis is the positive axis of x, are

$$y^2 = \frac{a^2 \sin^4\varphi}{\cos\varphi \cos(c + \varphi)}, \quad x^2 = \frac{a^2 \sin^2\varphi \cos\varphi}{\cos(c + \varphi)}.$$

Then $$2y \cdot \frac{dy}{d\varphi} = a^2 \sin^3\varphi \cdot \frac{(1 + 3\cos^2\varphi)\cos(c+\varphi) + \sin\varphi\cos\varphi\sin(c+\varphi)}{\cos^2\varphi\cos^2(c+\varphi)},$$

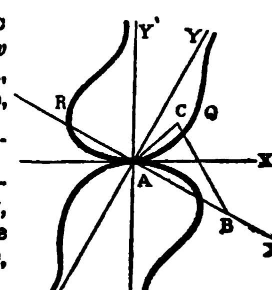
$$2x \cdot \frac{dx}{d\varphi} = a^2 \sin\varphi \cdot \frac{(3\cos^2\varphi - 1)\cos(c+\varphi) + \sin\varphi\cos\varphi\sin(c+\varphi)}{\cos^2(c+\varphi)},$$

$$\frac{dy}{dx} = \tan\varphi \cdot \frac{(3\cos^2\varphi + 1)\cos(c+\varphi) + \sin\varphi\cos\varphi\sin(c+\varphi)}{(3\cos^2\varphi - 1)\cos(c+\varphi) + \sin\varphi\cos\varphi\sin(c+\varphi)},$$

$$\frac{d^2y}{dx^2} = \frac{2\cos^{\frac{1}{2}}(c+\varphi)}{a\cos^{\frac{3}{2}}\varphi} \cdot \frac{12\cos^2\varphi - \{1 + \cos^2 c - \sin\varphi\cos\varphi\tan(c+\varphi)\}^2}{\{3\cos^2\varphi - 1 + \sin\varphi\cos\varphi\tan(c+\varphi)\}^3}.$$

I. When $c < 90°$.

Then when $\varphi = 0°$, $v = 0$, $\frac{dy}{dx} = 0$, or the curve passes through A, touching the angular axis AX′, which makes the angle c with the given line AX. While φ increases v increases also, and the branch AQY is described, so that when $\varphi = 90° - c$, $v = \infty$, $y = \infty$, $x = \infty$, and $\frac{dy}{dx} = \cot c$, so that the line AY, perpendicular to AX is an assymptote to the curve. Between $\varphi = 90° - c$ and $\varphi = 90°$, v is imaginary, so that no part of the curve is within the angle YAY′. When $\varphi = 90°$, $v = \infty$, $y = \infty$, $x = 0$, and $\frac{dy}{dx} = \infty$, so that the line AY′, perpendicu-



lar to AX′ is an assymptote to the curve. While φ increases from 90° to 180°, v continues to decrease; until, when $\varphi=\pi, v=0, x=0, y=0, \frac{dy}{dx}=0$; or the curve passes again through A, AX′ being a common tangent to both branches and precisely the same variations are gone through from $\varphi=\pi$ to $\varphi=2\pi$. The greatest distance of the branch ARY′ from its assymptote is when x is a *max.* or $\frac{dx}{d\varphi}=0$, or

$$(3\cos^2\varphi-1)\cos(c+\varphi)+\sin\varphi\cos\varphi\sin(c+\varphi)=0,$$
$$\text{or}\qquad \tan^3\varphi-\tan\varphi+2\cot c=0.$$

Only one of the the roots of this equation is real while $\cot^2 c>\frac{1}{27}$, and it may be calculated by putting

$$\tan^2 c=27\sin^2 2\omega;$$

$$\text{Then}\qquad \tan\varphi=-\frac{1}{\sqrt{3}}(\tan^{\frac{1}{3}}\omega+\cot^{\frac{1}{3}}\omega),$$

or if we take Ω so that $\tan\omega=\tan^3\Omega$, and consequently

$$\tan c=\frac{3\sqrt{3}.\sin^3 2\Omega}{3\cos^2 2\Omega+1},$$

$$\text{then}\qquad \cot\varphi=-\tfrac{1}{2}\sqrt{3}\sin 2\Omega,$$

$$\text{and}\qquad x=-\sqrt{\frac{8s}{3\sqrt{3}\sin^3 2\Omega}}.\sqrt{1-\frac{21\sin^2 2\Omega}{(4+3\sin^2 2\Omega)^2}}.$$

While c varies from 0° to $\tan^{-1}\sqrt{27}$, Ω varies from 0° to $\frac{1}{4}\pi$ and x varies from ∞ to $\sqrt{\frac{32s}{21\sqrt{3}}}$, so that the branch approaches its assymptote as c increases.* At the latter limit, or when $c=\tan^{-1}\sqrt{27}$, both the other roots of the equation are $\varphi=\frac{1}{6}\pi$, which is between the limits $\frac{1}{2}\pi-c$ and $\frac{1}{2}\pi$, and therefore gives no point in the curve. When $\tan^2 c>27$, take ψ so that

$$\cos 3\psi=-3\sqrt{3}.\cot c,$$

$$\text{then}\qquad \tan\varphi=\frac{2}{\sqrt{3}}.\cos\psi,\text{ or }=\frac{2}{\sqrt{3}}\cos(\tfrac{2}{3}\pi-\psi,)\text{ or }=\frac{2}{\sqrt{3}}\cos(\tfrac{2}{3}\pi+\psi).$$

While c varies between $\tan^{-1}\sqrt{27}$ and $\frac{1}{2}\pi$, ψ varies from $\frac{1}{3}\pi$ to $\frac{1}{6}\pi$, and

$$\tan\varphi\text{ varies from }\frac{1}{\sqrt{3}},\text{ or }\frac{1}{\sqrt{3}},\text{ or}-\frac{2}{\sqrt{3}};\text{ to }1,\text{ or }0,\text{ or }-1;$$

∴ φ varies from $\frac{1}{6}\pi$, or $\frac{1}{6}\pi$, or *nearly* 131°; to $\frac{1}{4}\pi$, or 0°, or $\frac{3}{4}\pi$: the two first roots are always within the limits $\frac{1}{2}\pi-c$ and $\frac{1}{2}\pi$, and give no points in the curve, the third gives a point whose distance x from the assymptote varies from $\sqrt{\frac{32s}{21\sqrt{3}}}$, to $\sqrt{s}$, which it becomes when $c=\frac{1}{2}\pi$.

In the same manner, if d be the distance of any point in the branch AQY from its assymptote, we shall have

$$d^2=v^2\cos^2(c+\varphi)=a^2.\sin\varphi\tan\varphi\cos(c+\varphi);$$

which is greatest when

* In fact, the point of greatest distance from the curve to the assymptote, decreases from c = 0° to c = 70½° nearly, it then increases until c = 112° nearly, and afterward decreases again.

$$\tan^3\varphi - 3\tan\varphi - 2\cot c = 0,$$

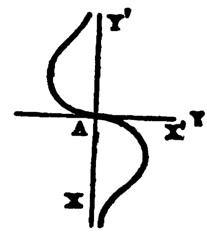
and $$\tan\varphi = \cot^{\frac{1}{3}}\tfrac{1}{2}c - \tan^{\frac{1}{3}}\tfrac{1}{2}c:$$

or if $\tan\frac{1}{2}c = \tan^3\chi$, then $\tan\varphi = 2\cot 2\chi$, and

$$\text{the } max.\ d^2 = 8a\cot^2 2\chi \cdot \frac{1+\cos^2 2\chi}{1+3\cos^2 2\chi}.$$

Now, while c varies from 0° to $\frac{1}{2}\pi$, χ varies from 0° to $\frac{1}{4}\pi$, and d^2 varies from ∞ to 0; hence this branch continually approaches its assymptote, as c increases, until

II. When $c = 90°$,

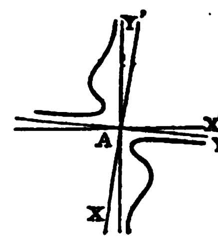


and then both the assymptote and curve are confounded with the line AX'; the curve becomes then of the third order, its equation, referred to the original axes, AX and AY, being

$$y(y^2+x^2)=2ax,$$

and its greatest distance from its assymptote is $\sqrt{a}$.

III. When $c > 90°$.



Then $\varphi = 0°$, is included between the impossible limits $\varphi=\frac{1}{2}\pi-c$ and $\varphi=\frac{1}{2}\pi$, so that the curve is included within the angle made by AY' and the prolongation of AY, and its vertical, these lines being still assymptotes to the curve. Hence the curve no longer passes through A, its nearest approach to A being when v is a *min.*, or when

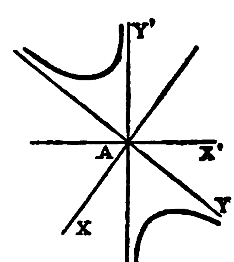
$$\tan\varphi = 2\cot c, \text{ and } v = \frac{2a}{\sin c}\cdot\sqrt{-\cos c}.$$

At this point also $\dfrac{dy}{dx} = -\frac{1}{2}\tan c$, which is sufficient to indicate that there is no cusp there. In the preceding investigation for finding the maximum distance of the curve from its assymptote, while c varies from $\frac{1}{2}\pi$ to $\tan^{-1} - \sqrt{27}$, ψ varies from $\frac{1}{2}\pi$ to 0°, also

$$\tan\varphi \text{ varies from } 1, \text{ or } 0, \text{ or } -1;\ \text{to } \frac{2}{\sqrt{3}}, \text{ or } \frac{-1}{\sqrt{3}}, \text{ or } \frac{-1}{\sqrt{3}};$$

and φ varies from $\frac{1}{4}\pi$, or π, or $\frac{3}{4}\pi$; to nearly 49°, or $\frac{5}{6}\pi$, or $\frac{5}{6}\pi$.

The first root gives no point in the curve, but the others give two points, at one of which the distance is a maximum and at the other a minimum. These points approach each as c increases, and when $c = \tan^{-1} - \sqrt{27}$, they coincide with each other; when $c > \tan^{-1} - \sqrt{27}$, the curve continually recedes from AY' while φ varies from $_2\pi$ to $\frac{3}{2}\pi - c$ its two limits, and the curve is always convex to AX'.



There are evidently points of inflexion in the first three forms of the curve, between the point of greatest distance of any branch from its assymptote and its infinite

extent; these points may be determined in any particular case from the roots of the equation

$$12\cos^2\varphi - \{1 + \cos^2\varphi - \sin\varphi\cos\varphi\tan(c+\varphi)\}^2 = 0,$$

which is of the sixth degree in $\tan\varphi$.

—— We are obliged to Mr. L. Abbott and Professor Peirce for their complete solutions to this question.

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(76). QUESTION X. *By Richard Tinto, Esq., Greenville, Ohio.*

If a given cone of revolution be cut by planes, so that the principal parameter of all the sections shall be equal to a given line; it is required to find the surface to which these planes shall all be tangent.

FIRST SOLUTION. *By Mr. J. B. Henk, Harvard University.*

Let ν be the angle formed with the axis by the generating line of the cone, i the angle formed by the cutting plane with the generating line, and a the distance of their point of intersection from the vertex. The expression for the parameter is then

$$2a\sin i\tan\nu = \text{a constant} = p,$$

therefore

$$a\sin i = \tfrac{1}{2}p\cot\nu = \text{const.}$$

But $a\sin i$ is the length of a perpendicular let fall from the vertex on the cutting plane; and therefore a sphere, described with its centre at the vertex and radius $= \frac{1}{2}p\cot\nu$, would be touched by the cutting plane in all its positions.

SECOND SOLUTION. *By Mr. O. Root.*

Let the intersections of the cone's surface with a plane through its axis, be axes of co-ordinates, then will the equation of the intersection of this section with the cutting plane be

$$y = b - ax \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Let $2\theta =$ cone's vertical angle, $n =$ given parameter $= 4p\sin^2\theta$, and A, B the transverse and conjugate diameters of the section; then from (1),

$$A = \left\{b^2 + \frac{b^2}{a^2} - \frac{2b^2}{a}\cos 2\theta\right\}^{\frac{1}{2}},\ B = \left\{\frac{4b^2}{a}\sin^2\theta\right\}^{\frac{1}{2}};$$

therefore

$$\frac{B^2}{A} = \frac{4b\sin^2\theta}{\{a^2 + 1 - 2a\cos 2\theta\}^{\frac{1}{2}}} = n = 4p\sin^2\theta,$$

and $b = p\sqrt{a^2 + 1 - 2a\cos 2\theta}$, then (1) becomes

$$y = p\sqrt{a^2 + 1 - 2a\cos 2\theta} - ax,$$

or

$$(y + ax)^2 = p^2(a^2 + 1 - 2a\cos 2\theta) \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Differentiating (2), supposing a only to vary,

$$x(y + ax) = p^2(a - \cos 2\theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Eliminating a between these two equations

$$y^2 + 2xy\cos 2\theta + x^2 = p^2\sin^2 2\theta \quad . \quad . \quad . \quad . \quad (4),$$

which is the equation of a circle, and therefore the surface is a sphere.

(77). QUESTION XI. *By James F. Macully, Esq., New-York.*

Required the sum of the infinite series

$$\frac{\sin^3 \frac{1}{4}\theta}{\cos^2\theta}+\frac{4\sin^3\frac{1}{16}\theta}{\cos^2\frac{1}{4}\theta}+\frac{16\sin^3\frac{1}{64}\theta}{\cos^2\frac{1}{16}\theta}+\&c. \ldots .$$

FIRST SOLUTION. *By Dr. Strong.*

Let s denote the sum of n terms of the series, and using Δ for the characteristic of finite differences, and putting $4^{-n}=y$, we get

$$\Delta s=\frac{4^n\sin^3 4^{-n-1}\theta}{\cos^2 4^{-n}\theta}=\frac{\sin^3\frac{1}{4}y\theta}{y\cos^2 y\theta}=\frac{1}{4y}[3\sin\tfrac{1}{4}y\theta-\sin\tfrac{3}{4}y\theta]\sec^2 y\theta \,.\, (1).$$

We shall suppose $\theta<\frac{1}{2}\pi$, then, (see Cagnoli's Trigon. pp. 53, 55, 57),

$$\tan A=A+\frac{A^3}{3}+\frac{2A^5}{3.5}+\frac{17A^7}{3^2 5.7}+\frac{62A^9}{3^2.5.7.9}+\&c.$$

$$\sec^2 A=1+A^2+\frac{2A^4}{3}+\frac{17A^6}{3^2.5}+\frac{62A^8}{3^2.5.7}+\frac{1382A^{10}}{3^2.5^2.7.9}+\&c. \quad (2)$$

$$\sin A=A-\frac{A^3}{2.3}+\frac{A^5}{2.3.4.5}-\frac{A^7}{2.3.4.5.6.7}+\&c. \quad . \quad . \quad . \quad . \quad (3).$$

Hence

$$\frac{1}{4y}\{3\sin\tfrac{1}{4}y\theta-\sin\tfrac{3}{4}y\theta\}=-\frac{3}{4^4.2.3}\left[y^2\theta^3(1-3^2)-\frac{y^4\theta^5}{4^2}\left(\frac{1-3^4}{4.5}\right)+\&c.\right]$$

$$=A''(ay^2\theta^3+by^4\theta^5+cy^6\theta^7+\&c.) \quad . \quad . \quad (4),$$

and $\quad \sec^2 y\theta=1+y^2\theta^2+Ay^4\theta^4+By^6\theta^6+\&c. \quad . \quad . \quad (5);$

where $A''=\dfrac{-3}{4^4.2.3}, a=1-3^2, b=\dfrac{3^4-1}{4^3.5}, c=\dfrac{1-3^6}{4^5.5.6.7}, d=\dfrac{3^8-1}{4^7.5\ldots 9}$, &c.

and $A=\frac{2}{3}$, $B=\dfrac{17}{3^2.5}$, $C=\dfrac{62}{3^2.5.7}$, &c.

Hence substituting (4) and (5) in (1), and putting

$$A'=a+b,\ B'=Aa+b+c,\ C'=Ba+Ab+c+d,\ \&c.$$

$$\Delta s=A''\{ay^2\theta^3+A'y^4\theta^5+B'y^6\theta^7+C'y^8\theta^9+\&c.\} \quad . \quad . \quad (6).$$

Now, using Σ as the sign of finite integrals, since $y=4^{-n}$, or $y^2=4^{-2n}$, we get $\Delta y^2=4^{-2n-2}-4^{-2n}$, or $4^{-2n}=y^2=\dfrac{\Delta y^2}{4^{-2}-1}$, therefore $\Sigma y^2=\dfrac{y^2}{4^{-2}-1}$,

and $\Sigma y^4=\dfrac{y^4}{4^{-4}-1}$, &c., and we get from (6),

$$s=A''\left[\frac{a\theta^3y^2}{4^{-2}-1}+\frac{A'\theta^5y^4}{4^{-4}-1}+\frac{B'\theta^7y^6}{4^{-6}-1}+\&c.\right]+C'' \quad . \quad . \quad (7),$$

where C'' denotes the correction of the integral. If we put $n=2$, or $y=4^{-2}$, and make $\theta=4\theta'$, we find

$$s=\frac{\sin^3\theta'}{\cos^2 4\theta'}+\frac{4\sin^3\frac{1}{4}\theta'}{\cos^2\theta'}=4A''\left[\frac{a\theta'^3}{1-4^2}+\frac{A'\theta'^5}{1-4^4}+\frac{B'\theta'^7}{1-4^6}+\&c.\right]+C''$$

$$\text{or } C''=\frac{\sin^3\theta'}{\cos^2 4\theta'}+\frac{4\sin^3\frac{1}{4}\theta'}{\cos^2\theta'}+4A''\left[\frac{a\theta'^3}{4^2-1}+\frac{A'\theta'^5}{4^4-1}+\frac{B'\theta'^7}{4^6-1}+\&c.\right]$$

$$\text{and } s = \frac{\sin^3\theta'}{\cos^2 4\theta'} + \frac{4\sin^3\frac{1}{4}\theta'}{\cos^2\theta'}$$

$$+ 4A''\left[\frac{a\theta'^3(1-y^2)}{4^2-1} + \frac{A'\theta'^5(1-y^4)}{4^4-1} + \frac{B'\theta'^7(1-y^6)}{4^6-1} + \&c.\right] (8).$$

If n is infinite, $y = 0$, and $s = c''$.

If $\theta = \frac{1}{2}\pi$, the first term is infinite, and will represent the sum.

SECOND SOLUTION. *By Professor C. Avery, Hamilton College.*

$$\text{Let} \quad \frac{\sin^3\frac{1}{4}\theta}{\cos^2\theta} + \frac{4\sin^3\frac{1}{16}\theta}{\cos^2\frac{1}{4}\theta} + \frac{16\sin^3\frac{1}{64}\theta}{\cos^2\frac{1}{16}\theta} + \&c. \ldots\ldots = A + B,$$

where A = the sum of a few of the first terms of the series, and B that of the remaining terms. Let A = sum of the first n terms, n being taken so that the arc $\frac{1}{4^{n+1}}\theta$ does not differ much from its sine, nor cos $\frac{1}{4^n}\theta$ from unity; as the series is very convergent, n will generally be very small, and the sum easily obtained by adding the several terms; then

$$B = 4^n . \frac{\theta^3}{4^{3n+3}} + 4^{n+1} . \frac{\theta^3}{4^{3n+6}} + 4^{n+2} . \frac{\theta^3}{4^{3n+9}} + \&c.$$

$$= \frac{\theta^3}{4^{2n+3}}\left(1 + \frac{1}{4^3} + \frac{1}{4^6} + \&c.\right)$$

$$= \frac{\theta^3}{63.4^{2n}}, \text{ nearly.}$$

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(78). QUESTION XII. *By* ——.

The co-ordinates of five points in space, are

1. 2, —1, 3;
2. 3, —2, 5;
3. 1, 5, —2;
4. —3, 0, 7;
5. —7, 4, —1.

It is required to find the volume of the polyedron which has its angles at these points.

FIRST SOLUTION. *By Mr. J. B. Henk.*

Change the origin of co-ordinates to the first point, and the ordinates become

(0,0,0),(1,—1, 2),(—1, 6,—5),(—5,1, 4),(—9, 5,—4).

The polyedron may be divided into two triangular pyramids, each having its vertex at the origin. But according to the formula of M. Cauchy, a triangular pyramid, having its vertex at the origin and the points of its base represented by $x_0y_0z_0$, $x_1y_1z_1$, $x_2y_2z_2$, has for its solidity

$$\tfrac{1}{6}(x_0y_1z_2 - x_0y_2z_1 + x_1y_2z_0 - x_1y_0z_2 + x_2y_0z_1 - x_2y_1z_0),$$

the points being so taken, that a radius vector, having one end fixed at the origin of co-ordinates, and passing over the faces of the solid angle

at the point, may, by a motion of direct rotation, touch the three points in the order of the indices of the co-ordinates, 0, 1, 2. Thus, for the base of the first pyramid we shall take the fourth, second and third points in order, and for the second pyramid the third, fifth and fourth points in order; so that

$$\begin{array}{lll|lll} \multicolumn{3}{c|}{\text{for the first pyramid}} & \multicolumn{3}{c}{\text{for the second pyramid}} \\ x_0=-5, & x_1=1, & x_2=-1, & x_0=-1, & x_1=-9, & x_2=-5 \\ y_0=1, & y_1=-1, & y_2=6, & y_0=6, & y_1=5, & y_2=1, \\ z_0=4, & z_1=2, & z_2=-5. & z_0=-5, & z_1=-4, & z_2=4. \end{array}$$

Hence, first pyramid $=\frac{1}{6}(-25+60+24+5-2-4)=9\frac{2}{3}$,
and second pyramid $=\frac{1}{6}(-20-4+45+216+120-125)=38\frac{2}{3}$,
and the whole polyedron $=9\frac{2}{3}+38\frac{2}{3}=48\frac{1}{3}$.

SECOND SOLUTION. *By Mr. Geo. R. Perkins, Clinton Liberal Institute.*

The polyedron may have six triangular faces, or it may have one quadrangular and four triangular faces. To determine which, we must find the equations of all the different planes made to pass through every three of the five points; then if, among these equations we find two alike, it will prove one of the faces to be a quadrilateral. In this way it is found that the plane through the 1st, 2d, and 4th points, is the same as that through the 1st, 2d, and 5th, its equation being

$$3x+7y+2z-5=0.$$

Therefore these four points are in one plane; the polyedron is a quadrangular pyramid having its vertex at the 3d point, and its volume is found by multiplying the area of its quadrilateral base by $\frac{1}{3}$ of the height or perpendicular through the 3d point upon the above plane. By dividing the base into two triangles whose sides are given from the co-ordinates of their angular points, we easily find its area $=5\sqrt{62}$, and the perpendicular upon it from the vertex is found by the usual formulas $=\dfrac{29}{\sqrt{62}}$: hence the solidity $=5\sqrt{62}\times\dfrac{29}{3\sqrt{62}}=48\frac{1}{3}$.

THIRD SOLUTION. *By Mr. L. Abbott, Niles, N. Y.*

Denote the five points in space respectively by (1), (2), (3), (4), (5). Then the equation of a plane through the points (2), (3), 5 is found to be

$$y+z=3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c),$$

which is perpendicular to the plane of $y\,z$. If lines be drawn through the points (1) and (4) perpendicular to the plane of $x\,y$, they will pierce the plane at points which have severally $z=4$, $z=3$, while the points themselves have $z=3$, and $z=7$, therefore they are on different sides of the plane c, which must hence divide the polyedron into two triangular pyramids, whose common base is the triangle (2), (3), (5), and their vertices are at (1) and (4). The three sides of the triangle are found to be $\sqrt{102}$, $\sqrt{172}$, $\sqrt{66}$, and its area $=29\sqrt{2}$; the perpendicular from (1) upon (c) $=\dfrac{1}{\sqrt{2}}$, and the perpendicular from (4) upon (c) $=\dfrac{4}{\sqrt{2}}$. Hence the sum of the two pyramids, or the whole polyedron,

$$=\frac{29\sqrt{2}}{3}\left(\frac{1}{\sqrt{2}}+\frac{4}{\sqrt{2}}\right)=\frac{5.29}{3}=48\tfrac{1}{3}.$$

(79). QUESTION XIII. *By* ——.

Find the sum of the reciprocals of the radii, the sum of the radii, and the sum of the areas, of the n tangent circles described as in Question (50). See page 245 of the Mathematical Miscellany, Number IV.

SOLUTION. *By the Editor.*

Taking the value of r_x given in the equation (20), page 247,

$$r_x = \frac{2(s-r)(R-s)}{R-r-d\cos 2(x\beta+\theta)}$$

$$= \frac{(R-r)^2-d^2}{2(R-r)-2d\cos 2(x\beta+\theta)}$$

$$= \frac{P}{k+2\cos 2(x\beta+\theta)},$$

by restoring the value of s, and putting

$$P = \frac{d^2-(R-r)^2}{d},\ k = 2.\frac{r-R}{d}\quad . \ : \ . \ . \ . \ . \ . \ . \quad (1).$$

Using the symbol Σu_x to represent the sum of n terms of the series

$$u_1+u_2+u_3+\ldots\ldots u^n$$

$$P\Sigma\left(\frac{1}{r_x}\right) = \Sigma\{k+2\cos 2(x\beta+\theta)\}$$

$$= kn+2\Sigma\cos 2(x\beta+\theta)$$

$$= kn+\frac{2\sin n\beta\cos(\overline{n+1}.\beta+2\theta)}{\sin\beta}.$$

But, by equation (24),

$$\sin n\beta = \sin i\pi = 0,$$

therefore

$$\Sigma\left(\frac{1}{r_x}\right) = \frac{nk}{P} = \frac{2n(r-R)}{d^2-(R-r)^2}$$

$$= \frac{n}{2}\left(\frac{1}{r}-\frac{1}{R}\right)\cot^2\frac{i\pi}{n}\quad . \ . \ . \ . \ . \ . \quad (2),$$

by substituting the value of d given in equation (25), which is the sum of the reciprocals of the radii; and when the given circles are placed without each other

$$\Sigma\left(\frac{1}{r_x}\right) = \frac{n}{2}\left(\frac{1}{r}+\frac{1}{R}\right)\cot^2\frac{i\pi}{n}.$$

Now if we put

$$\frac{1}{k+2\cos s} = \tfrac{1}{2}a_0+a_1\cos s+a_2\cos 2s+a_3\cos 3s+\&c.,$$

multiply both members of the equation by $k+2\cos s$, and assimilate the terms by the formula.

$$2\cos s\cos is = \cos(i-1)s+\cos(i+1)s,$$

and then equate the co-efficients of like terms, we have

$$a_1+\tfrac{1}{2}ka_0-1=0,$$
$$a_2+ka_1+a_0=0,$$
$$a_3+ka_2+a_1=0,$$
$$\&c.$$
$$a_x+ka_{x-1}+a_{x-2}=0.$$

If we take $a_x = h^x$, this last equation will give

$$h^2 + kh + 1 = 0,$$

and $$h = -\tfrac{1}{2}k \pm \sqrt{\tfrac{1}{4}k^2 - 1}.$$

Hence, if $$-\tfrac{1}{2}k + \sqrt{\tfrac{1}{4}k^2 - 1} = h^{-1} = \frac{\mathrm{R} - r + \sqrt{(\mathrm{R}-r)^2 - d^2}}{d} \quad . \quad . (3)$$

then $$-\tfrac{1}{2}k - \sqrt{\tfrac{1}{4}k^2 - 1} = h,$$

and the complete value of a_x will be

$$a_x = c_1 h^x + c_2 h^{-x},$$

c_1 and c_2 being constant quantities, depending on k. Then

$$\frac{1}{k + 2\cos s} = c_1(\tfrac{1}{2} + h\cos s + h^2\cos 2s + h^3\cos 3s + \&c.)$$
$$+ c_2(\tfrac{1}{2} + h^{-1}\cos s + h^{-2}\cos 2s + h^{-3}\cos 3s + \&c.);$$

But since either h or h^{-1} is greater than unity, one of these series must be divergent, and we must have the corresponding constant $= 0$. Let $c_2 = 0$, then since $a_0 = c_1$, and $a_1 = c_1 . h$, the first of the equations of condition becomes, after writing for k its value $-\dfrac{h^2+1}{h}$,

$$hc_1 - \frac{h^2+1}{2h}.c_1 - 1 = 0,$$

or $$c_1 = \frac{2h}{h^2 - 1}.$$

$$\therefore \frac{1}{k + 2\cos s} = \frac{2h}{h^2-1}(\tfrac{1}{2} + h\cos s + h^2\cos 2s + h^3\cos 3s + \&c.) (4).$$

Hence the sum of the radii of the given circles is

$$\Sigma(r_x) = \mathrm{P}\Sigma\left(\frac{1}{k + 2\cos 2(x\beta+\theta)}\right)$$
$$= \frac{2\mathrm{P}h}{h^2-1}.\Sigma\{\tfrac{1}{2} + h\cos 2(x\beta+\theta) + h^2\cos 4(x\beta+\theta) + h^3\cos 6(x\beta+\theta) + \&c.\}$$
$$= \frac{2\mathrm{P}h}{h^2-1}\{\tfrac{1}{2}n + h\Sigma\cos 2(x\beta+\theta) + h^2\,\Sigma\cos 4(x\beta+\theta) + \&c.\}.$$

But, as is well known,

$$\Sigma\cos p(x\beta+\theta) = \frac{\sin\tfrac{1}{2}pn\beta\cos p(\overline{\tfrac{1}{2}n+1}\beta+\theta)}{\sin\tfrac{1}{2}p\beta},$$

and any term in the preceding series may be represented by

$$u_{y+1} = h^y.\frac{\sin ny\beta\cos y(\overline{n+1}\beta + 2\theta)}{\sin y\beta}$$
$$= h^y\{\cot y\beta\sin ny\beta\cos y(n\beta + 2\theta) - \sin ny\beta\sin y(n\beta + 2\theta)\}.$$

But $n\beta = i\pi$, and $\sin ny\beta = \sin iy\pi = 0$, so that the second term will always disappear; and the first also, except in the cases where

$$y = mn,$$

m being any integer, and then the first term within the brackets becomes

$$\frac{\sin ny\beta\cos y(n\beta + 2\theta)}{\tan y\beta} = \frac{0}{0};$$

and by differentiating the two terms of this vanishing fraction, and then

putting $y = mn$, recollecting that $\sin mn\beta = 0$, and $\cos mn\beta = \pm 1$, we find its value

$$= n \cos^2 mn\beta \cos 2mn(n\beta + \theta)$$
$$= n \cos 2mn\theta;$$

then the series for $\Sigma(r_x)$ becomes

$$\Sigma(r_x) = \frac{2\text{P}nh}{h^2 - 1}\{\tfrac{1}{2} + h^n \cos 2n\theta + h^{2n} \cos 4n\theta + h^{3n} \cos 6n\theta + \&c.\}.$$

But we have seen above that

$$\tfrac{1}{2} + h \cos s + h^2 \cos 2s + \&c. = \frac{h^2-1}{2h} \cdot \frac{1}{k + 2\cos s} = \tfrac{1}{2} \cdot \frac{1 - h^2}{1 - 2h\cos s + h^2};$$

$$\therefore \tfrac{1}{2} + h^n \cos 2n\theta + h^{2n} \cos 4n\theta + \&c. = \tfrac{1}{2} \cdot \frac{1 - h^{2n}}{1 - 2h^n \cos 2n\theta + h^{2n}},$$

and $$\Sigma(r_x) = \frac{\text{P}nh}{h^2 - 1} \cdot \frac{1 - h^{2n}}{1 - 2h^n \cos 2n\theta + h^{2n}} \quad . \quad . \quad . \quad (5).$$

In the case where one circle is within the other, since $d < \text{R} - r$, put

$$d = (\text{R} - r) \sin 2\omega,$$

so that $$d^2 = (\text{R} - r)^2 \sin^2 2\omega = (\text{R} - r)^2 - 4\text{R}r \tan^2 \frac{i\pi}{n},$$

and $$\cos 2\omega = \frac{2\sqrt{\text{R}r}}{\text{R} - r} \cdot \tan \frac{i\pi}{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Then $$h = \frac{\text{R} - r - \sqrt{(\text{R} - r)^2 - d^2}}{d}$$
$$= \frac{1 - \cos 2\omega}{\sin 2\omega}$$
$$= \tan \omega,$$

and $$\text{P} = -(\text{R} - r)\frac{\cos^2 2\omega}{\sin 2\omega}$$
$$= -2\sqrt{\text{R}r} \tan \frac{i\pi}{n} \cot 2\omega;$$

hence, if Ω be such an angle that

$$\tan \tfrac{1}{2}\Omega = \tan {}^n\omega,$$

$$\Sigma(r_x) = n\sqrt{\text{R}r} \cdot \tan \frac{i\pi}{n} \cdot \frac{\cos \Omega}{1 - \sin \Omega \cos 2n\theta} \quad . \quad . \quad . \quad . \quad (7).$$

In the case where one circle is without the other, and $d > \text{R} + r$, put

$$\text{R} + r = d \cos \psi,$$

so that $$(\text{R} + r)^2 = \cos^2 \psi \left\{(\text{R} + r)^2 + 4\text{R}r \tan^2 \frac{i\pi}{n}\right\},$$

and $$\tan \psi = \frac{2\sqrt{\text{R}r}}{\text{R} + r} \cdot \tan \frac{i\pi}{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Then $$\text{P} = d \sin^2 \psi = 2\sqrt{\text{R}r} \sin \psi \tan \frac{i\pi}{n},$$

$$h = \frac{-(\text{R} + r) - \sqrt{(\text{R} + r)^2 - d^2}}{d}$$
$$= -\cos \psi - \sin \psi \sqrt{-1},$$
$$= -e^{\psi\sqrt{-1}}$$

$$\therefore\ \Sigma(r_x) = n\sqrt{\text{R}r}\,.\tan\frac{i\pi}{n}.\frac{\sin n\psi}{\cos n\psi - (-1)^n\cos 2n\theta}\ .\quad .\ (9).$$

Equation (9) will not give the numerical sum of the radii in the last case since, as we have seen (page 249) some of them are negative.

The area of one of the circles is

$$r_x^2\pi = \text{P}^2\pi\{k + 2\cos 2(x\beta + \theta)\}^{-2}.$$

But
$$\frac{dr_x}{dk} = -\text{P}\{k + 2\cos 2(x\beta + \theta)\}^{-2},$$

therefore
$$r_x^2\pi = -\text{P}\pi.\frac{dr_x}{dk},$$

and
$$\Sigma(r_x^2\pi) = -\text{P}\pi\Sigma\left(\frac{dr_x}{dk}\right)$$

$$= -\text{P}\pi.\frac{d.\Sigma(r_x)}{dk}$$

$$= \text{P}\pi h(h - h^{-1})^{-1}.\frac{d.\Sigma(r_x)}{dh}.$$

Hence, putting the value of $\Sigma(r_x)$ in (5) into the form

$$\Sigma(r_x) = \text{P}n(h - h^{-1})^{-1}(h^{-n} - h^n)(h^n + h^{-n} - 2\cos 2n\theta)^{-1},$$

$$\frac{d\Sigma(r_x)}{dh} = -\text{P}nh^{-1}(h - h^{-1})^{-2}(h + h^{-1})(h^{-n} - h^n)(h^n + h^{-n} - 2\cos 2n\theta)^{-1}$$

$$- \text{P}n^2h^{-1}(h - h^{-1})^{-1}(h^{-n} + h^n)(h^n + h^{-n} - 2\cos 2n\theta)^{-1}$$

$$+ \text{P}n^2h^{-1}(h - h^{-1})^{-1}(h^{-n} - h^n)^2(h^n + h^{-n} - 2\cos 2n\theta)^{-2}$$

$$= -\frac{h + h^{-1}}{h - h^{-1}}.h^{-1}\Sigma(r_x) - \frac{h^{-n} + h^n}{h^{-n} - h^{-n}}.nh^{-1}\Sigma(r_x) + \frac{h - h^{-1}}{\text{P}h}.\{\Sigma(r_x)\}^2,$$

$$\therefore \Sigma(r_x^2\pi) = \frac{-\text{P}\pi}{h - h^{-1}}\left\{\frac{h + h^{-1}}{h - h^{-1}} + n.\frac{h^{-n} + h^n}{h^{-n} - h^n}\right\}\Sigma(r_x) + \pi\{\Sigma(r_x)\}^2\ .\quad .\ (10).$$

So that, when one of the given circles is placed within the other,

$$\Sigma(r_x^2\pi) = \pi n\text{R}r\tan^2\frac{i\pi}{n}\left[\frac{\sec 2\omega\cos\Omega - n}{1 - \sin\Omega\cos 2n\theta} + \frac{n\cos^2\Omega}{(1 - \sin\Omega\cos 2n\theta)^2}\right]$$

and, when one is without the other,

$$\Sigma(r_x^2\pi) = \pi n\text{R}r\tan^2\frac{i\pi}{n}\left[\frac{\cot\psi\sin n\psi - n\cos n\psi}{\cos n\psi - (-1)^n\cos 2n\theta} + \frac{n\sin^2 n\psi}{(\cos n\psi - (-1)^n\cos 2n\theta)^2}\right].$$

Corollary. If the first tangent circle be so placed that (page 247)

$$2n\theta = (k + \tfrac{1}{2})\pi,\ \text{or}\ \tan\tfrac{1}{2}\phi_1 = c\tan(2i + k + \tfrac{1}{2})\frac{\pi}{n},$$

k being any integer, then

$$\Sigma(r_x) = n\sqrt{\text{R}r}.\tan\frac{i\pi}{n}\cos\Omega,$$

$$\text{or}\quad = n\sqrt{\text{R}r}\tan\frac{i\pi}{n}\tan n\psi;$$

$$\text{and}\ \Sigma(r_x^2\pi) = \pi n\text{R}r\tan^2\frac{i\pi}{n}(\sec 2\omega\cos\Omega - n\sin^2\Omega),$$

$$\text{or}\quad = \pi n\text{R}r\tan^2\frac{i\pi}{\pi}(\cot\psi\tan n\psi - n + n\tan^2 n\psi).$$

(80). QUESTION XIV. (*Communicated by Professor Peirce.*)
From Talbot's Researches in the Integral Calculus, Phil. Trans. Lond., 1836.

1°. Find such an equation between x and y that

$$\int dx\sqrt{1+x^n}+\int dy\sqrt{1+y^n}$$

may be expressed algebraically; n being either 3, 4 or 6.*

2°. Find two such equations between x, y and z that

$$\int dx\sqrt{1+x^n}+\int dy+\sqrt{1+y^n}+\int dz\sqrt{1+z^n}$$

may be expressed algebraically; n being 3, 4, 5, 6, 8, or 10.

FIRST SOLUTION. *By Professor B. Peirce.*

PART I.

1. When $n=3$, make

$$1+x^3=(1+vx)^2=1+2vx+v^2x^2,$$
$$\text{or}\qquad x^2-v^2x-2v=0,$$

whence, $\quad x+y=v^2,\quad xy=-2v;$

and, eliminating v,

$$4(x+y)=x^2y^2.$$

We have also

$$dx\sqrt{1+x^3}+dy\sqrt{1+y^3}=d(x+y)+\tfrac{1}{2}vd(x^2+y^2)$$
$$=2v^4dv+4vdv,$$

so that $\int dx\sqrt{1+y^3}+\int dx\sqrt{1+y^3}=\frac{2}{5}v^5+2v^2+c$

$$=\tfrac{1}{80}x^5y^5+2(x+y)+c.$$

2. When $n=4$, make $x^2=t$, and we have

$$dx\sqrt{1+x^4}=\tfrac{1}{2}dt\sqrt{\frac{1+t^2}{t}}.$$

Make $\quad \dfrac{1+t^2}{t}=v^2,\ \text{or}\ t^2-v^2t+1=0;$

whence $\quad x^2+y^2=v^2,\quad yx=1,$

the latter of which is the only necessary equation between x and y because v is arbitrary. We have also

$$dx\sqrt{1+x^4}+dy\sqrt{1+y^4}=\tfrac{1}{2}vd(x^2+y^2)=v^2dv;$$

so that $\int dx\sqrt{1+x^4}+\int dy\sqrt{1+y^4}=\frac{1}{3}v^3+c=\frac{1}{3}(x^2+y^2)^{\frac{3}{2}}+c.$

3. When $n=6$, make $x^2=t$, and we have

$$dx\sqrt{1+x^6}=\tfrac{1}{2}dt\sqrt{\frac{1+t^3}{t}}.$$

Make $\quad \dfrac{1+t^3}{t}=(v-t)^2,\ \text{or}\ t^2-\tfrac{1}{2}vt+\dfrac{1}{2v}=0;$

whence $\quad x^2+y^2=\tfrac{1}{2}v,\quad x^2y^2=\dfrac{1}{2v};$

therefore $\quad 4x^2y^2(x^2+y^2)=1.$

We have also,

$$dx.\sqrt{1+x^6}+dy\sqrt{1+y^6}=\tfrac{1}{2}vd.(x^2+y^2)-\tfrac{1}{4}d.(x^4+y^4)$$
$$=\tfrac{1}{8}vdv-\frac{dv}{4v^2},$$

---

* This was written "5" instead of "6" in Number 4; a press error for which we have to beg the indulgence of our correspondents.

so that $\int dx\sqrt{1+x^6}+\int dy\sqrt{1+y^6}=\frac{1}{16}v^2+\frac{1}{2v}+c$

$$=\tfrac{1}{4}(x^2+y^2)^2+x^2y^2+c.$$

PART II.

1. When $n=3$; make

$$1+x^3=(v-v'x)^2, \text{ or } x^3-v'^2x+2vv'x-(v^2-1)=0;$$

whence
$$x+y+z=v'^2,$$
$$xy+xz+yz=2vv',$$
$$xyz=v^2-1;$$

and eliminating v and v',

$$(xy+xz+yz)^2=4(1+xyz)(x+y+z),$$

which is the only equation between x, y, z involved in the preceding, and with which may be combined any other equation necessary to render the integral exact, when necessary. We have then

$$dx\sqrt{1+x^3}+dy\sqrt{1+y^3}+dz\sqrt{1+z^3}=vd.(x+y+z)-\tfrac{1}{2}v'd.(x^2+y^2+z^2)$$
$$=-2v'^4dv'+2v'^2dv+4vv'dv'$$

$$\therefore \int dx\sqrt{1+x^3}+\int dy\sqrt{1+y^3}+\int dz\sqrt{1+z^3}=-\tfrac{2}{5}v'^5+2v'^2v+c$$
$$=(x+y+z)^{\frac{1}{2}}[xy+xz+yz-\tfrac{2}{5}(x+y+z)^2)]+c.$$

2. When $n=4$; make $x^2=t$, and we have

$$dx\sqrt{1+x^4}=\tfrac{1}{2}dt\sqrt{\frac{1+t^2}{t}}.$$

Make $\dfrac{1+t^2}{t}=\left(\dfrac{v-t}{v'}\right)^2$, or $t^3-(2v+v'^2)t^2+v^2t-v'^2=0$;

whence
$$x^2+y^2+z^2=2v+v'^2,$$
$$x^2y^2+x^2z^2+y^2z^2=v^2,$$
$$x^2y^2z^2=v'^2;$$

and eliminating v and v',

$$(x^4+y^2+z^2-x^2y^2z^2)^2=4(x^2y^2+x^2z^2+y^2z^2).$$

We have, also,

$$dx\sqrt{1+x^4}+dy\sqrt{1+y^4}+dz\sqrt{1+z^4}=\frac{vd.(x^2+y^2+z^2)-\frac{1}{2}d.(x^4+y^4+z^4)}{2v'}$$
$$=-v'^2dv'-v'dv-vdv',$$

and $\int dx\sqrt{1+x^4}+\int dy\sqrt{1+y^4}+\int dz\sqrt{1+z^4}=c-vv'-\tfrac{1}{3}v'^3$
$$=c+\tfrac{1}{6}x^3y^3z^3-\tfrac{1}{2}xyz(x^2+y^2+z^2).$$

3. When $n=5$; make

$$1+x^5=(1+vx^2)^2, \quad \text{or} \quad x^3-v^2x^2-2v=0.$$

Whence
$$x+y+x+z=v^2,$$
$$xy+xz+yz=0,$$
$$xyz=2v;$$

and the two equations between x, y, z, are

$$xy+xz+yz=0,$$
$$4(x+y+z)=xyz.$$

We have also

$$dy\sqrt{1+x^5}+dy\sqrt{1+y^5}+dz\sqrt{1+z^5}=d.(x+y+z)+\tfrac{1}{2}vd.(x^2+y^2+z^2)$$
$$=2v^5dv+4vdv,$$

so that $\int dx\sqrt{1+x^5}+\int dy\sqrt{1+y^5}+\int dz\sqrt{1+z^5}=\tfrac{2}{7}v^7+2v^2+c$
$$=\tfrac{1}{448}x^7y^7z^7+\tfrac{1}{2}x^2y^2z^2+c.$$

4. When $n=6$; make $x^2=t$, and we have

$$dx\sqrt{1+x^6}=\tfrac{1}{2}dt\sqrt{\frac{1+t^3}{t}},$$

Make $\dfrac{1+t^3}{t}=(v-v't)^2$, or $t^3+\dfrac{2vv'}{1-v'^2}.t^2-\dfrac{v^2}{1-v'^2}.t+\dfrac{1}{1-v'^2}=0$,

whence
$$x^2+y^2+z^2=\frac{2vv'}{v'^2-1},$$
$$x^2y^2+x^2z^2+y^2z^2=\frac{v^2}{v'^2-1},$$
$$x^2y^2z^2=\frac{1}{v'^2-1},$$

and eliminating v and v',
$$(x^2+y^2+z^2)^2=4(x^2y^2+x^2z^2+y^2z^2)(1+x^2y^2z^2).$$

We have then
$$\int dx\sqrt{1+x^6}+\int dy\sqrt{1+y^6}+\int dz\sqrt{1+z^6}$$
$$=\tfrac{1}{2}\int vd.(x^2+y^2+z^2)-\tfrac{1}{4}\int v'd.(x^4+y^4+z^4)$$
$$=\tfrac{1}{2}v(x^2+y^2+z^2)-\tfrac{1}{4}v'(x^4+y^4+z^4)-\tfrac{1}{4}\int\{2(x^2+y^2+z^2)dv-(x^4+y^4+z^4)dv'\}$$
$$=\frac{v^2v'^3-3v'^2}{2(v'^2-1)^2}-\tfrac{1}{2}\int\Big\{\frac{v'}{v'^2-1}.2vdv+v^2d.\Big(\frac{v'}{v'^2-1}\Big)\Big\}$$
$$=c-\frac{v^2v'}{(v'^2-1)^2}$$
$$=c-xyz(x^2y^2+x^2z^2+y^2z^2)\sqrt{1+x^2y^2z^2}.$$

5. When $n=8$; make $x^2=t$, we have

$$dx\sqrt{1+x^8}=\tfrac{1}{2}dt\sqrt{\frac{1+t^4}{t}}.$$

Make $\dfrac{1+t^4}{t}=\Big(\dfrac{1\pm t\sqrt{2}+t^2}{v}\Big)^2$, or $t^3-(v^2\mp\sqrt{2})t^2+(1\pm v^2\sqrt{2})t-v^2=0$;

whence
$$x^2+y^2+z^2=v^2\mp\sqrt{2},$$
$$x^2y^2+x^2z^2+y^2z^2=1\pm v^2\sqrt{2},$$
$$xyz=v;$$

and, eliminating v,
$$x^2+y^2+z^2=x^2y^2z^2\mp\sqrt{2},$$
$$x^2y^2+x^2z^2+y^2z^2=1\pm x^2y^2z^2.\sqrt{2}.$$

We have, then,
$$dx\sqrt{1+x^8}+dy\sqrt{1+y^8}+dz\sqrt{1+z^8}=v^4dv\mp3\sqrt{2}v^2dv+dv,$$
$$\int dx\sqrt{1+x^8}+\int dy\sqrt{1+y^8}+\int dz\sqrt{1+z^8}=\tfrac{1}{5}v^5\mp\sqrt{2}v.^3+v+c$$
$$=\tfrac{1}{5}x^5y^5z^5\mp\sqrt{2}.x^3y^3z^3+xyz+c.$$

6. When $n=10$, make $x^2=t$, and we have

$$dx\sqrt{1+x^{10}}=\tfrac{1}{2}dt.\sqrt{\frac{1+t^5}{t}}.$$

Make
$$\frac{1+t^5}{t}=(v+t^2)^2,\text{ or }t^3+\tfrac{1}{2}vt-\frac{1}{2v}=0.$$

whence $x^2+y^2+z^2=0$, $\quad x^2y^2+x^2z^2+y^2z^2=\tfrac{1}{2}v$, $\quad x^2y^2z^2=\dfrac{1}{2v}$,

and, eliminating v,

$$x^2+y^2+z^2=0,$$
$$4x^2y^2z^2(x^2y^2+x^2z^2+y^2z^2)=1,$$

which necessarily involve imaginary quantities. We have then

$$dx\sqrt{1+x^{10}}+dy\sqrt{1+y^{10}}+dz\sqrt{1+z^{10}}=\tfrac{1}{2}d.(x^6+y^6+z^6)=d.\left(\frac{1}{2v}\right),$$

$$\textstyle\int dx\sqrt{1+x^{10}}+\int dy\sqrt{1+z^{10}}+\int dz\sqrt{1+z^{10}}=\frac{1}{2v}+c=x^2y^2z^2+c.$$

Imaginary quantities would be avoided by making

$$\frac{1+t^5}{t}=v^2[1-\tfrac{1}{2}(1\pm\sqrt{5})t+t^2]^2,$$

$$\text{or } t^3-\frac{v^2+1}{2(v^2-1)}.(1\pm\sqrt{5})t^2+\frac{2v^2-1\mp\sqrt{5}}{2(v^2-1)}t-\frac{1}{v^2-1}=0;$$

whence

$$x^2+y^2+z^2=\tfrac{1}{2}(1\pm\sqrt{5}).\frac{v^2+1}{v^2-1},$$
$$x^2y^2+x^2z^2+y^2z^2=1+\frac{1\mp\sqrt{5}}{2(v^2-1)},$$
$$x^2y^2z^2=\frac{1}{v^2-1};$$

or, eliminating v,

$$x^2+y^2+z^2=\tfrac{1}{2}(1\pm\sqrt{5})(2x^2y^2z^2+1),$$
$$x^2y^2+x^2z^2+y^2z^2=1+\tfrac{1}{2}(1\mp\sqrt{5})x^2y^2z^2,$$

We have then

$$\textstyle\int dx\sqrt{1+x^{10}}+\int dy\sqrt{1+y^{10}}+\int dz\sqrt{1+z^{10}}$$
$$=\tfrac{1}{3}xyz\sqrt{1+x^2y^2z^2}\,\{4(2\pm\sqrt{5})x^4y^4z^4+(11\pm4\sqrt{5})x^2y^2z^2+3\}+c.$$

7. *We will now give another method for obtaining the sum of a required number of integrals, by first obtaining the sum for a greater number;* and it will easily be understood by the following application to the last example.

Make

$$\frac{1+t^5}{t}=(v-v't+t^2)^2,$$

$$\text{or}\quad t^4-\left(\tfrac{1}{2}v'+\frac{v}{v'}\right)t^3+vt^2-\frac{v^2}{2v'}t+\frac{1}{2v'}=0,$$

of which there being four roots, we have

$$x^2+y^2+z^2+u^2=\tfrac{1}{2}v'+\frac{v}{v'},$$
$$x^2y^2+x^2z^2+x^2u^2+y^2z^2+y^2u^2+z^2u^2=v,$$
$$x^2y^2z^2+x^2y^2u^2+x^2z^2u^2+y^2z^2u^2=\frac{v^2}{2v'},$$
$$x^2y^2z^2u^2=\frac{1}{2v'},$$

or, eliminating v and v', and using S as a sign to express the sum of all the quantities of the same kind,

$$4S(x^2).x^2y^2z^2u^2=1+8S(x^2y^2).x^4y^4z^4u^4,$$
$$S(x^2y^2z^2)=\{S(x^2y^2)\}^2.x^2y^2z^2u^2.$$

We have also

$$\begin{aligned}\int S(dx\sqrt{1+x^{10}}) &= \tfrac{1}{2}\int v dS(x^2) - \tfrac{1}{4}\int v' d.S(x^4) + \tfrac{1}{6}\int d.S(x^6)\\ &= \tfrac{1}{2}vS(x^2) - \tfrac{1}{4}v'S(x^4) + \tfrac{1}{6}S(x^6) - \tfrac{1}{4}\int[2S(x^2)dv - S(x^4)dv']\\ &= \tfrac{1}{2}vS(x^2) - \tfrac{1}{4}v'S(x^4) + \tfrac{1}{6}S(x^6)\\ &\quad - \tfrac{1}{4}\int\left[v'dv + \frac{2vdv}{v'} - \tfrac{1}{4}v'^2dv' + vdv' - \frac{v^2dv'}{v'^2}\right]\\ &= \tfrac{1}{2}vS(x^2) - \tfrac{1}{4}v'S(x^4) + \tfrac{1}{6}S(x^6) - \tfrac{1}{4}vv' - \frac{v^2}{4v'} + \tfrac{1}{48}v'^3 + \text{c}\\ &= \tfrac{1}{6}S(x^6) + \tfrac{1}{2}S(x^2y^2z^2) - \frac{S(x^4)}{8x^2y^2z^2u^2} + \frac{1}{384x^6y^6z^6u^6} + \text{c}.\end{aligned}$$

We may now take $u =$ a const., and we shall have left two equations between three variables x, y, z; and the integral corresponding to u may be considered as included in the arbitrary constant which is added to complete the sum of the other three integrals. This method will be found very useful in such problems as *finding the sum of two or three arcs of a curve.*

Second Solution. *By Dr. T. Strong.*

The given expressions, by integration by parts, are easily changed to

$$\frac{2}{n+2}\left\{x\sqrt{1+x^n} + y\sqrt{1+y^n}\right\} + \frac{n}{2(n+2)}\left\{\int\frac{dx}{\sqrt{1+x^n}} + \int\frac{dy}{\sqrt{1+y^n}}\right\} \quad . \ (1),$$

$$\frac{2}{n+2}\left\{x\sqrt{1+x^n} + y\sqrt{1+y^n} + z\sqrt{1+z^n}\right\} + \frac{n}{2(n+2)}\left\{\int\frac{dx}{\sqrt{1+x^n}} + \int\frac{dy}{\sqrt{1+y^n}} + \int\frac{dz}{\sqrt{1+z^n}}\right\} \quad . \ (2).$$

Put

$$\int\left(\frac{dx}{\sqrt{1+x^n}} + \frac{dy}{\sqrt{1+y^n}}\right) = c = \textit{const.},$$

$$\int\left(\frac{dx}{\sqrt{1+x^n}} + \frac{dx}{\sqrt{1+y^n}} + \frac{dx}{\sqrt{1+z^n}}\right) = c' = \textit{const.},$$

and (1) and (2) will be expressed in an algebraic form, as required; then

$$\frac{dx}{\sqrt{1+x^n}} + \frac{dy}{\sqrt{1+y^n}} = 0 \ . \ (3), \qquad \frac{dx}{\sqrt{1+x^n}} + \frac{dy}{\sqrt{1+y^n}} + \frac{dz}{\sqrt{1+z^n}} = 0 \ . \ . \ (4),$$

from which we are to find the relations between x, y, and between x, y, z; which relations we suppose are to be expressed algebraically when, in (3), $n = 3, 4$ or 6, and in (4), $n = 3, 4, 5, 6, 8$, or 10.

The case of $n = 6$, reduces to that of $n = 3$; for when $n = 6$, (3) becomes

$$\frac{dx}{\sqrt{1+x^6}} + \frac{dy}{\sqrt{1+y^6}} = \frac{dx}{x^3\sqrt{1+x^{-6}}} + \frac{dy}{y^3\sqrt{1+y^{-6}}} = 0,$$

therefore, if for x^{-2}, y^{-2}, we put x and y, the equation becomes

$$\frac{dx}{\sqrt{1+x^3}} + \frac{dy}{\sqrt{1+y^3}} = 0,$$

and a similar reduction applies to (4); again, when $n=8$ or 10 in (4), if we change x^2, y^2, z^2 into x, y, z, we shall have

$$\frac{dx}{\sqrt{x+x^5}}+\frac{dy}{\sqrt{y+y^5}}+\frac{dz}{\sqrt{z+z^5}}=0, \text{ or } \frac{dx}{\sqrt{x+x^6}}+\frac{dy}{\sqrt{y+y^6}}+\frac{dz}{\sqrt{z+z^6}}=0 \quad (5)$$

To find the algebraic integral of (3) in the cases specified, we shall use the method given by Lacroix, *Calcul Diff. et Int.*, Vol. 2, pp. 475, 476. Imagine x and y to be functions of the independent variable t, and that $dt=$ const., then assume

$$\frac{dx}{dt}=\sqrt{1+x^n}, \frac{dy}{dt}=-\sqrt{1+y^n} \quad \ldots \ldots \ldots (6),$$

and (3) is satisfied; therefore $\frac{dx^2}{dt^2}=1+x^n$, $\frac{dy^2}{dt^2}=1+y^n$, or

$$\frac{d^2x}{dt^2}=\frac{n}{2}.x^{n-1}, \frac{d^2y}{dt^2}=\frac{n}{2}.y^{n-1} \quad \ldots \ldots \ldots (7).$$

Put $\quad x+y=p,\ x-y=q,$ or $x=\frac{1}{2}(p+q), y=\frac{1}{2}(p-q) \quad . . (8),$

then $$\frac{d^2p}{dt^2}=\frac{n}{2^n}\{(p+q)^{n-1}+(p-q)^{n-1}\} \quad \ldots (9);$$

also $$\frac{dpdq}{dt^2}=\frac{dx^2}{dt^2}-\frac{dy^2}{dt^2}=x^n-y^n=\frac{1}{2^n}\{(p+q)^n-(p-q)^n\} \quad . . (10).$$

Therefore

$$\frac{qd^2p-dp\,dq}{dt^2}=\frac{nq}{2^n}\{(p+q)^{n-1}+(p-q)^{n-1}\}-\frac{1}{2^n}\{(p+q)^n-(p-q)^n\},$$

and

$$d.\left(\frac{dp^2}{q^2dt^2}\right)=\frac{dp}{2^{n-1}q^3}[nq\{(p+q)^{n-1}+(p-q)^{n-1}\}-(p+q)^n+(p-q)^n] (11).$$

If $n=3$, (11) becomes $d.\left(\frac{dp^2}{q^2dt^2}\right)=dp$, whose integral gives

$$\frac{dp^2}{q^2dt^2}=p+c, \text{ or } \frac{dp}{dt}=q\sqrt{p+c} \quad \ldots \ldots (12),$$

where $c=$ the arbitrary constant; but $p=x+y$, $q=x-y$, $\frac{dp}{dt}=\frac{dx}{dt}+\frac{dy}{dt}$ $=\sqrt{1+x^3}-\sqrt{1+y^3}$, therefore

$$\sqrt{1+x^3}-\sqrt{1+y^3}=(x-y)\sqrt{x+y+c} \quad . . . (13),$$

which is the required algebraical integral when $n=3$, and by changing x and y into x^{-2} and y^{-2} we shall have the algebraic integral when $n=6$.

Again, if $n=4$, (11) becomes $d\left(\frac{dp^2}{q^2dt^2}\right)=2pdp$, whose integral gives

$$\frac{dp}{dt}=\sqrt{1+x^4}-\sqrt{1+y^4}=(x-y)\sqrt{(x+y)^2+c'} \quad . . (14),$$

for the algebraic integral; if $n=2$, (11) becomes $d\left(\frac{dp^2}{q^2dt^2}\right)=0$, whose integral gives $\frac{dp}{dt}=qc''$, or $\sqrt{1+x^2}-\sqrt{1+y^2}=(x-y)c''$. (15), for the algebraic integral. But the integral can be exhibited in another

form; for, by (7), we have, when $n=2$, $\frac{d^2x}{dt^2}=x$, $\frac{d^2y}{dt^2}=y$, hence $\frac{yd^2x-xd^2y}{dt^2}=0$, or $d\left(\frac{ydx-xdy}{dt}\right)=0$, whose integral gives

$$y.\frac{dx}{dt}-x.\frac{dy}{dt}=y\sqrt{1+x^2}+x\sqrt{1+y^2}=\text{A}=\text{const.} \quad . \quad (16),$$

for the algebraic integral, and it has been found after the method of proving Kepler's first law in Astronomy. If we integrate the equati on

$$\frac{dx}{\sqrt{1+x^2}}+\frac{dy}{\sqrt{1+y^2}}=0, \text{ we get}$$

$$\{x+\sqrt{1+x^2}\}\{y+\sqrt{1+y^2}\}=xy+\sqrt{(1+x^2)(1+y^2)}+x\sqrt{1+y^2}+y\sqrt{1+x^2}=const.$$

or since, by (16), we have $y\sqrt{1+x^2}+x\sqrt{1+y^2}=$ const., therefore

$$xy+\sqrt{1+x^2}.\sqrt{1+y^2}=\text{B}=\text{const.} \quad . \quad . \quad . \quad . \quad (17),$$

which is yet another form of the algebraic integral, when $n=2$.

Now, see Le Gendre's *Fonctions Elliptiques*, p. 33, Vol I., it is evident that we can assume z such an algebraic function of x and y that the integral of $\frac{dx}{\sqrt{1+x^n}}+\frac{dy}{\sqrt{1+y^n}}+\frac{dz}{\sqrt{1+z_n}}=0$, shall be algebraic, when $n=3$, 4 or 6; or if we please we can change $\frac{dx}{\sqrt{1+x^n}}, \frac{dy}{\sqrt{1+y^n}}, \frac{dz}{\sqrt{1+z^n}}$, in these cases, into elliptic functions, and thence obtain the algebraic integral of $\frac{dx}{\sqrt{1+x^n}}+\frac{dy}{\sqrt{1+y^n}}=0$, after the method of Lacroix, given vol. 2, p. 481, &c., whence the algebraic integral of $\frac{dx}{\sqrt{1+x^n}}+\frac{dy}{\sqrt{1+y^n}}+\frac{dz}{\sqrt{1+z^n}}=0$, can be found by the method proposed by him at p. 490; see also Le Gendre, p. 32.

But we can apply the beautiful Theorem of Abel to this question. For an account of this Theorem we shall refer to the second supplement to Le Gendre's Elliptic Functions, p. 163; for the method of applying the Theorem to elliptic functions, we shall refer to Le Gendre's third supplement, p. 181, &c., where he has obtained the well known algebraic integrals in the case of elliptic functions, see p. 193, &c.; he has also applied it to an expression of the form $\frac{dx}{\sqrt{1+x^5}}+\frac{dy}{\sqrt{1+y^5}}+$ &c. $=0$, and has obtained integrals corresponding to the algebraic integrals in elliptic functions, see p. 207, &c. It is evident the theorem can be applied to every case of this question, and that we shall obtain equations corresponding to the required algebraic integrals. We shall not stop to make the application, on account of the length of the process, but shall content ourselves with referring to Le Gendre, as above. We will however apply the Theorem to one or two simple cases for the purpose of

showing its use; to this end it will be necessary to state the Theorem :—

Suppose then $\psi x = \int \frac{fx dx}{(x-a)\sqrt{\varphi x}}$, where fx and φx denote integral functions of x, also put $\varphi x = \varphi' x \times \varphi'' x$; then assume

$$\left.\begin{array}{l} \theta x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots a_n x^n, \\ \theta' x = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots\ldots c_m x^m, \end{array}\right\} \quad . \quad (a),$$

where a_0, a_1, a_2, &c., c_0, c_1, c_2, &c., are independent of x, but they may be functions of the same independent variable, which may be denoted by y; but the co-efficients contained in $\varphi' x$, $\varphi'' y$, as well as a, in $x - a$, are supposed to be independent of y. Then if we assume the equation

$$(\theta x)^2 \varphi' x - (\theta' x)^2 \varphi'' x = (x - x_1)(x - x_2)(x - x_3) \ldots (x - x_p) \quad . \quad (b),$$

so as to be identical, we shall have

$$e_1 \psi(x_1) + e_2 \psi(x_2) + \ldots e_p \psi(x_p) = \frac{-fa}{\sqrt{\varphi a}} \log \frac{\theta a \sqrt{\varphi' a} + \theta' a \sqrt{\varphi'' a}}{\theta a \sqrt{\varphi' a} - \theta' a \sqrt{\varphi'' a}} + \mathrm{c} + \Pi(\mathrm{X}) (c),$$

where $e_1 e_2$, &c, are either $+1$ or -1, according to the terms to which they are applied, c = an arbitrary constant, and $\Pi(\mathrm{X})$ denotes the co-efficient of x^{-1}, in the expansion of the function

$$\mathrm{X} = \frac{fx}{(x-a)\sqrt{\varphi x}} . \log \frac{\theta x \sqrt{\varphi' x} + \theta' x \sqrt{\varphi'' x}}{\theta x \sqrt{\varphi' x} - \theta' x \sqrt{\varphi'' x}},$$

according to the descending powers of x. If $fx = x - a$, we shall have

$\psi x = \int \frac{dx}{\sqrt{\varphi x}}$ . . (d), and (c) becomes

$$e_1 \psi(x_1) + e_2 \psi(x_2) + \&c. = c = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad (e).$$

If $\varphi x = 1 + x^2$, and we put $\varphi' x = 1 + x^2$, $\varphi'' x = 1$, and $\theta x = a_0$, $\theta' x = c_0 + c_1 x$, (b) becomes $\quad a_0^2 (1 + x^2) - (c_0 + c_1 x)^2 = (x - x_1)(x - x_2)$, $\quad$ which being identical, gives $a_0^2 - c_1^2 = 1$, $a_0^2 - c_0^2 = x_1 x_2$, $2c_0 c_1 = x_1 + x_2$, therefore $c_0^2 - c_1^2 = 1 - x_1 x_2$, and we have

$$(c_0^2 - c_1^2)^2 + 4c_0^2 c_1^2 = (c_0^2 + c_1^2)^2 = (1 - x_1 x_2)^2 + (x_1 + x_2)^2$$
$$= (1 + x_1^2)(1 + x_2^2)$$
$$\therefore \quad c_0^2 + c_1^2 = \sqrt{(1 + x_1^2)(1 + x_2^2)},$$

but

$$(a_0^2 - c_1^2) + (a_0^2 - c_0^2) = 2a_0^2 - (c_0^2 + c_1^2) = 2a_0^2 - \sqrt{(1 + x_1^2)(1 + x_2^2)}$$
$$= 1 + x_1 x_2,$$

or we have $\quad 2a_0^2 - 1 = x_1 x_2 + \sqrt{(1 + x_1^2)(1 + x_2^2)}$, which, by putting $2a_0^2 - 1 = \mathrm{B}$ = const., and putting $x_1 = x$, $x_2 = y$ becomes $\quad xy + \sqrt{(1 + x^2)(1 + y^2)} = \mathrm{B}$, and agrees with (17), which is the algebraic integral of

$$\frac{dx}{\sqrt{1 + x^2}} + \frac{dy}{\sqrt{1 + y^2}} = 0.$$

Again, if $\varphi x = 1 + x^3$, and we put $\varphi' x = 1 + x^3$, $\varphi'' x = 1$, $\theta x = a_0$, $\theta' x = c_0 + c_1 x$, we shall have

$$a_0^2 (1 + x^3) - (c_0 + c_1 x)^2 = x(x - x_1)(x - x_2)$$

which must be identical, therefore we get $a_0^2 = 1 = c_0^2$, $x_1 x_2 = -2c_1 c_0 = -2c_1$, $x_1 + x_2 = c_1^2$; therefore $4(x_1 + x_2) = x_1^2 x_2^2$; or changing x_1, x_2 into x and y, we shall have $x^2 y^2 = 4(x + y)$, which is a particular algebraic

integral of the equation $\frac{dx}{\sqrt{1+x^3}}+\frac{dy}{\sqrt{1+y^3}}=0$, which can also be obtained by the first method given in the solution of this question. For, put $c=0$, in (13), and we have $\sqrt{1+x^3}-\sqrt{1+y^3}=(x-y)\sqrt{x+y}$; also, by the first of (10), we have $\frac{dp}{dt}\cdot\frac{dq}{dt}=x^3-y^3$, or since $\frac{dp}{dt}=\sqrt{1+x^3}-\sqrt{1+y^3}=(x-y)\sqrt{x+y}$, and $\frac{dq}{dt}=\sqrt{1+x^3}+\sqrt{1+y^3}$,

we shall have $\frac{x^3-y^3}{(x-y)\sqrt{x+y}}=\frac{x^2+xy+y^2}{\sqrt{x+y}}=\sqrt{1+x^3}+\sqrt{1+y^3}$,

therefore $\frac{x^2+xy+y^2}{\sqrt{x+y}}+(x-y)\sqrt{x+y}=2\sqrt{1+x^3}$, or $\frac{2x^2+xy}{\sqrt{x+y}}=2\sqrt{1+x^3}$,

or we have $(2x^2+xy)^2=4(x+y)(1+x^3)$, and by reduction, $x^2y^2=4(x+y)$, as above. Since the arbitrary constant $=0$, we may call this the singular algebraic integral.

If we yet suppose $\varphi x=1+x^3$, and put $\varphi' x=1+x$, $\varphi'' x=1-x+x^2$, $\theta x=a_0+a_1x$, $\theta' x=c_0$, our identical equation becomes

$$\begin{aligned}(a_0+a_1x)^2(1+x)-c_0^2(1-x+x^2)&=(x-x_1)(x-x_2)(x-x_3)\\&=x^3-x^2(x_1+x_2+x_3)\\&\quad+x(x_1x_2+x_2x_3+x_1x_3)-x_1x_2x_3,\end{aligned}$$

hence we get $a_1^2=1$, $a_0^2-c_0^2=-x_1x_2x_3$, $a_1^2+2a_0a_1-c_0^2=-(x_1+x_2+x_3)$, $2a_0a_1+a_0^2+c_0^2=x_1x_2+x_1x_3+x_2x_3$, which give $a_1=1$, $2(a_0+a_0^2)=x_1x_2+x_1x_3+x_2x_3-x_1x_2x_3$, and $1+2a_0-a_1^2=-(x_1+x_2+x_3)+x_1x_2x_3$, or if we change x_1, x_2, x_3 into x, y, z, we shall have

$$2(a_0+a_0^2)=xy+xz+yz-xyz \quad . \quad . \quad . \quad . \quad . \quad (1'),$$

$$2-(a_0-1)^2=xyz-(x+y+z) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2'),$$

which we may take for the two algebraic integrals required in the second part of the question when $n=3$; and it is evident that we may proceed in a similar way to find the algebraic integrals of the equation

$\frac{dx}{\sqrt{1+x^n}}+\frac{dy}{\sqrt{1+y^n}}+\frac{dz}{\sqrt{1+z^n}}+\frac{dp}{\sqrt{1+p^n}}+$ &c. $=0$, for any integral positive values of n, and however many terms the equation may consist of, provided the number of terms is not less than two.

It may be observed that, by (1') and (2') we shall have $xyz-2(x+y+z)+xy+xz+yz=6a_0+2=c=$ const. which corresponds to the algebraic integral of $\frac{dx}{\sqrt{1+x^3}}+\frac{dy}{\sqrt{1+y^3}}+\frac{dz}{\sqrt{1+z^3}}=0$, and if we put $z=0$, we have $xy-2(x+y)=c$, which corresponds to

$$\frac{dx}{\sqrt{1+x^3}}+\frac{dy}{\sqrt{1+y^3}}=0.$$

—— We have given this solution entire, being desirous of affording those of our readers who have not access to the voluminous works of

Lacroix and Legendre, an opportunity of seeing the methods they employed, and of contrasting the latest improvements of these methods by Abel, with those of Mr. Talbot, as generalized by Prof. Peirce in a subsequent article of this Number.

Since the above Solution was put to press, we have received another solution of most of the cases in the question from Dr. Strong, in which he uses the method of Mr. Talbot, and from which we extract his method of finding

$$\int dx\sqrt{1+x^3}+\int dy\sqrt{1+y^3}=\tfrac{2}{5}x\sqrt{1+x^3}+\tfrac{2}{5}y\sqrt{1+y^3}+\tfrac{3}{10}\int\left[\frac{dx}{\sqrt{1+x^3}}+\frac{dy}{\sqrt{1+y^3}}\right],$$

algebraically.

"Since $\dfrac{dx}{\sqrt{1+x^3}}=\dfrac{dx}{x+1}.\sqrt{\dfrac{x+1}{x^2-x+1}}$; put $\dfrac{x^2-x+1}{x+1}=v$, or

$$x^2-(1+v)x+1-v=0;$$

therefore, let x, y be the roots of this equation, and we have by the theory of equations, $x+y=1+v$, $xy=1-v$; therefore

$$x+y+xy=2,$$

which is the sought algebraic equation, and we shall have

$$\int\frac{dy}{\sqrt{1+y^3}}+\int\frac{dx}{\sqrt{1+x^3}}=c=\text{const.}$$

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(81). QUESTION XV. *By* ψ.

A vertical cylinder is revolving uniformly about its axis, which is fixed; it is required to determine the motion of a particle of matter in the cylindric surface, supposing it to begin to move from a given point in the surface, with a given velocity, in a given direction. The point is confined to the surface and subjected to the power of gravitation, and the friction varies directly as the pressure and as the square of the velocity of the particle.

SOLUTION. *By Professor B. Peirce.*

Let R = the radius of the cylinder,
g = gravity,
s = the arc of the curve,
θ = the angle which s makes with the vertical,
φ = the angle which the line drawn from the axis to the moving point makes with the plane of xz;

and we have

$$x=R\cos\varphi,\qquad y=R\sin\varphi,$$
$$dz=ds\,.\cos\theta,\qquad Rd\varphi=ds\,.\sin\theta.$$

Let, now, ρ = the radius of curvature of the path of the moving body,
I = the angle which the plane of this path makes with the tangent plane of the cylinder,
p = the pressure arising from the centrifugal force against the cylinder,

$v = \frac{ds}{dt} =$ the velocity of the body, and we have

$$\sin \text{I} = \text{R}\varrho \frac{d\varphi^2}{ds^2},$$

$$p = \frac{v^2}{\varrho} \sin \text{I} = \text{R}.\frac{d\varphi^2}{dt^2} = \frac{v^2}{\text{R}} \sin^2\theta;$$

the unit of pressure being that which corresponds to

$$\text{I} = 90^\circ, \quad v = 1, \quad \varrho = 1.$$

Let, again, $n =$ the velocity of rotation of the cylinder,

$p\text{f}(v) =$ the friction corresponding to a velocity v, and a pressure p. We have, then, for the force of friction in the present case, resolved, in the directions dz and $\text{R}d\varphi$,

$$\frac{v^2 \cos\theta \sin^2\theta}{\text{R}\sqrt{v^2 - 2vn\sin\theta + n^2}}.\,\text{f}(\sqrt{v^2 - 2vn\sin\theta + n^2}) = \text{v}v\cos\theta,$$

and
$$\frac{v^2 \sin^2\theta(n - v\sin\theta)}{\text{R}\sqrt{v^2 - 2vn\sin\theta + n^2}}.\,\text{f}(\sqrt{v^2 - 2vn\sin\theta + n^2}) = \text{v}(n - v\sin\theta),$$

by putting
$$\text{v} = \frac{v^2\sin^2\theta}{\text{R}\sqrt{v^2 - 2vn\sin\theta + n^2}}.\,\text{f}(\sqrt{v^2 - 2vn\sin\theta + n^2}).$$

The equations of motion are, then,

$$\text{R}\frac{d^2\varphi}{dt^2} - \text{v}(n - v\sin\theta) = 0,$$

$$\frac{d^2z}{dt^2} - g + \text{v}(v\cos\theta) = 0;$$

which become by substituting the values of dz and $\text{R}d\varphi$,

$$\frac{dv}{dt}\sin\theta + v\cos\theta\frac{d\theta}{dt} - \text{v}(n - v\sin\theta) = 0,$$

$$\frac{dv}{dt}\cos\theta - v\sin\theta\frac{d\theta}{dt} - g + \text{v}(v\cos\theta) = 0;$$

and if we put
$$n - \text{R}\frac{d\varphi}{dt} = n - v\sin\theta = v',$$

$$\frac{dz}{dt} = v \cos \theta = v'';$$

we have
$$\text{v} = \frac{(n - v')^2}{\text{R}\sqrt{v'^2 + v''^2}}\,\text{f}(\sqrt{v'^2 + v''^2})$$

$$(1) \quad \ldots\ldots \quad \frac{dv'}{dt} + \text{v}v' = 0,$$

$$(2) \quad \ldots\ldots \quad \frac{dv''}{dt} - g + \text{v}v'' = 0.$$

If follows from the equation (1) that v' continually approaches zero as t increases, and if the body started from a state of rest in which we have $v' = n$, the pressure would become zero, and there would strictly be no friction, so that the body would fall vertically. But if we consider the

friction in this case as very small, the path of the body will at first be nearly vertical and will become less and less so till the point determined by the equation

$$\frac{d\theta}{dt}=0, \quad \text{or} \quad g\tan\theta - \mathrm{v}n = 0,$$

at which there is a point of contrary flexure. Beyond this point v' continues to diminish until it becomes infinitely small or zero, after which it does not increase. When v' is so small that its square may be neglected, we have

$$\mathrm{v} = \frac{n^2 - 2nv'}{\mathrm{R}v''}\mathrm{f}(v''),$$

$$\frac{dv'}{dt} + \frac{n^2 v'}{\mathrm{R}v''}\mathrm{f}(v'') = 0,$$

$$\frac{dv''}{dt} - g + \frac{n^2 - 2nv'}{\mathrm{R}}\mathrm{f}(v'') = 0;$$

and, in determining v', v'' may be considered a function of t, determined by the equation

$$\frac{dv''}{dt} - g + \frac{n^2}{\mathrm{R}}\mathrm{f}(v'') = 0.$$

Hence $$\log. v' = -\frac{n^2}{\mathrm{R}}\int\frac{\mathrm{f}(v'')}{v''}dt;$$

and this value of v', substituted in the equation for determining v'', gives to find the value of $\delta v''$, to be added to its value already obtained in order to complete it,

$$\frac{d\delta v''}{dt} - \frac{2nv'}{\mathrm{R}}\mathrm{f}(v'') + \frac{n^2}{\mathrm{R}}\cdot\frac{d.\mathrm{f}(v'')}{dv''}\cdot\delta v'' = 0.$$

If we apply this to the case in which

$$\mathrm{f}(v) = av^2$$

we have $$\frac{dv''}{dt} - g + \frac{an^2}{\mathrm{R}}v''^2 = 0,$$

$$\frac{dv'}{v'} = -\frac{an^2 v''}{\mathrm{R}}dt = -\frac{an^2 v'' dv''}{\mathrm{R}g - an^2 v''^2};$$

$$v'^2 = g - \frac{an^2}{\mathrm{R}}v''^2,$$

or neglecting v'^2, $$v'' = \frac{1}{n}\sqrt{\frac{\mathrm{R}g}{a}},$$

v' is sensibly constant;

$$\delta v'' = \frac{v'}{n}\sqrt{\frac{\mathrm{R}g}{a}} - \mathrm{A}e^{-2nt\sqrt{\frac{ag}{\mathrm{R}}}},$$

A being a constant to be determined by the value of v'' at the time when v' first assumes its present value.

In the *usual case of friction*,

$$f(v) = av,$$

we have
$$V = \frac{a}{R}(n - v')^2,$$

$$\frac{dv'}{dt} + \frac{av'}{R}(n - v')^2 = 0,$$

$$\frac{dv''}{dt} - g + \frac{av''}{R}(n - v')^2 = 0;$$

whence, by integration,

$$\frac{1}{n(n - v')} - \frac{1}{n^2}\log.\frac{n - v'}{v'} = \frac{at}{R} + C.$$

We have also, by eliminating dt,

$$\frac{av''dv' - av'dv''}{v'^2} - \frac{Rgdv'}{v'^2(n - v')^2} = 0;$$

the integral of which is

$$\frac{av''}{v'} + \frac{Rg}{n - v'}\left(\frac{2}{n^2} - \frac{1}{nv'}\right) - \frac{2Rg}{n^3}\log.\frac{n - v'}{v'} = \text{const.} = C;$$

so that the problem admits of a complete solution in this case.

When v' and v'' are found, we have

$$z = \int v''dt, \quad R\varphi = nt - \int v'dt;$$

the arbitrary constants being included in the sign of integration.

Corollary. If v' were nothing at the beginning, it would always remain so, and we should have

$$V = \frac{n^2}{Rv''}f(v''),$$

$$\frac{dv''}{dt} - g + \frac{n^2}{R}f(v'') = 0;$$

whence
$$t + C = \int\frac{Rdv''}{Rg - n^2 f(v'')}.$$

If
$$f(v) = av + bv^2,$$

we have $t + C = \dfrac{R}{n\sqrt{n^2a^2 + 4Rbg}}.\log.\dfrac{2nbv'' + na + \sqrt{n^2a^2 + 4Rbg}}{2nbv'' + na - \sqrt{n^2a^2 + 4Rbg}}.$

When t is very great, we have nearly

$$2nbv'' + na - \sqrt{n^2a^2 + 4Rbg} = e^{-\frac{n}{R}t\sqrt{n^2a^2 + 4Rbg}};$$

and when
$$t = \infty$$

$$v'' = \frac{\sqrt{n^2a^2 + Rbg} - na}{2nb}.$$

List of Contributors and of Questions answered by each. The figures refer to the number of the Questions, as marked in Number IV., Article XV.

LYMAN ABBOTT, Niles, N. Y., ans. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12.
ALFRED, ans. 1, 4, 5, 6, 7.
Prof. C. AVERY, Hamilton College, N. Y., ans. all the Questions.
P. BARTON, Jun., Duanesburgh, N. Y., ans. 1, 2, 3, 4, 5, 6, 7, 9, 12.
B. BIRDSALL, New-Hartford, Oneida Co., N. Y., ans. 1, 2, 3, 4, 5, 6, 7, 8.
J. BLICKENSDERFER, Jun., Roscoe, Ohio, ans. 5.
J. V. CAMPBELL, St. Paul's College, ans. 1.
Prof. M. CATLIN, Hamilton College, N. Y., ans. all the Questions.
B. F. CHAPMAN, Hamilton College, N. Y., ans. 1.
J. B. HENK, Harvard University, Cambridge, ans. 1, 2, 4, 5, 6, 7, 10, 12.
ROBT. S. HOWLAND, St. Paul's College, N. Y., ans. 1.
J. F. MACULLY, Teacher of Mathematics, New-York, ans. 1, 2, 3, 4, 9, 11.
GEO. R. PERKINS, Clinton Liberal Institute, N. Y., ans. 12.
Prof. B. PEIRCE, Harvard University, ans. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 14, 15.
PETRARCH, New-York, ans. 1.
P. ans. 4, 9.
ψ. ans. 15.
O. ROOT, Principal of the Syracuse Academy, N. Y., ans. all the Questions.
Prof. T. STRONG, LL. D., New-Brunswick, N. J., ans. all the Questions.
RICHARD TINTO, Greenville, Ohio, ans. 10.
N. VERNON, Frederick, Md., ans. 1, 3, 4, 5, 8.

*** All communications for Number VII., which will be published on the first day of May, 1839, must be post paid, addressed to the Editor, St. Paul's College, Flushing, L. I.; and must arrive before the first day of February, 1839. New Questions must be accompanied with their solutions.

☞ Mr. LYMAN ABBOTT wishes a situation in a School or College as Teacher of Mathematics. His communications to the Miscellany place his mathematical attainments beyond question; and the fact that he is a graduate of Hamilton College is testimony sufficient as to the excellence of his Education.

NEW BOOKS.

"Elements of Trigonometry, Plane and Spherical. By the Rev. C, W. Hackley, Professor of Mathematics in the University of the city of New-York."

The fourth volume of the Translation of Laplace's Mécanique Céleste, the last sheets of which were proof-read a few days before Dr. Bowditch's lamented death, will be issued as soon as a copious index to the work, as far as translated, can be prepared. Our readers will also be glad to hear that Professor Peirce is engaged in translating the fifth and last volume of this important work, to be printed uniformly with the previous volumes.

Our correspondents are requested to make the following substitutions and corrections in Number V., pp, 327 and 328.

(89). QUESTION VIII. *By Professor B. Peirce, Harvard University.*

Prove that, if all the roots of the equation

$$x^n - \text{A}x^{n-2} + \text{B}x^{n-3} - \&c. = 0,$$

are real, we shall have

$$n(n-1)(3\text{B})^2 < (n-2)^2(2\text{A})^3.$$

(92.) QUESTION XI. *By J. F. Macully. Esq.*

Required the value of n terms of the continued product

$$(1 + 2\cos\theta)(1 + 2\cos 3\theta)(1 + 2\cos 9\theta)\ \ldots\ldots$$

(94). QUESTION XIII. *By Professor Peirce.*

Find the curve which is its own involute.

ARTICLE XXV.

NEW QUESTIONS TO BE ANSWERED IN NUMBER VIII.

Their Solutions must arrive before August 1st, 1839.

(98). QUESTION I. *By an Engineer.*

a, b, c, d are four points on a hill which is to be reduced to a level of 10 feet below a; the surface nearly coincides with planes drawn through $a, b, c,$ and through $a, c, d.$ It is requred to find the quantity of earth to be removed from this part of the hill; the relative position of the points being given, as below:

Stations.	Bearing.	Distance.	Elevation.
a to b	S. 23° 17′E.	51 feet 3 in.	— 5° 25′
b to c	S. 54° 38′W.	79 " 10 "	+ 8° 37′
c to d	N. 10° 15′W.	63 " 5 "	+ 10° 9′
d to a	———	———	———

(99). QUESTION II. *By Wm. Lenhart, Esq., York, Penn.*

Show how to find those integers whose cubes terminate with the three digits 048.

(100). QUESTION III. *Generalized from Peirce's Algebra.*

n men play together on the condition that he who loses shall give to all the rest as much as they already have. They play n games, and each loses in his turn, after which it is found that they have given sums of money. How much had each when they began to play?

(101). QUESTION IV. *By P.*

A hemisphere and cone are fastened with their equal bases together. It is required to find the height of the cone, so that the whole solid may be in equilibrium on any point of the curve surface of the hemisphere.

(102). QUESTION V. *By* ψ.

Given the equation

$$y^4 - 9y^3x + 2x^2 = 0;$$

to express y in a series of monomials, arranged 1°. according to the ascending, and 2°. according to the descending powers, of x.

(103). QUESTION VI. *By* ——.

It is required to place a given parabola so as to touch a given line at a given point in it, and to intersect a second given line at a given angle.

(104). QUESTION VII. *By Mr. G. R. Perkins, Clinton Liberal Institute.*

Given the sum of the squares, and the sum of the fourth powers of four lines drawn from a point to the four vertices of a regular tetraedron, to find the side of the tetraedron.

(105). QUESTION VIII. *By Wm. Lenhart, Esq.*

It is required to find n numbers such that their sum increased by the sum of their cubes shall be equal to the sum of n other numbers increased by the sum of their cubes.

(106). QUESTION IX. *By J. F. Macully, Esq., New-York.*

It is required to find the sum of the series

$$\frac{1+4\cos^4\theta}{\cos^2 2\theta\cos^2\theta}+\frac{1}{4^2}\cdot\frac{1+4\cos^4\frac{1}{4}\theta}{\cos^2\frac{1}{2}\theta\cos^2\frac{1}{4}\theta}+\frac{1}{4^4}\cdot\frac{1+\cos^4\frac{1}{16}\theta}{\cos^2\frac{1}{8}\theta\cos^2\frac{1}{16}\theta}+\&c.$$

(107). QUESTION X. *By P.*

It is required to solve question (75) when, instead of the area and vertical angle, there are given the area and the side opposite the fixed extremity of the base.

(103). QUESTION XI. *From Legendre's Theorie des Nombres, Vol. 2., p. 144.*

(Communicated by Mr. Geo. R. Perkins.)

A	B	C	D
E	F	G	H
I	K	L	M
N	O	P	Q

"In a square, divided into 16 spaces, as in the adjoining figure, inscribe 16 numbers A, B, C, ... Q, which will satisfy the following conditions:

1°. That the sum of the squares of the numbers may be equal in each of the four horizontal lines, also equal in each of the four vertical lines, and in the two diagonals.

2°. That the sum of the products, taken two and two, such as AE + BF + CG + DH may be equal to nothing with regard to the first two horizontal lines, as well as with regard to any two horizontal lines whatever, and that this may be the same also with regard to any two vertical lines.

(109). QUESTION XII. *By* ——.

It is required to find the locus of the centres, and the *envelope*, of all the spheres that can be made to touch the surface of a given sphere, and also two planes, given in position.

(110). QUESTION XIII. *By ψ.*

See Dr. Bowditch's Commentary on the Mécanique Céleste, Vol. I. page 304, equations (i). It is required to be determined whether these equations cannot be reduced to the forms given in page 313, equations (H), in a more simple manner than has been done in that admirable work.

(111). QUESTION XIV. *By Professor B. Peirce, Harvard University.*

Calling the evolute of a curve its first evolute, the evolute of the first evolute the second evolute, that of the second evolute the third evolute, and that of the third evolute the fourth evolute; to find a curve whose fourth evolute is the curve placed in a position parallel to its original one; *i. e.* one in which the equation is the same when referred to rectangular axes parallel in the one case to those in the other.

(112). QUESTION XV. *By the same Gentleman.*

Integrate the equations

$$\frac{d^2y}{dx^2} + \frac{A}{x}.\frac{dy}{dx} - B^2\ x^n y = 0,$$

$$\frac{d^2y}{dx^2} + A.\frac{dy}{dx} - B^2\ e^{nx} y = 0;$$

in which A, B and n are constants, and e is the base of the Naperian system of logarithms.

ARTICLE XXVI.

ANOTHER SOLUTION OF QUESTION (51).

Extracted from Maclaurin's Fluxions, by V.

Put $x^4+ax^2+b=0$, or $x^4+b=-ax^2$;
therefore $x^4+2x^2\sqrt{b}+b=x^2\ (2\sqrt{b}-a)$,
and $x^2+\sqrt{b}=\pm x\sqrt{2\sqrt{b}-a}$;
therefore $x^4+ax^2+b=(x^2-x\sqrt{2\sqrt{b}-a}+\sqrt{b})\ (x^2+x\sqrt{2\sqrt{b}-a}+\sqrt{b})$,
which is the result found in the solutions of Number V. when $a^2-4b<0$.

ARTICLE XXVII.

SOLUTION OF A PROBLEM.

By Professor J. H. Harney, Louisville, Ken.

Let x be one of the rectangular co-ordinates of a body, φ any function of x which expresses the sum of all the particles of the body, multiplied respectively by any function, f, of their distances from the plane of the other co-ordinates. It is required to demonstrate, by the method employed by Lagrange in his *Theorie des Fonctions*, that $d\varphi=fdm$; m being the mass of the body, and also a function of x.

SOLUTION.

Let x become $x+h$, then φ becomes

$$\varphi-\frac{d\varphi}{dx}.h+\frac{d^2\varphi}{dx^2}.\frac{h^2}{2}+\&c.;$$

then each particle between x and $x+h$, multiplied respectively by the given function of its distance will be

$$\frac{d\varphi}{dx}.h+\frac{d^2\varphi}{dx^2}.\frac{h^2}{2}+\&c.$$

When x becomes $x+h$, m becomes

$$m+\frac{dm}{dx}.h+\frac{d^2m}{dx^2}.\frac{h^2}{2}+\&c.;$$

then the mass between x and h, is

$$\frac{dm}{dx}.h+\frac{d^2m}{dx^2}.\frac{h^2}{2}+\&c.$$

Suppose f to be an increasing function, we shall have

$$f.\frac{dm}{dx}.h+f.\frac{d^2m}{dx^2}.\frac{h^2}{2}+\&c.<\frac{d\varphi}{dx}.h+\frac{d^2\varphi}{dx^2}.\frac{h^2}{2}+\&c.,$$

for the first expression is the whole of the mass, multiplied by the smallest value of the function f. For a similar reason

$$\left(\frac{dm}{dx}.h+\frac{d^2m}{dx^2}.\frac{h^2}{2}+\&c.\right)\left(f+\frac{df}{dx}.h+\&c.\right)>\frac{d\varphi}{dx}.h+\frac{d^2\varphi}{dx^2}.\frac{h^2}{2}+\&c.,$$

or

$$f.\frac{dm}{dx}.h+\left(\frac{2dfdm}{dx^2}+f.\frac{d^2m}{dx^2}\right).\frac{h^2}{2}+\&c.>.\frac{d\varphi}{dx}.h+\frac{d^2\varphi}{dx^2}.\frac{h^2}{2}+\&c.,$$

consequently

$$\frac{dfdm}{dx^2}h^2+\&c.>\left(\frac{d\varphi}{dx}-f.\frac{dm}{dx}\right)h+\left(\frac{d^2\varphi}{dx^2}-f.\frac{d^2m}{dx^2}\right).\frac{h^2}{2}+\&c.$$

This last expression could not be true if $\frac{d\varphi}{dx}$ were greater than $f.\frac{dm}{dx}$, for we should then have a series commencing with the second power greater than one commencing with the first. Neither could $\frac{d\varphi}{dx}$ be less than $f.\frac{dm}{dx}$, for that would, for some values of h, render the whole expression on the right of the sign $>$ negative, which cannot be, since it results from subtracting a less quantity from a greater; hence

$$\frac{d\varphi}{dx}=f.\frac{dm}{dx},$$

and $\qquad d\varphi=fdm.$ Q. E. D.

ARTICLE XXVIII.

ON THE ORTHOGRAPHIC PROJECTION OF THE CIRCLE.

By Dr. T. Strong, New-Brunswick, N. J.

Imagine any plane to pass through the centre of a circle, and suppose the circle to make the angle φ with the plane, then 1°. The orthographic projection of the circle on the plane is an ellipse, whose greater axis is that diameter of the circle which is the common section of the two planes, and its lesser axis is the projection of that diameter which is perpendicular to the common section of the two planes, also the eccentricity equals the radius of the circle multiplied by the sine of the angle made by the two planes. Let the common section of the planes be taken for the axis of x, the origin being at the centre of the circle; put $\text{A} =$ the radius of the circle, $\text{B} = \text{A} \cos\varphi$. Then, evidently, B is the projection of the radius of the circle which is perpendicular to the common section of the two planes, and if Y denotes any ordinate in the circle to the common section of the two planes, and y its projection, we evidently have

$$y = \text{Y} \cos\varphi;$$

but, by the nature of the circle

$$\text{Y}^2 = \text{A}^2 - x^2,$$

$$\therefore\ y^2 = (\text{A}^2 - x^2)\cos^2\varphi = (\text{A}^2 - x^2)\frac{\text{B}^2}{\text{A}^2},$$

$$\text{or} \qquad \text{A}^2 y^2 + \text{B}^2 x^2 = \text{A}^2\text{B}^2 \quad \ldots\ldots\ldots \quad (1),$$

which shows the projection to be an ellipse, whose semiaxes are A and B, and its eccentricity $= \sqrt{\text{A}^2 - \text{B}^2} = \text{A} \sin\varphi$.

Again, put $c = \text{A}\sin\varphi$, then set off the distance c from the origin, on the axis of x, both on the negative and positive side, and denote the points thus found by F, f; let r, r' denote lines drawn from F, f, to the extremity of y, and we get,

$$\begin{aligned} r^2 &= (c+x)^2 + y^2 = c^2 + 2cx + x^2 + \text{A}^2\cos^2\varphi - x^2\cos^2\varphi \\ &= \text{A}^2 + 2cx + x^2\sin^2\varphi \\ &= \text{A}^2 + 2cx + \frac{c^2}{\text{A}^2}x^2, \end{aligned}$$

$$\text{or,} \qquad r = \text{A} + \frac{cx}{\text{A}};$$

$$\text{similarly} \qquad r' = \text{A} - \frac{cx}{\text{A}};$$

and, by addition $\quad r + r' = 2\text{A}$,

which also shows the projection to be an ellipse, whose foci are F and f, hence we obtain the same results as from (1).

2°. Any two perpendicular diameters in the circle are projected into a pair of conjugate diameters of the ellipse.

For let R′, R denote any two radii of the circle at right angles to each other, and suppose R′, R are projected into B′, A′: also let Y′ denote any ordinate in the circle to R, and X′ its distance from the centre of the circle;

and put y', x' for the projection of Y′, X′; then, by the principles of orthographic projection, we obtain

$$\frac{R'}{B'} = \frac{Y'}{y'}, \text{ and } \frac{R}{A'} = \frac{X'}{x'},$$

$$\text{or} \quad Y' = \frac{R'y'}{B'}, \text{ and } X' = \frac{Rx'}{A'}.$$

But

$$R^2 = R'^2 = X'^2 + Y'^2,$$

$$\therefore \frac{y'^2}{B'^2} + \frac{x'^2}{A'^2} = 1,$$

$$\text{or} \quad A'^2y'^2 + B'^2x'^2 = A'^2B'^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2);$$

which is the well known equation of the ellipse when referred to any two conjugate diameters, as axes of co-ordinates. Again, put $\pi = 3{,}14159$, &c., then let $\psi =$ the angle which R makes with the greater axis of the ellipse; then $\psi \pm \frac{1}{2}\pi =$ the angle which R′ makes with the same axis, and

$$A'^2 = R^2 \cos^2\psi + R^2 \sin^2\psi \cos^2\varphi,$$

$$B'^2 = R'^2 \cos^2(\psi \pm \tfrac{1}{2}\pi) + R'^2 \sin^2(\psi \pm \tfrac{1}{2}\pi) \cos^2\varphi$$

$$= R^2 \sin^2\psi + R^2 \cos^2\psi \cos^2\varphi$$

$$\therefore A'^2 + B'^2 = R^2 + R^2 \cos^2\varphi = A^2 + B^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

which shows that the sum of the squares of any two conjugate diameters equals the sum of the squares of the axes of the ellipse, as is well known. We may also remark that since any two perpendicular diameters divide the circle into four quadrants, their projections will divide the ellipse into four equal elliptic quadrants; also that, since tangents drawn through the four vertices of any perpendicular diameters form squares, which are all equal to each other, the projections of those squares will be parallelograms formed by tangents drawn through the four vertices of any two conjugate diameters, therefore all such parallelograms are equal to each other, and each equal to the rectangle of the two axes of the ellipse.

Since, also, the area of the circle $= R^2\pi$,

the area of the ellipse $= R^2\pi \cos\varphi = AB\pi$.

3°. To draw tangents to the ellipse. Let us take the perpendicular radii R′, R′ as before. Y′ being ordinately applied to R in the circle, and X′ its distance from the centre, also x' being the projection of X′. Imagine a tangent to the circle to be drawn through the extremity of Y′ and to cut R produced at the distance D from the centre. Then, by a well-known property of the tangent to the circle, we have the equation

$$R^2 = D \cdot X'.$$

Let D′ denote the projection of D; then it is evident that the projection of the tangent to the circle will be a tangent to the ellipse; also, since A′ is the projection of R, we have, from the principles of the projection,

$$\frac{R}{A'} = \frac{X'}{x'}, \text{ and } \frac{R}{A'} = \frac{D}{D'};$$

$$\text{therefore} \quad \frac{R^2}{A'^2} = \frac{DX'}{D'x'} = \frac{R^2}{D'x'};$$

$$\text{and} \quad A'^2 = D'x' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

which enables us to draw the tangent. Again, if we take the equation of the tangent to the circle under the form

$$YY' + XX' = R^2,$$

Y', X' being the co-ordinates of the point of contact, and y', x' their projections; Y, X the co-ordinates of any point in the tangent, and y, x their projections; then, since A', B' are the respective projections of R and R',

$$\frac{R'}{B'}=\frac{Y'}{y'}=\frac{Y}{y}, \text{ and } \frac{R}{A'}=\frac{X'}{x'}=\frac{X}{x},$$

therefore
$$Y'Y=\frac{R'^2yy'}{B'^2}, \text{ and } X'X=\frac{R^2xx'}{A'^2},$$

and
$$Y'Y+X'X=\frac{R'^2yy'}{B'^2}+\frac{R^2xx'}{A'^2}=R^2,$$

or
$$A'^2yy'+B'^2xx'=A'^2B'^2 \quad \ldots\ldots\ldots \quad (5),$$

which is the equation of the tangent referred to any pair of conjugate diameters as axes of co-ordinates. It is evident by (4) that tangents to the ellipse through the extremity of y', and y' produced to cut the ellipse again, intersect the semi-diameter A' produced at the distance D' from the centre of the ellipse, which also follows immediately from the circumstance that tangents to the circle drawn through the extremities of the double ordinate to any diameter intersect the diameter produced in the same point: we may also remark, since it is a known property of the circle that if any secant is drawn through the point of intersection of any two tangents to cut the chord which joins their points of contact, it will be divided harmonically by the chord and that point of its intersection with the circumference which is between the chord and tangents, therefore the same property obtains in the ellipse, as is evident from what has been done above.

4°. To prove that right lines r, r' drawn from the foci F, f, to any point in the perimeter of the ellipse make equal angles with the tangent to the ellipse at that point. Let the tangent cut the greater axis at the distance D from the centre, and let x be the distance of the ordinate to the point of contact from the centre of the ellipse; then, by (4), $D=\frac{A^2}{x}$; also

$$D+c=\frac{A^2}{x}+c, \text{ and } D-c=\frac{A^2}{x}-c$$

are the distances of the foci from the point of intersection of the tangent with the axis, $\therefore$ we shall have

$$D+c:D-c::A+\frac{cx}{A}:A-\frac{cx}{A}::r:r',$$

hence the tangent bisects the angle formed by r', and r produced, (see Simpson's Euclid, B. 6, prop. A,) and the proposition is evident.

We might go on in the same way to obtain all the known properties of the ellipse, but as we have said enough to show the spirit of the method, we shall here leave the subject.

ARTICLE XXIX.

AN ACCOUNT OF MR. TALBOT'S "RESEARCHES IN THE INTEGRAL CALCULUS,"

Published in the Philosophical Transactions, London, 1836, 1837;

WITH A MORE GENERAL SOLUTION OF THE PRINCIPAL PROBLEM.

By Professor Benjamin Peirce, Harvard University, Cambridge.

The object of Mr. Talbot's labors is to find the sum of such integrals as

$$\int\varphi(x)dx+\int\varphi(y)dy+\&c.,$$

in which $\varphi(x)$ represents a known function of x, and x, y, &c., are connected together by equations to be determined. He has succeeded in determining the equations by which this sum can be reduced to the form

$$\int \mathbf{V}dv,$$

in which $\mathbf{V}$ is a function of v, whenever we have

$$\varphi(x)=\frac{\mathbf{X}}{\mathbf{X}'}\cdot\psi\left(\frac{\mathbf{X}''}{\mathbf{X}'''}\right),$$

in which $\mathbf{X}, \mathbf{X}', \mathbf{X}''$, and $\mathbf{X}'''$ are entire polynomials, and ψ denotes any function whatever.

The general principle upon which Mr. Talbot has proceeded may be stated as follows. If there are $n-1$ symmetrical equations between the n quantities x, y, &c., if v is a symmetrical function of x, y, &c., and if, in the partial differential co-efficient $\left(\frac{dv}{dx}\right)$, the values of y, z, &c., obtained from the given equations in terms of x, are substituted, the result being denoted by $\varphi(x)$, we shall have

$$dv=\varphi(x)dx+\varphi(y)dy+\&c..$$

This principle is too obvious to require demonstration, but it is unnecessarily cramped by the condition that there should be $n-1$ equations; for it is often the case that a less number is sufficient to reduce $\left(\frac{dv}{dx}\right)$ to a function of x; this case may, however, be included in the general one, by regarding the deficient equations as identical ones. Thus if we had

$$v=\mathrm{f}(x)+\mathrm{f}(y)+\&c.,$$

no equation would be required; and if we have

$$v=\mathrm{f}'[\mathrm{f}(x)+\mathrm{f}(y)+\&c.],$$

the single equation

$$\mathbf{S}.\mathrm{f}(x)=\mathrm{A},$$

is sufficient, A being an arbitrary constant, and $\mathbf{S}$ denoting the sum of all similar functions of x, y, z, &c. Again, if

$$v=a\mathbf{S}.\mathrm{f}(x)\mathrm{f}(y),$$

the equation

$$\mathbf{S}.\mathrm{f}(x)=\mathrm{A},$$

is sufficient; and if

$$v=\mathrm{f}'[a\mathbf{S}.\mathrm{f}(x)\mathrm{f}(y)],\ \text{or}\ =a\mathbf{S}.\mathrm{f}(x)\mathrm{f}(y)\mathrm{f}(z),$$

the two equations

$$\mathbf{S}\,.\,f(x)f(y) = A,$$
$$\mathbf{S}\,.\,f\;(x) \;= B$$

in which A and B are arbitrary constants, are sufficient. Also, if

$$v = af(x).f(y).f(z)\;\ldots\ldots,$$

the single equation

$$v = A,$$

is sufficient.

With regard now to the general problem *to find such equations between* x, y, z, &c., *that*

$$\mathbf{S}\,.\!\int\!\varphi(x)dx\!\int\!\mathbf{S}\,.\,\varphi(x)dx,$$

may be obtained algebraically or by means of circular or logarithmic functions.

Solution. Let v, v', v'', &c., be symmetrical functions of x, y, z, &c. Put then

$$\varphi x = \pi(v,\, v',\, v'',\, \&c.\;\ldots\ldots x)$$

π denoting any function whatever such, that

$$\int\pi(v,\, v',\, v'',\, \&c.\;\ldots\ldots x)dx$$

can be exactly obtained relatively to x, when we regard v, v', v'', &c., as constants; and let us put this integral relative to x

$$\int\pi(v,\, v',\, v'',\;\ldots\ldots x)dx = {}'\pi(v,\, v',\, v''\ldots\ldots\ldots x),$$

and we have, for the complete integral,

$$\int\pi(v,v',v'',\ldots x)dx = {}'\pi(v,\,v',\,v'',\ldots x) - \int\left(\frac{d.{}'\pi(v,v',v''\ldots x)}{dv}\right)dv$$
$$-\int\left(\frac{d.{}'\pi(v,v',v'',\ldots x)}{dv}\right)dv'$$
$$-\;\&c.$$

$$\text{Hence}\int\mathbf{S}.\varphi(x)dx = \mathbf{S}.\pi'(v,v',v'',\ldots x) - \int dv.\,\mathbf{S}\left(\frac{d.{}'\pi(v,v',v'',\ldots x)}{dv}\right)$$
$$-\int dv'.\,\mathbf{S}.\left(\frac{d.{}'\pi(v,v',v''\ldots x)}{dv'}\right)$$
$$-\;\&c.$$

$$= \mathbf{U} - \int(\mathbf{V}dv + \mathbf{V}'dv' + \mathbf{V}''dv'' + \&c.).$$

in which
$$\mathbf{U} = \mathbf{S}.{}'\pi(v,v',v'',\ldots x),$$
$$\mathbf{V} = \mathbf{S}.\left(\frac{d.{}'\pi(v,v',v'',\ldots x)}{dv}\right).$$
&c.

Now since x, y, z, &c., are roots of the equation

$$\varphi(x) - \pi(v,v',v'',\ldots x) = 0,$$

if we take them so as to be equal to all the roots of this equation, and put

$$\varphi'(x) = \frac{d\,.\,\varphi(x)}{dx},\; \pi'(v,v',v'',\ldots x) = \left(\frac{d.\pi(v,v',v'',\ldots x)}{dx}\right),$$

and use Cauchy's notation, as employed in his Residuary Analysis, we have

$$\mathrm{U}=\mathcal{E}\frac{'\pi(v,v'\ .\ .\ .\ .\ x).(\varphi'(x)-\pi'(v,\ v'\ .\ .\ .\ .x))}{((\varphi(x)-\pi(v,v',\ .\ .\ .\ .\ x)))},$$

$$\mathrm{V}=\mathcal{E}\frac{\left(\frac{d.'\pi(v,\ v',\ .\ .\ .\ .\ x)}{dv}\right).(\varphi'(x)-\pi'(v,\ v',\ .\ .\ .\ .\ x))}{((\varphi(x)-\pi(v,\ v',\ .\ .\ .\ .\ x)))},$$

&c., &c.,

and these formulas admit of the various simplifications which Cauchy has pointed out in his article on "finding the sum of similar func tions of the roots of an equation."

The values of V,V', &c., being thus found, the problem is completely resolved whenever

$$\int(\mathrm{V}dv+\mathrm{V}'dv'+\text{\&c.})$$

is an exact integral, and no other equation will be required between x, y, z, &c., than those involved in the equation of which they are roots. But when the integral is not complete, it can often be rendered so by introducing some new equations between v, v,' &c. A striking example of this occurs whenever the part

$$\int(\mathrm{V}dv+\mathrm{V}'dv'+\text{\&c.}),$$

is an exact integral relative to the quantities v, v', &c., whose differentials occur in it, but the remainder

$$\int(\mathrm{W}dw+\mathrm{W}'dw'+\text{\&c.}),$$

is not an exact integral; for by putting

$$w=\mathrm{A},\ w'=\mathrm{A}',\ \text{\&c.},$$

in which A, A', &c, are arbitrary constant quantities, the required integral is reduced to that part of it which is exact.

Corollary 1. Whenever the equation of which x, y, &c, are roots, contains a factor which is independent of v, v', &c, this factor is to be suppressed, because all its roots, being constant, can only lead to constant integrals, whose sum may be included in the arbitrary constant which is added to complete the remaining integrals.

Corollary 2. Whenever the equation of which x, y, &c, are roots can be reduced to an algebraic one of the n^{th} degree, the number of its roots x, y, &c, will be n, and if it is written in the form

$$x^n+ax^{n-1}+bx^{n-2}+\text{\&c.}=0,$$

a, b, &c., being functions of v, v', &c., we have

$$\mathrm{S}.x=-a,\qquad \mathrm{S}.xy=b,\qquad \mathrm{S}.xyz=-c,\ \text{\&c.},$$

and if $\pi(v,\ v',\ .\ .\ .\ .\ x)$ is a fraction whose numerator and denominator are entire algebraic functions of x, the value of

$$\mathrm{S}.\varphi(x)dx$$

can easily be found without having recourse to Cauchy's notation.

Corollary 3. When we have

$$\varphi(x)=\frac{\mathrm{X}}{\mathrm{X}'}.\psi\left(\frac{\mathrm{X}''}{\mathrm{X}'''}\right),$$

in which $\mathrm{X},\mathrm{X}',\mathrm{X}'',\mathrm{X}'''$ are entire algebraic functions of x, we have only to suppose

$$\pi(v,\ v'\ .\ .\ .\ x)=\frac{\mathrm{X}}{\mathrm{X}'}.\psi\left(\frac{\mathrm{R}}{\mathrm{R}'}\right),$$

in which $\mathbf{R}$ and $\mathbf{R}'$ are entire algebraic functions relatively to x, and contain v, v', &c., in order to obtain the equation

$$\mathbf{R}'\,\mathbf{X}''-\mathbf{R}\mathbf{X}'''=0,$$

of which x, y, &c. are roots, and if

$$\psi\left(\frac{\mathbf{R}}{\mathbf{R}'}\right)$$

is a fraction whose terms are both entire functions of x, both the conditions of the preceding corollary are satisfied.

Corollary 4. By taking

$$\frac{\mathbf{R}}{\mathbf{R}'}=v,$$

we have
$$\psi\left(\frac{\mathbf{R}}{\mathbf{R}'}\right)=\psi(v),$$

and obtain
$$\int \mathbf{S}\,.\,\varphi(x)dx=\int\psi(v).\,\mathbf{S}\,.\,\frac{\mathbf{X}dx}{\mathbf{X}'}.$$

And if
$$\mathbf{S}\,.\,\frac{\mathbf{X}dx}{\mathbf{X}'}=0,$$

we have
$$\mathbf{S}\int\varphi(x)dx=\text{const.}$$

Corollary 5. Whenever the function ψ is the fractional power $\frac{m}{m'}$, we may take

$$\frac{\mathbf{R}}{\mathbf{R}'}=\left(\frac{\mathbf{R}''}{\mathbf{R}'''}\right)^{m'},$$

in which case we have

$$\psi\left(\frac{\mathbf{R}}{\mathbf{R}'}\right)=\left(\frac{\mathbf{R}}{\mathbf{R}'}\right)^{\frac{m}{m'}}=\left(\frac{\mathbf{R}''}{\mathbf{R}'''}\right)^{m}.$$

Corollary 6. If, now, between the n equations of Corollary 2,

$$\mathbf{S}\,.\,x=-a,\ \mathbf{S}\,.\,xy=b,\ \text{\&c.},$$

the quantities v, v', &c. are eliminated, whose number we will suppose to be n', the number of resulting equations, combined with those which will be necessary to reduce

$$\mathbf{V}\,dv+\mathbf{V}'\,dv'+\text{\&c.},$$

to an exact differential, which we will call n'', will be $n-n'+n''$.

These $n-n'+n''$ equations will often contain arbitrary constant quantities, which could not be eliminated with safety, since they may be implicitly contained in the arbitrary constant quantity which is added to complete the integral, and might thus affect the value of the result.

Having thus generalized the method of Mr. Talbot, which seems to me one of the most fruitful sources for future investigations into the Integral Calculus, I shall now trace it through some of his examples and shall give many of the investigations the more general form which I have here employed.

Ex. 1. Let

$$\varphi(x)=\frac{1}{\sqrt{1-x^2}}=(1-x^2)^{-\frac{1}{2}}.$$

Take, then, in Corollary 5,

$$m=-1,\ m'=2,$$
$$\mathbf{R}''=v+v'x,\ \mathbf{R}'''=1;$$

and the equation for x, y, &c., is

$$1-x^3=(v+v'x)^2=v^2+2vv'x+v'^2x^2,$$

or
$$x^3+v'^2x^2+2vv'x+v^2-1=0.$$

Whence, the number of these roots is, in general, 3, and we have

$$\mathbf{S}\,.\,x=-v'^2,\ \mathbf{S}\,.\,xy=2vv',\ xyz=1-v^2.$$

$$\mathbf{S}\,\varphi(x)dx=\mathbf{S}\frac{dx}{v+vx}=\frac{v^2d.\,\mathbf{S}.x+vv'd.\,\mathbf{S}xy+v'^2d.\,xyz}{v^3+v'v'\mathbf{S}.x+vv'^2\mathbf{S}xy+v'^3xyz.}=0,$$

whence
$$\int\frac{dx}{\sqrt{1-x^3}}+\int\frac{dy}{\sqrt{1-y^3}}+\int\frac{dz}{\sqrt{1-z^3}}=\text{const.},$$

whenever we have the equation

$$4(xyz-1)\,\mathbf{S}\,.\,x=(\mathbf{S}\,.\,xy)^2.$$

If we take $xyz=1$, we have $\mathbf{S}.\ xy=0$,
which is Mr. Talbot's result.

We might also, for a simple case, take

$$\mathbf{S}\,.\,x=0,\ \mathbf{S}\,.\,xy=0;$$

or, we might take

$$z=\text{const.}=a,$$

in which case we should get the sum of the two integrals,

$$\int\frac{dx}{\sqrt{1-x^3}}+\int\frac{dy}{\sqrt{1-y^3}}=\text{const.},$$

whenever x and y satisfy the equation

$$4\,(axy-1)\,(a+x+y)=[a\,(x+y)+xy]^2.$$

But here it must be observed that the value of the arbitrary constant a affects that which is added to complete the integral.

When we put $a=1$,
this equation is reduced to

$$xy-(x+y)=2,$$

which is Mr. Lubbock's equation, quoted by Mr. Talbot.

When we put $a=0$;
we have $x^2y^2+4\,(x+y)=0.$

Ex. 2. Let

$$\varphi(x)=\frac{x}{\sqrt{1-x^3}}.$$

By the substitutions of the preceding example we have

$$\mathbf{S}\,.\,\varphi(x)dx=-2dv';$$

whence
$$\int\frac{xdx}{\sqrt{1-x^3}}+\int\frac{ydy}{\sqrt{1-y^3}}+\int\frac{2dz}{\sqrt{1-z^3}}=-2v'+\text{const.}$$
$$=\text{const.}-2\sqrt{-(x+y+z)}.$$

Applications to the Circle.

Ex. 3. Let

$$\varphi(x)=\frac{1}{1+x^2}=(1+x^2)^{-1},$$

and make, in Corollary 5,

$$m=-1,\ m'=1,\ \mathbf{R}''=v+v'x,\ \mathbf{R}'''=1;$$

and the equation for x, y, &c., is

$$x^2-v'x+1-v=0:$$

whence the number of roots is two, and we have

$$\mathbf{S}.\ x=v',\ xy=1-v,$$

$$\mathbf{S}.\ \varphi(x)\,dx=\frac{vdv'-v'dv}{v^2+v'^2}=d\left(\frac{v'}{v}\right)\left(1+\frac{v'^2}{v^2}\right)^{-1}.$$

Hence by integration

$$\text{arc. tan } x+\text{arc. tan } y=\text{arc. tan }\frac{v'}{v}+\text{const.}$$

or $\quad \tan(\theta+\theta')=\dfrac{v'}{v}=\dfrac{\mathbf{S}.\ x}{1-xy}=\dfrac{\tan\theta+\tan\theta'}{1-\tan\theta\tan\theta'},$

the constant being so taken as to give, when $y=0$,

$$\text{arc. tan } x=\text{arc. tan } x.$$

This integral is reduced to Mr. Talbot's particular case when $v=0$.

Again, make

$$\mathbf{R}''=v+v'x,\ \mathbf{R}'''=v''+x,$$

and we shall obtain the well-known theorem

$$\tan(\theta+\theta'+\theta'')=\frac{\tan\theta+\tan\theta'+\tan\theta''-\tan\theta\tan\theta'\tan\theta''}{1-(\tan\theta\tan\theta'+\tan\theta\tan\theta''+\tan\theta'\tan\theta'')},$$

of which Mr. Talbot's results are but particular cases.

Ex. 4. Let

$$\varphi(x)=(1-x^2)^{-\frac{1}{2}},$$

and make

$$m=-1,\ m'=2,\ \mathbf{R}''=1+v'x,\ \mathbf{R}'''=1+vx;$$

and the equation for x, y, z, &c., is

$$x^3+\frac{2}{v}x^2+\left(\frac{v'^2+1}{v^2}-1\right)x-2.\ \frac{v-v'}{v^2}=0.$$

whence, in general, the number of roots is 3, and we have

$$\mathbf{S}.\ x=-\frac{2}{v},\ \mathbf{S}.\ xy=\frac{v'^2+1}{v^2}-1,\ xyz=\frac{2(v-v')}{v^2};$$

$$\mathbf{S}.\ \varphi(x)dx=0;$$

whence $\qquad \theta+\theta'+\theta''=\text{const.},$

if $\sin\theta$, $\sin\theta'$, $\sin\theta''$ are the roots of the equation in x, it being remembered that, in this equation, v and v' are arbitrary, and any change in their values does not even affect the value of the constant which is the value of the sum of the three arcs.

Mr. Talbot's result, which is a particular case of this, and involves only one arbitrary quantity, is obtained from it by making

$$v'=0,\ \frac{2}{v}=r.$$

Application to the Parabola.

Ex. 5. "If," as Mr. Talbot says, "the tangent at the vertex of a parabola be taken for the axis of abscissæ, and the semiparameter $=1$, and

if x be the abscissa, the equation of the curve is

$$2y=x^2$$

and the arc. $=\int dx\sqrt{1+x^2}=\frac{1}{2}x\sqrt{1+x^2}+\frac{1}{2}\log.(x+\sqrt{1+x^2})$,"
which arc he denotes as arc x. If now we make

$$m=1,\ m'=2,\ \mathbf{R}''=v+v'x+x^2, \mathbf{R}'''=v;$$

we have the equation

$$x^3+2v'x^2+(2v+v'^2-v^2)x+2vv'=0;$$

whence the number of roots is 3, and we find

$$\mathbf{S}\varphi(x)\,dx=-2vdv'-2v'dv=d(-2vv')=d.\,xyz,$$

so that arc. $x+$ arc. $y+$ arc. $z=x\,y\,z$,
for it appears by trial that the constant which is to be added is zero.

This is the same with Mr. Talbot's second theorem, of which his first theorem is a case, obtained by supposing $v=1$.

Application to the Ellipse.

Ex. 6. Let

$$\varphi(x)=\sqrt{\frac{1-e^2x^2}{1-x^2}},$$

if we make

$$m=1,\ m'=2,\ \mathbf{R}''=1+vx,\ \mathbf{R}'''=1+v'x,$$

we have the equation

$$(v^2-e^2v'^2)\,x^3-2\,(e^2v'-v)\,x^2+(v'^2-v^2-e^2+1)\,x-2\,(v-v')=0;$$

whence the number of roots is three, and we find

$$\mathbf{S}.\,x=2.\,\frac{e^2v'-v}{v^2-e^2v'^2},\ \mathbf{S}.\,xy=\frac{v'^2-v^2-e^2+1}{v^2-e^2v'^2}, xyz=2.\,\frac{v-v'}{v^2-e^2v'^2};$$

$$\mathbf{S}.\,\varphi(x)dx=\frac{2e^2(1+2vv'-v'^2)dv'-2e^2(1+v'^2)dv}{(1-v'^2)(v^2-e^2v'^2)}+\frac{4e^2(v'dv-vdv')}{(1-v'^2)(v^2-e^2v'^2)^2}.$$

If now we put

$$\frac{v}{v'}=\text{const.}=a,$$

we have $$\mathbf{S}.\,\varphi(x)dx=\frac{2e^2(a-1)}{e^2-a^2}.\,\frac{dv}{v^2},$$

whence arc. $x+$ arc. $y+$ arc. $z=\dfrac{2e^2(1-a)}{(e^2-a^2)v}=e^2xyz+$ const.,
when x, y, z are roots of the equation

$$x^3-\frac{2}{v}.\,\frac{ae^2-1}{1-a^2e^2}.\,x^2+\left(a^2-1+\frac{1-e^2}{v^2}\right).\,\frac{x}{1-a^2e^2}-\frac{2}{v}.\frac{1-a}{1-a^2e^2}=0,$$

in which v is entirely arbitrary, and a only an arbitrary constant. If we put $a=0$, we obtain Mr. Talbot's first example.

By a similar process we might generalize his other two theorems on the ellipse. Instead of this, we shall put

$$x=\sqrt{t},$$

whence $$dx.\sqrt{\frac{1-e^2x^2}{1-x^2}}=\tfrac{1}{2}dt\sqrt{\frac{1-e^2t}{t-t^2}}.$$

Make now

$$\varphi(x)dx=\tfrac{1}{2}dt.\sqrt{\frac{1-e^2t}{t-t^2}}, \mathbf{R}''=v,\ \mathbf{R}'''=v'+t;$$

and we have the equation

$$e^2t^3-(v^2+1-2v'e^2)t^2+(v'e^2-2)v't-(v'^2-v^2)=0,$$

which has three roots. Hence

$$\mathbf{S}.x^2=\mathbf{S}.t=\frac{v^2+1}{e^2}-2v',$$

$$\mathbf{S}.x^2y^2=\mathbf{S}.tt'=v'^2-\frac{2v'}{e^2},$$

$$x^2y^2z^2=tt't''=\frac{v'^2-v^2}{e^2};$$

$$\mathbf{S}.\varphi(x)dx=dv,$$

and $\int\mathbf{S}.\varphi(x)dx=v+\text{const.}=\text{arc.}\,x+\text{arc.}\,y+\text{arc.}\,z,$

in which the equation for finding x, y, z, has two arbitrary quantities

—— For other examples, see solution to Question (60), page 383.

ARTICLE XXX.

ANOTHER SOLUTION TO QUESTION (50).

By Dr. T. Strong, Rutger's College, New-Brunswick, N. J.

Put R′, R for the greater and lesser radii of the given circles, and suppose that the smaller circle is wholly within the other; put 2D= the distance of their centres, R′+R=2A, $\frac{R'-R}{2A}=e'$, $\frac{D}{A}=e$,; d, d' the distances of the centre of any tangent circle from the centres of the circles radii R, R′, $r=$ the radius of the tangent circle, then because it touches the circle rad. R, externally, and the other internally, we get $d=R+r$, $d'=R'-r$, $\therefore d+d'=R'+R=2A$, hence the locus of the centres of the tangent circles is an ellipse whose foci are at the centres of the given circles, and whose greater axis $=2A$, and the equation of the ellipse when referred to rectangular axes is $y^2=(1-e^2)(2Ax-x^2)$, (1), the origin of the co-ordinates being at that vertex which is nearest to the circumferences of the given circles, the axis of x being the greater axis of the ellipse, and x, y are the co-ordinates of the centre of the tangent circle; let x', y', denote the co-ordinates of the centre of another tangent circle rad. r' supposed to touch the circle rad. r externally, then we shall have $(r+r')^2=(x'-x)^2+(y'-y)^2$. (2), where we shall suppose $x'>x$; we also have $d^2=(R+r)^2=\{x-A(1-e')\}^2+y^2$, (3),.

Put $x=A(1-\cos u)$, $x'=A(1-\cos u')$, $\therefore u'>u$, and (1) gives $y=A\sqrt{1-e^2}\sin u$, $y'=A\sqrt{1-e^2}\sin u'$, also (3) gives $r=A(e'-e\cos u)$, $r'=A(e'-e\cos u')$, (4), we also get by (2), $(e'-e\cos u).(e'-e\cos u')=(1-e^2)\sin^2\frac{u'-u}{2}$, (5), put $\frac{1-e^2}{(e'+e)^2}=b^2$, $\frac{e'-e}{e'+e}=c^2$, (6), and (5) is easily changed to $b^2\left(\cot\frac{u}{2}-\cot\frac{u'}{2}\right)^2=\left(1+c^2\cot^2\frac{u}{2}\right).\left(1+c^2\cot^2\frac{u'}{2}\right)$, (7), in the same way if $u''>u'$ corresponds to another tangent circle rad. r'' which

touches the circle rad. r', we have $b^2\left(\cot\frac{u'}{2}-\cot\frac{u''}{2}\right)^2=\left(1+c'^2\cot^2\frac{u'}{2}\right).$ $\left(1+c^2\cot^2\frac{u''}{2}\right)$, (8), and so on for any number of successive tangent circles, which touch each other. Put $c\cot\frac{u}{2}=\cot x$, $c\cot\frac{u'}{2}=\cot x'$, &c. (a), then by (7), $\sin^2(x'-x)=\frac{c^2}{b^2}=\frac{e'^2-e^2}{1-e^2}=\sin^2(x''-x')=\sin^2(x'''-x'')=$ &c. by (x), and by (a) since $u<u'<u''<$ &c. we have $x\angle x'\angle x''\angle$ &c., hence we have $\sin(x'-x)=\sin(x''-x')=\sin x'''-x'')=$ &c. $=\sqrt{\frac{e'^2-e^2}{1-e^2}}$, (b), put $\pi=3.14159$, &c. t, t', t'', &c. any positive integers, o being included, then if we assume $x'-x=2t\pi+v$, we shall have $x''-x'=2t'\pi+v$, $x'''-x''=2t''\pi+v$, and so on, $\therefore$ by addition $x_n-x=(t+t'+t''+$ &c.$)\,2\pi+nv$, (c), or since x_n corresponds to the $n+1^{th}$ tangent circle which is to coincide with the first, we have $x_n-x=2\mathrm{T}\pi$, $\therefore$ put $\mathrm{T}-(t+t'+t''+$ &c.$)=a$, and we get by (c), $v=\frac{2a\pi}{n}$, (d),. Since $\sin(x'-x)=\sin v$, we get $\sin^2\frac{2a\pi}{n}=\frac{e'^2-e^2}{1-e^2}$, which gives $e^2=$ $\dfrac{e'^2-\sin^2\frac{2a\pi}{n}}{\cos^2\frac{2a\pi}{n}}$, or by restoring the values of e, e', we shall have $4\mathrm{D}^2=(\mathrm{R}'-\mathrm{R})^2-4\mathrm{R}'\mathrm{R}\tan^2\frac{2a\pi}{n}$, (e), this equation is easily adapted to the case when the circles fall wholly without each other by making R' negative, and we get $4\mathrm{D}^2=(\mathrm{R}'+\mathrm{R})^2+4\mathrm{R}'\mathrm{R}\tan^2\frac{2a\pi}{n}$, (f), in which case we have $e>e'$, and $c=\sqrt{-1}\times\sqrt{\frac{e-e'}{e+e'}}$, $\cot\frac{u}{2}=\dfrac{\varepsilon^{\frac{-u}{2}}+\varepsilon^{\frac{u}{2}}}{\varepsilon^{\frac{-u}{2}}-\varepsilon^{\frac{u}{2}}}\times\sqrt{-1}$ where ε denotes the hyperbolic base, $\therefore c\cot\frac{u}{2}=\cot x=\sqrt{\frac{e-e'}{e+e'}}\times\frac{\varepsilon^u+1}{\varepsilon^u-1}$, also $\cot x'=\sqrt{\frac{e-e'}{e+e'}}\times\frac{\varepsilon^{u'}+1}{\varepsilon^{u'}-1}$, and so on, (g), and it may be observed that the angles u, u', &c. correspond to what is called the eccentrick anomaly in astronomy.

Now by assuming a, so as to satisfy (e) and (f) we find $2\mathrm{D}$ the distance of the centres of the given circles in the two cases mentioned above, (see Mis. p. 248,) $\therefore v=\frac{2a\pi}{n}$ is known, then by assuming u we find x by the first of (a), or (g), according to the first or second case of the question then we have $x'=2t\pi+x+v$, $x''=2t'\pi+x'+v$, &c. (h), which will enable us to find u', u'', &c.

CONTENTS

OF THE

FIRST VOLUME.

JUNIOR DEPARTMENT.

SENIOR DEPARTMENT.

2

USEFUL TABLES

RELATING TO

CUBE NUMBERS.

CALCULATED AND ARRANGED

BY

WILLIAM LENHART,

YORK, PENN.

DESIGNED TO ACCOMPANY HIS GENERAL INVESTIGATION OF THE EQUATION

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2),$$

PUBLISHED IN THE MATHEMATICAL MISCELLANY, VOL. I., PAGE 114; AND BY HIM,
THROUGH HIS FRIEND, PROFESSOR C. GILL,
PRESENTED TO THE
LIBRARY OF ST. PAUL'S COLLEGE, FLUSHING, LONG ISLAND,
MAY 4TH, 1837.

"There are few difficulties which will not yield to perseverance."

NEW-YORK:
PRINTED BY WILLIAM OSBORN,
88 William-street.

1838.

Besides the Tables given here, the manuscript copy compiled with so much labor and care, by Mr. Lenhart, includes a Table,

"Containing a variety of Numbers between 1 and 100,000, and the roots, not exceeding two places of figures, of two cubes, to whose difference the numbers are respectively equal;" together with a Table,

"Exhibiting the roots of three cubes to satisfy the indeterminate equation

$$x^3 + y^3 + z^3 = A,$$

for all values of A, from 1 to 50, inclusive."

Both these Tables are extremely curious, and are open to the inspection of all who may wish to consult them. They are lodged in the library of *St. Paul's College.*

ERRATA.

Page 1, for "1391," read 1395.
2, for "3382," read 3383.
3, the asterisk should be placed before 5425 instead of 5404.
" opposite 5977, for the numerator " 194," read 154.
" " 6122, " " 1125," read 1129.
" 9364 and 9343 should be transposed.
4, opposite 11115, for the numerator " 561," read 569.
" " 15996, for the denominator " 4," read 3.
5, $21428 = 21^3 + 23^3$ is omitted.
8, 49247 is divisible by 11^3, and may be omitted.
" put 62517 in its proper place.
9, opposite 64790, for the denominator " 7," read 14.
" " 65906, for the numerator " 101," read 201.

TABLE I.

EXHIBITING A VARIETY OF NUMBERS, BETWEEN 1 AND 100,000, AND THE ROOTS, NOT EXCEEDING FOUR PLACES OF FIGURES, OF TWO CUBES OF WHICH THEY ARE COMPOSED:

$$\text{thus } 6 = \left(\tfrac{17}{21}\right)^3 + \left(\tfrac{37}{21}\right)^3,$$

$$9 = 2^3 + 1^3.$$

Numbers.	Numerators of the Roots.	Denominators of the Roots.	Numbers.	Numerators of the Roots.	Denominators of the Roots.	Numbers.	Numerators of the Roots.	Denominators of the Roots.	Numbers.	Numerators of the Roots.	Denominators of the Roots.
6	17 . 37	21	183	13 . 14	3	539	3 . 8		1007	378 . 629	67
7	4 . 5	3	201	11 . 16	3	546	2267 .2647	381	* 20	2357 .6823	687
*9	1 . 2		* 203	32 . 229	39	553	25 . 47	6	27	3 . 10	
12	19 . 89	39	209	701 . 971	182	559	6 . 7		“	31 . 33	4
13	2 . 7	3	217	1 . 6		579	2 . 25	3	36	15 . 17	2
15	397 . 683	294	“	123 . 125	26	589	134 . 145	21	51	29 . 35	4
19	3 . 5	2	* “	543 .1441	244	601	23 . 49	6	55	16 . 29	3
“	1 . 8	3	218	217 . 935	156	628	11 . 25	3	57	11 . 61	6
** “	33 . 92	35	219	10 . 17	3	630	1961 .3709	453	* 99	27 . 37	4
20	1 . 19	7	223	67 . 509	84	637	5 . 8		1108	5 . 31	3
26	53 . 75	28	254	437 . 587	104	651	1 . 26	3	* 22	3439 .6659	669
28	1 . 3		259	7115 .8437	1548	657	7 . 17	2	32	13 . 19	2
30	107 . 163	57	* 273	8 . 19	3	** “	56 . 163	19	71	25 . 39	4
33	523 .1853	582	278	3 . 13	2	658	39 . 55	7	83	635 . 886	93
35	2 . 3		279	1505 .2488	407	665	5409 .5576	793	90	487 . 703	73
37	18 . 19	7	287	121 . 248	39	683	323 . 360	49	1205	14 . 31	3
** “	40 . 303	91	289	90 . 199	31	719	179 . 540	61	28	323 . 649	63
43	1 . 7	2	308	109 . 199	31	721	19 . 53	6	41	8 . 9	
62	7 . 11	3	309	7 . 20	3	730	1 . 9		“	270 . 971	91
63	127 . 248	65	316	1977 .2023	370	737	2 . 9		47	2816 .6059	365
65	1 . 4		323	71 . 252	37	754	2701 .3331	422	61	22 . 75	7
* “	197 . 323	86	325	97 . 128	21	793	4 . 9		“	771 . 854	95
70	17 . 53	13	330	1349 .1621	273	“	17 . 55	6	67	23 . 41	4
84	323 . 433	111	335	452 . 763	117	823	142 . 983	105	“	3464 .4411	465
* 86	5 . 13	3	341	5 . 6		829	393 . 607	70	70	341 . 379	42
89	36 . 53	13	* 370	3 . 7		845	22 . 23	3	73	7 . 65	6
91	3 . 4		387	2096 .4495	637	851	8 . 199	21	95	13 . 32	3
*** “	23 . 94	21	399	5 . 22	3	854	5 . 9		1324	11 . 21	2
98	355 . 669	152	“	328 . 401	63	** “	291 . 685	74	30	269 .1061	97
105	3527 .4033	1014	407	4 . 7		855	7 . 8		“	4861 .5779	614
107	17 . 90	19	420	1567 .2213	327	873	5 . 19	2	* 30	887 .4873	444
110	251 . 629	134	422	1 . 15	2	884	127 . 757	79	32	1 . 11	
117	484 . 545	133	433	35 . 37	6	889	2 . 125	13	39	2 . 11	
124	83 . 205	42	“	181 . 252	37	901	361 . 540	61	* “	331 . 596	57
“	607 .1441	296	436	17 . 19	3	905	19 . 26	3	43	7 . 10	
126	1 . 5		441	11 . 13	2	916	7 . 29	3	49	359 . 990	91
“	127 .1007	201	449	126 . 323	43	919	216 . 703	73	58	3 . 11	
“	3961 .5111	1156	453	4 . 23	3	“	955 . 989	126	* 78	161 . 687	62
133	2 . 5		457	31 . 41	6	946	499 .1085	114	79	55 . 142	13
134	7 . 9	2	468	5 . 7		961	13 . 59	6	86	4787 .7687	741
143	15 . 73	14	“	1171 .3041	399	962	79 . 129	14	87	21 . 43	4
153	5183 .5833	1302	477	70 . 89	13	973	346 . 905	93	“	630 . 757	79
** 163	73 . 90	19	481	29 . 43	6	“	5919 .9460	1027	93	5 . 67	6
171	20 . 37	7	485	2991 .3664	539	981	19 . 308	31	94	4 . 11	
180	719 . 901	183	497	16 . 55	7	988	3187 .4717	518	1421	25 . 236	21
182	5 . 11	2	* 523	3275 .3286	513	995	17 . 28	3	* 26	1559 .1753	186
** “	1 . 17	3	532	13 . 23	3	1001	1 . 10		59	527 . 804	77

1463	305 . 388	39	2223	101 . 547	42	3383	327 . 761	52	4247	24 . 113	7
"	46 . 647	57	31	533 . 939	76	3400	1007 .2393	163	4303	51 . 74	5
82	25 . 29	3	61	4 . 13		09	19 . 44	3	"	22 . 309	19
"	3869 .9469	849	67	629 .1638	127	39	73 .3366	223	38	17 . 31	2
1505	11 . 34	3	2317	31 . 32	3	"	4 . 15		62	5 . 49	3
21	1 . 23	2	31	10 . 11		* 45	763 .1622	111	73	1170 .3203	199
31	19 . 45	4	47	11 . 53	4	58	71 . 73	6	81	161 . 176	13
33	2483 .2620	279	59	29 . 34	3	"	739 .2719	181	83	271 .1190	73
47	6 . 11		2413	6 . 13		65	71 . 94	7	96	929 .1087	78
* 54	23 . 31	3	76	5 . 27	2	73	9 . 14		4417	2651 .3028	219
88	1 . 35	3	83	15 . 176	13	74	23 . 25	2	38	1599 .3473	219
91	179 . 208	21	2527	26 . 37	3	3506	67 . 77	6	39	7 . 16	
95	2861 .9899	854	40	7 . 13		21	1633 .2894	201	45	128 . 507	31
96	5167 .9197	831	46	41 . 93	7	27	1007 .2520	169	53	49 . 76	5
*1603	87 . 142	13	55	4 . 41	3	47	3 . 61	4	4514	49 . 95	6
12	9 . 23	2	** 74	191 . 337	26	"	1041 .1156	91	35	7 . 33	2
21	631 . 990	91	2611	9 . 55	4	54	65 . 79	6	37	48 . 77	5
29	110 . 433	37	* "	25 . 38	3	96	173 . 871	57	71	39 . 614	37
57	1 . 71	6	34	13 . 41	3	3689	1808 .2323	171	4627	47 . 78	5
"	461 . 870	77	39	124 . 253	19	98	61 . 83	6	99	1980 .2719	181
65	413 . 547	52	"	626 .2767	201	3709	1819 .1890	151	4706	47 . 97	6
73	662 .1489	129	47	1009 .1639	127	"	1617 .3296	221	"	11 . 15	
99	17 . 47	4	66	257 . 767	56	18	7 . 15		"	249 .2647	158
1703	58 . 73	7	"	937 .1313	105	23	143 . 316	21	11	13 . 50	3
10	7289 .8101	813	74	129 . 253	19	24	1 . 31	2	23	46 . 79	5
29	1 . 12		2709	8 . 13		79	1336 .2393	163	45	1276 .2009	129
"	9 . 10		35	13 . 27	2	82	13 . 109	7	78	81 . 943	56
37	934 .1253	117	45	1 . 14		87	17 . 46	3	4825	9 . 16	
91	289 . 308	31	55	52 . 93	7	94	59 . 85	6	19	230 . 319	21
98	163 . 301	26	* 63	1115 .2878	209	3815	9 . 31	2	67	2795 .7253	436
1801	594 .1207	103	2821	23 . 40	3	29	1265 .3658	237	4921	2 . 17	
20	613 .1207	103	69	5 . 14		43	559 . 905	62	"	121 . 122	9
35	554 . 661	63	"	59 . 92	7	50	1691 .2159	157	"	698 .1499	91
42	19 . 35	3	99	7 . 57	4	58	7 . 47	3	** "	829 .1502	92
43	8 . 11		"	409 . 527	42	87	8 . 15		23	40 . 41	3
53	5 . 12		2907	449 . 520	43	3906	19 . 29	2	27	119 . 124	9
"	792 .1061	97	"	286 . 683	49	"	209 . 541	35	33	44 . 81	5
70	233 .1637	133	** 14	331 . 515	39	07	1 . 63	4	"	118 . 125	9
91	15 . 49	4	18	2917 .3257	273	"	62 . 63	5	40	3 . 17	
"	81 . 167	14	26	9 . 13		13	61 . 64	5	51	116 . 127	9
* 92	157 . 239	21	45	2 . 43	3	"	71 . 488	31	63	115 . 128	9
95	8 . 37	3	47	22 . 41	3	"	118 .1213	77	77	38 . 43	3
*1925	39 . 236	19	61	3583 .5441	412	*.*"	1732 .3299	219	"	4 . 17	
55	6623 .9017	806	77	848 .1213	93	25	12 . 13		93	113 . 130	9
67	1111 .1418	129	"	1562 .4999	351	43	59 . 66	5	5011	112 . 131	9
80	8011 .9809	903	90	1297 .1693	133	61	630 .3331	211	31	37 . 44	3
* 81	8229 .9383	916	94	11 . 43	3	"	937 .3024	193	39	5 . 17	
96	25 . 7	2	3015	106 . 299	21	67	58 . 67	5	47	43 . 82	5
2007	3561 .4631	416	31	197 . 236	19	97	16 . 47	3	53	110 . 133	9
13	32 . 265	21	41	1404 .1637	133	"	57 . 68	5	"	3917 .7819	474
*.*15	19 . 21	2	43	1170 .1873	139	99	2609 .3223	234	69	260 . 973	57
45	7 . 38	3	52	3 . 29	2	4033	56 . 69	5	74	509 . 553	39
60	9 . 11		59	11 . 12		"	1513 .2520	169	77	109 . 134	9
71	7 . 12		* 95	117 . 508	35	34	55 . 89	6	5111	2087 .3024	193
"	395 . 973	78	3155	1 . 44	3	75	11 . 14		29	6 . 17	
2107	13 . 51	4	97	10 . 13		97	1 . 16		31	107 . 136	9
"	2706 .3419	305	3211	5 . 59	4	4123	3 . 16		38	43 . 101	6
14	239 . 657	52	15	11 . 29	2	"	54 . 71	5	59	241 . 463	28
35	17 . 23	2	41	20 . 43	3	42	83 . 221	14	61	106 . 137	9
51	71 . 177	14	49	182 . 901	61	77	53 . 72	5	67	42 . 83	5
81	841 . 887	84	3325	751 .1049	78	78	53 . 91	6	"	2251 .2357	168
98	1 . 13		* 58	2547 .3203	245	4221	5 . 16		83	1511 .3672	217
2205	2 . 13		*.*67	1151 .1513	114	31	813 .1384	91	93	35 . 46	3
13	953 .1260	109	83	2 . 15		37	52 . 73	5	5227	104 . 139	9

5236	2033 . 3203	199	6253	85 . 158	9	7300	63 . 737	38	8703	20 . 61	3
57	11 . 52	3	62	89 . 113	7	24	1913 . 2087	130	10	31 . 59	3
63	103 . 140	9	92	141 . 343	19	53	1189 . 7067	364	29	9 . 20	
"	1099 . 3814	221	93	13 . 16		* 63	69 . 130	7	39	1241 . 1672	91
93	41 . 84	5	6309	431 . 1672	91	72	157 . 231	13	* 49	1169 . 4215	206
5301	34 . 47	3	27	28 . 53	3	78	3329 . 4049	241	8827	1 . 62	3
41	101 . 142	9	"	919 . 1190	73	90	37 . 53	3	"	59 . 184	9
78	41 . 103	6	"	557 . 979	56	7415	1 . 39	2	* "	22 . 103	5
83	100 . 143	9	35	3 . 37	2	71	71 . 172	9	55	3 . 1262	61
*5404	1007 . 1297	84	"	417 . 488	31	"	15 . 16		92	11 . 145	7
* 25	8 . 17		43	34 . 91	5	77	28 . 97	5	"	601 . 647	38
73	98 . 145	9	55	96 . 109	7	7553	394 . 685	37	8918	53 . 75	4
5509	27 . 29	2	6403	83 . 160	9	73	70 . 173	9	46	197 . 229	13
"	755 . 6541	314	81	82 . 161	9	88	9 . 19		53	58 . 185	9
10	1819 . 3691	217	98	11 . 37	2	7609	4 . 59	3	57	41 . 436	21
14	1 . 53	3	*99	247 . 356	21	** 57	14 . 17		87	49 . 50	3
17	1862 . 2131	143	6517	33 . 92	5	77	23 . 58	3	9009	16 . 17	
21	97 . 146	9	6623	50 . 129	7	87	27 . 98	5	37	15 . 41	2
37	362 . 655	39	23	2142 . 4481	247	"	3529 . 4158	247	53	47 . 52	3
51	10 . 53	3	41	12 . 17		7733	1205 . 2458	129	73	21 . 104	5
63	39 . 86	5	43	80 . 163	9	"	193 . 214	13	74	19 . 125	6
"	1872 . 3691	217	59	3293 . 3366	223	48	287 . 309	19	81	19 . 62	3
71	32 . 49	3	97	32 . 93	5	77	268 . 843	43	* "	380 . 2647	127
72	13 . 15		6715	201 . 479	26	83	68 . 175	9	9106	229 . 235	14
79	2137 . 5036	291	27	79 . 164	9	7829	1026 . 6803	343	09	2736 . 6373	313
5623	95 . 148	9	51	25 . 132	7	59	10 . 19		19	46 . 53	3
34	13 . 35	2	86	43 . 131	7	74	25 . 119	6	97	4409 . 4788	277
42	9 . 17		90	43 . 47	3	91	67 . 176	9	9206	51 . 77	4
61	890 . 1007	67	6813	26 . 55	3	7902	545 . 751	42	11	56 . 187	9
77	25 . 31	2	"	884 . 1387	79	03	26 . 99	5	23	3442 . 5781	247
"	94 . 149	9	20	2339 . 4481	247	43	4 . 419	21	62	1 . 21	
"	1858 . 5441	309	23	3060 . 3763	229	66	3093 . 6009	314	63	1279 . 2497	124
5707	38 . 87	5	18	31 . 113	6	8001	1 . 20		69	2 . 21	
33	31 . 50	3	67	2 . 19		** "	22 . 59	3	9310	29 . 61	3
59	1541 . 2446	147	78	2651 . 2837	182	11	2071 . 5940	301	25	4 . 21	
89	55 : 772	43	83	31 . 94	5	27	3 . 20		31	11 . 20	
91	92 . 151	9	86	3 . 19		29	17 . 39	2	64	3565 . 4211	234
95	74 . 231	13	6901	77 . 166	9	"	13 . 18		43	55 . 188	9
5833	1 . 18		03	7 . 1409	74	8101	2971 . 5130	271	86	5 . 21	
"	2161 . 3672	217	07	4196 . 7783	429	"	2011 . 4848	247	9431	3420 . 6011	301
51	91 . 152	9	10	41 . 49	3	13	65 . 178	9	49	43 . 56	3
57	37 . 88	5	23	4 . 19		17	3510 . 4607	259	69	4446 . 5023	283
5906	37 . 107	6	58	501 . 635	38	83	367 . 1136	57	9506	17 . 127	6
57	5 . 18		91	76 . 167	9	90	11 . 19		42	49 . 79	4
77	12 . 127	7	7033	1447 . 3422	183	** "	397 . 1523	76	49	1330 . 1853	97
"	89 . 194	9	49	197 . 227	14	98	63 . 65	4	83	19 . 106	5
6013	23 . 33	2	63	3659 . 5422	309	8227	64 . 179	9	9603	14 . 19	
"	36 . 89	5	67	61 . 130	7	46	61 . 67	4	13	53 . 190	9
39	185 . 706	39	75	6 . 19		58	23 . 121	6	20	957 . 1123	62
43	88 . 155	9	83	25 . 56	3	97	17 . 8280	409	41	1157 . 1331	74
6111	29 . 52	3	85	202 . 343	19	*8323	575 . 614	37	52	1601 . 3871	186
19	14 . 15		7107	254 . 367	21	** 42	59 . 69	4	9709	545 . 823	42
22	871 . 1125	70	10	13 . 17		53	24 . 101	5	51	52 . 191	9
46	1697 . 1705	117	18	4611 . 5215	323	8407	2 . 61	3	66	1029 . 3083	146
67	3775 . 4154	273	41	820 . 917	57	09	1577 . 1798	105	73	8 . 21	
75	7 . 18		54	29 . 115	6	61	62 . 181	9	79	41 . 58	3
81	8 . 55	3	63	11 . 18		86	57 . 71	4	9809	3869 . 5940	301
"	2651 . 4208	247	77	74 . 169	9	8541	88 . 131	7	47	18 . 107	5
"	86 . 157	9	89	19 . 37	2	81	61 . 182	9	91	17 . 64	3
88	305 . 307	21	7202	7 . 19		87	23 . 102	5	99	2969 . 6930	331
94	35 . 109	6	31	5 . 58	3	* "	12 . 19		9926	575 . 843	43
6202	379 . 507	31	73	29 . 96	5	8602	2591 . 6011	301	"	47 . 81	4
44	11 . 17		"	73 . 170	9	58	7 . 41	2	54	5 . 43	2
47	1530 . 4717	259	89	2159 . 5130	271	78	55 . 73	4	77	40 . 59	3

10003	4792 .8069	399	11627	34 . 65	3	13477	29 . 214	9	15433	66 . 167	
010	3637 .6373	313	667	107 . 109	6	483	6 . 119	5	542	29 . 99	4
033	50 . 193	9	673	901 .2990	133	498	11 . 23		"	15 . 23	
075	2486 .4489	219	676	53 . 55	3	510	19 . 71	3	561	17 . 22	
089	26 . 151	7	691	35 . 37	2	516	875 .3049	129	** "	719 . 820	39
117	17 . 108	5	727	13 . 68	3	538	1 . 143	6	572	233 . 311	14
"	2798 .9369	437	739	103 . 113	6	539	83 . 133	6	587	25 . 74	3
177	49 . 194	9	803	791 .2080	93	607	29 . 70	3	626	1 . 25	
197	13 . 20		811	101 . 115	6	663	28 . 215	9	652	3 . 25	
213	13 . 43	2	863	11 . 114	5	692	41 . 67	3	657	845 .2827	114
234	15 . 19		893	38 . 205	9	699	2348 .4141	183	689	4 . 25	
261	10 . 21		906	7 . 137	6	722	2267 .4483	195	732	125 . 151	7
"	49 . 282	13	914	4427 .7487	349	793	3292 .6257	273	769	50 . 87	3
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323	16 . 65	3	964	49 . 59	3	895	12 . 23		771	71 . 145	6
358	45 . 83	4	12027	97 . 119	6	949	5 . 24		841	17 . 226	9
381	4685 .8662	417	061	37 . 206	9	958	33 . 95	4	"	6 . 25	
393	16 . 109	5	171	95 . 121	6	14021	8923 .9104	471	939	377 . 382	19
403	259 . 549	26	175	2 . 23		023	15 . 22		965	233 . 282	13
406	115 . 127	7	194	3 . 23		041	26 . 217	9	996	35 . 73	
418	13 . 131	6	229	2039 .2067	112	069	28 . 71	3	16001	7307 .8694	403
423	717 . 772	43	252	47 . 61	3	167	7 . 24		003	49 . 68	3
439	38 . 61	3	292	5 . 23		173	4 . 121	5	021	13 . 24	
459	1260 .9199	421	293	5741 .6552	337	174	17 . 21		030	39 . 41	2
"	3266 .3295	189	"	221 . 416	57	187	79 . 137	6	051	16 . 227	9
471	47 . 196	9	341	757 .1006	49	213	67 . 166	7	137	8 . 25	
547	3060 .7487	349	350	4033 .8317	373	233	25 . 218	9	165	99 . 166	7
621	46 . 197	9	353	32 . 67	3	381	27 . 170	7	237	1371 .2725	112
649	1 . 22		383	6 . 23		427	8 . 73	3	263	5 . 76	3
675	3 . 22		403	35 . 208	9	527	3 . 122	5	"	16 . 23	
697	2224 .2843	147	410	567 . 793	38	"	4233 .6415	286	270	37 . 43	2
703	37 . 62	3	411	158 . 433	19	547	77 . 139	6	354	9 . 25	
731	541 . 782	39	434	5 . 139	6	590	17 . 73	3	371	1748 .3709	151
745	17 . 18		493	9 . 116	5	623	23 . 220	9	373	898 .1441	61
829	4241 .4463	248	* "	2437 .6212	273	687	6930 .7757	379	406	27 . 101	4
838	43 . 85	4	510	7 . 23		726	31 . 97	4	453	1353 .2743	112
863	1801 .1820	103	531	91 . 125	6	733	1870 .3041	133	477	14 . 229	9
891	3961 .6930	331	566	37 . 91	4	821	22 . 221	9	543	8263 .8280	409
898	11 . 133	6	577	34 . 209	9	833	58 . 59	3	549	47 . 70	3
927	983 .1024	57	580	5777 .6603	343	849	12 . 467	19	588	181 .1095	43
955	16 . 19		601	4284 .8317	373	859	19 . 20		"	4665 .5543	254
963	14 . 111	5	629	10 . 163	7	868	275 . 433	19	597	5 . 51	2
981	2510 .9657	437	635	1021 .1859	84	887	2 . 123	5	693	13 . 230	9
991	7 . 22		679	8 . 23		911	56 . 61	3	707	67 . 149	6
11083	43 . 200	9	747	89 . 127	6	"	14 . 23		709	23 . 76	3
115	391 . 561	28	749	31 . 68	3	987	505 .1636	67	739	3275 .4654	201
* 167	32 . 827	37	753	11 . 70	3	989	55 . 62	3	857	2576 .3043	139
241	14 . 67	3	817	8 . 117	5	15021	7 . 74	3	861	46 . 71	3
249	6177 .6679	362	844	2541 .2867	146	059	26 . 73	3	884	3155 .3277	158
257	13 . 112	5	845	13 . 22		106	629 . 865	39	911	4 . 77	3
297	35 . 64	3	913	17 . 20		132	37 . 71	3	937	7038 .9899	427
366	41 . 87	4	931	32 . 211	9	155	11 . 24		990	13 . 77	3
377	9 . 22		978	1 . 47	2	191	669 . 706	35	17019	2605 .3347	148
401	41 . 202	9	13069	9 . 47	2	219	1223 .3850	157	071	1277 .1819	78
411	2394 .9017	403	111	31 . 212	9	223	53 . 64	3	131	11 . 232	9
419	185 . 416	19	116	43 . 65	3	"	20 . 223	9	138	7939 .9199	421
431	124 .1509	67	137	365 . 418	21	237	2089 .2990	133	211	65 . 151	6
439	5774 .9601	455	147	7 . 118	5	253	1 . 124	5	243	66 . 335	13
458	13 . 21		167	10 . 23		275	1127 .9448	381	261	20 . 21	
557	11 . 45	2	204	857 .1191	56	"	469 . 706	31	290	9 . 181	7
"	12 . 113	5	238	35 . 93	4	281	93 . 320	13	318	25 . 103	4
563	40 . 203	9	251	85 . 131	6	339	73 . 143	6	353	10 . 233	9
590	23 . 67	3	293	10 . 71	3	353	3525 .5923	254	"	12 . 25	
"	577 . 673	35	357	16 . 21		427	19 . 224	9	422	1435 .3061	122

17470	33 . 47	2	19631	124 . 169	7	22659	667 .1574	57	25991	23 . 24	
507	19 . 22		684	1 . 27		681	9 . 28		26110	1 . 89	3
* 563	44 . 73	3	*+691	2 . 27		785	67 . 68	3	117	914 .1095	43
593	2455 .2458	119	692	739 . 761	35	806	1507 .7181	254	"	12 . 29	
603	3 . 26		721	16 . 25		827	47 . 169	6	143	29 . 88	3
647	1259 .1262	61	747	4 . 27		847	14 . 85	3	147	10 . 89	3
701	5 . 26		909	21 . 22		849	143 . 170	7	162	47 . 207	7
738	23 . 339	13	922	53 . 73	3	913	37 . 51	2	273	22 . 25	
803	8 . 235	9	20026	7 . 27		939	1133 .2144	79	307	29 . 298	7
"	5246 .6921	299	033	135 . 164	7	965	64 . 71	3	342	9 . 119	4
822	13 . 25		"	563 .1093	42	23068	153 . 163	7	364	19 . 89	3
919	7 . 26		091	55 . 161	6	093	1622 .1677	73	381	151 . 562	19
953	43 . 74	3	167	20 . 23		175	227 . 373	14	385	52 . 83	3
999	18 . 23		195	8 . 27		196	23 . 85	3	425	1132 .2643	91
18012	31 . 77	3	239	227 . 320	13	236	62 . 73	3	542	19 . 27	
031	7 . 236	9	243	2345 .2879	122	283	11 . 28		586	13 . 29	
167	1063 .3356	129	258	1385 .1509	67	415	61 . 74	3	787	37 . 179	6
261	2 . 79	3	293	38 . 79	3	634	61 . 545	19	827	1429 .3484	119
265	5384 .7261	309	"	254 .2645	97	639	13 . 86	3	830	21 . 59	2
278	23 . 105	4	342	19 . 109	4	779	16 . 27		837	21 . 26	
"	147 . 815	31	501	178 . 569	21	"	199 . 354	13	847	83 . 388	13
291	61 . 155	6	538	17 . 25		798	13 . 115	4	873	31 . 57	2
305	9 . 26		557	1341 .1355	62	865	59 . 76	3	881	1426 .3487	119
310	11 . 79	3	603	17 . 82	3	877	251 . 886	31	923	28 . 89	3
369	14 . 25		683	10 . 27		959	32 . 85	3	972	269 . 335	13
404	1715 .2137	93	"	19 . 24		969	35 . 53	2	27001	1 . 30	
430	31 . 49	2	739	53 . 163	6	24122	43 . 83	3	"	348 . 523	19
493	5 . 238	9	797	1 . 55	2	135	58 . 77	3	133	14 . 29	
557	758 .1893	73	839	37 . 80	3	149	13 . 28		343	7 . 30	
"	20 . 79	3	942	263 . 303	13	225	307 .1418	43	649	287 . 289	12
578	61 . 65	3	951	15 . 26		271	701 .1166	43	651	35 . 181	6
613	3 . 53	2	21014	11 . 27		274	425 . 529	21	652	143 . 145	6
"	2115 .2798	119	070	27 . 53	2	309	2071 .6032	211	657	95 . 97	4
631	2112 .2801	119	086	795 . 827	37	339	43 . 173	6	673	283 . 293	12
724	2671 .2765	129	190	7 . 83	3	390	1 . 29		683	20 . 27	
737	17 . 24		196	1135 .1893	73	397	2 . 29		684	47 . 49	2
746	59 . 67	3	203	465 .1166	43	453	4 . 29		686	7 . 121	4
779	263 . 538	21	223	541 . 675	28	490	123 . 187	7	697	281 . 295	12
783	1150 .5111	193	268	5741 .7347	302	514	5 . 29		748	139 . 149	6
811	41 . 76	3	329	43 . 45	2	535	1339 .3166	109	755	101 . 204	7
837	146 . 337	13	"	16 . 83	3	545	1157 .1403	56	764	15 . 29	
849	808 . 839	39	446	17 . 111	4	596	17 . 27		769	277 . 299	12
867	59 . 157	6	457	18 . 25		598	129 . 373	13	817	275 . 301	12
907	11 . 26		537	3400 .3779	163	605	6 . 29		844	137 . 151	6
963	1 . 80	3	593	41 . 47	2	661	31 . 86	3	873	91 . 101	4
19026	19 . 23		638	175 . 523	19	670	23 . 57	2	915	49 . 86	3
097	3269 .3963	172	691	1772 .3141	119	765	56 . 79	3	929	8 . 91	3
107	439 . 452	21	727	1769 .3144	119	822	103 .1079	37	937	271 . 305	12
129	730 .3923	147	756	25 . 83	3	886	21 . 25		** "	103 . 204	7
"	2062 .2591	111	835	3378 .3497	155	901	8 . 29		** 986	37 . 89	2
164	29 . 79	3	953	1 . 28		25046	11 . 117	4	28009	269 . 307	12
178	1093 .2921	111	979	3 . 28		118	9 . 29		"	7404 .8221	325
* 201	2 . 241	9	22009	35 . 82	3	131	41 . 175	6	089	89 . 103	4
217	19 . 80	3	077	5 . 28		198	29 . 557	19	092	17 . 91	3
279	40 . 77	3	085	1011 .2144	79	201	136 . 183	7	132	133 . 155	6
286	21 . 107	4	106	47 . 79	3	207	772 . 853	35	177	265 . 311	12
361	1882 .7289	273	107	49 . 167	6	298	41 . 85	3	"	668 . 815	31
369	1131 .1636	67	121	39 . 49	2	327	15 . 28		273	263 . 313	12
"	4724 .7443	299	363	5167 .5426	237	389	10 . 29		324	131 . 157	6
418	55 . 71	3	427	14 . 27		441	349 . 578	21	331	11 . 30	
431	2627 .3850	157	484	19 . 25		606	1641 .2017	79	489	259 . 317	12
441	1 . 242	9	489	17 . 26		613	632 .3027	103	538	725 . 777	31
565	151 . 369	14	598	15 . 113	4	675	176 .1799	61	548	43 . 53	2
630	29 . 51	2	633	34 . 83	3	935	53 . 82	3	582	455 . 467	19

21428 $= 21^3 + 23^3$

28609	257 . 319	12	30233	244 . 373	13	32409	73 . 119	4	34706	101 . 149	8
683	1657 .7904	259	249	79 . 113	4	426	111 . 139	5	* "	29 . 97	3
721	29 . 59	2	303	8 . 31		548	109 . 179	6	723	3207 .3652	133
737	85 . 107	4	331	22 . 27		581	139 . 204	7	821	1798 .9809	301
791	3407 .7561	254	340	241 . 579	19	686	499 . 563	21	831	129 . 214	7
804	127 . 161	6	361	1345 .1472	57	689	217 . 359	12	846	509 .1849	57
811	19 . 28		387	29 . 187	6	769	1 . 32		859	1 . 98	2
827	4373 .5236	199	411	122 . 205	7	* 779	138 . 205	7	865	717 .1118	37
853	7 . 92	3	439	153 . 190	7	786	109 . 141	5	873	203 . 373	12
861	401 .1118	37	457	235 . 341	12	795	3 . 32		909	5864 .6115	231
873	253 . 323	12	"	5365 .9062	309	* 858	31 . 95	3	35001	1924 .2069	77
29017	251 . 325	12	518	3 . 125	4	893	5 . 32		037	22 . 29	
054	1969 .2425	91	553	152 . 191	7	938	2399 .4495	147	044	101 . 187	6
078	5 . 123	4	673	233 . 343	12	977	215 . 361	12	089	128 . 215	7
"	1511 .2643	91	"	151 . 192	7	983	137 . 206	7	092	4273 .5915	201
* "	249 .3905	127	767	5 . 94	3	33111	7 . 32		113	19 . 98	3
085	47 . 88	3	791	10 . 31		124	107 . 181	6	191	51 . 53	2
092	125 . 163	6	799	150 . 193	7	166	15 . 31		217	67 . 125	4
097	1616 .2677	93	833	27 . 61	2	167	76 . 77	3	252	2391 .2645	97
169	83 . 109	4	897	77 . 115	4	193	136 . 207	7	306	99 . 151	5
197	13 . 30		931	149 . 194	7	194	107 . 143	5	308	25 . 27	
230	19 . 61	2	31012	115 . 173	6	201	25 . 26		353	127 . 216	7
* 302	17 . 29		065	44 . 91	3	209	25 . 63	2	411	65 . 88	3
329	247 . 329	12	069	148 . 195	7	219	5510 .5563	217	425	102 . 223	7
412	41 . 55	2	122	11 . 31		269	74 . 79	3	427	19 . 197	6
413	171 . 172	7	129	229 . 347	12	273	71 . 121	4	503	49 . 55	2
* 419	170 . 173	7	274	123 . 127	5	315	41 . 94	3	569	199 . 377	12
"	25 . 92	3	346	121 . 129	5	339	23 . 193	6	623	18 . 31	
"	3049 .9118	299	363	146 . 197	7	371	73 . 80	3	779	5429 .6550	231
431	169 . 174	7	369	227 . 349	12	409	135 . 208	7	819	64 . 89	3
449	24 . 25		466	119 . 131	5	497	9 . 32		828	465 .1231	35
451	31 . 185	6	492	113 . 176	6	577	211 . 365	12	835	38 . 97	3
473	167 . 176	7	519	145 . 198	7	579	265 . 596	19	849	23 . 65	2
497	245 . 331	12	"	12 . 31		631	134 . 209	7	899	125 . 218	7
503	166 . 177	7	561	1903 .2722	95	650	21 . 29		929	197 . 379	12
539	165 : 178	7	591	1599 .2914	97	659	141 . 206	7	938	1 . 33	
581	164 . 179	7	634	117 . 133	5	677	71 . 82	3	945	2 . 33	
603	1315 .2914	97	681	144 . 199	7	732	35 . 61	2	954	97 . 153	5
629	163 . 180	7	707	503 . 550	21	803	2 . 97	3	36001	411 . 562	19
637	3220 .6659	223	757	4 . 95	3	852	11 . 97	3	"	4 . 33	
665	303 . 377	14	785	43 . 92	3	859	19 . 30		062	5 . 33	
683	162 . 181	7	798	1117 .1329	49	867	4466 .6823	229	127	47 . 57	2
711	3590 .3637	147	814	1739 .2655	91	881	70 . 83	3	181	124 . 219	7
715	46 . 89	3	836	13 . 95	3	887	16 . 31		225	307 .1418	43
743	23 . 26		849	143 . 200	7	889	209 . 367	12	"	536 .1189	37
764	121 . 167	6	850	23 . 27		929	338 . 579	19	297	65 . 127	4
799	2 . 31		870	17 . 63	2	34021	1086 .3827	119	"	4499 .5626	195
818	3 . 31	.	873	223 . 353	12	067	6364 .9773	327	349	505 . 799	26
809	160 . 183	7	913	17 . 30		093	132 . 211	7	449	8 . 33	
855	4 . 31		988	13 . 31		"	1083 .3830	119	469	123 . 220	7
857	241 . 335	12	32004	37 . 59	2	099	20 . 97	3	484	97 . 191	6
881	159 . 184	7	006	1 . 127	4	* "	11 . 32		507	17 . 199	6
959	158 . 185	7	023	201 . 142	7	119	23 . 28		556	23 . 29	
979	3961 .6032	211	* 095	316 . 339	13	154	103 . 147	5	650	19 . 31	
30007	6 . 31		114	113 . 137	5	290	341 . 379	14	667	1673 .7246	219
043	157 . 186	7	123	172 . 181	7	333	131 . 212	7	673	193 . 383	12
049	239 . 337	12	137	221 . 355	12	"	277 . 835	26	731	691 . 781	28
133	156 . 187	7	"	2252 .2339	91	372	103 . 185	6	735	37 . 98	3
134	7 . 31		149	22 . 95	3	417	3309 .3550	133	737	62 . 91	3
148	119 . 169	6	"	6024 .9601	325	532	155 . 201	7	763	122 . 221	7
191	6605 .7696	291	203	141 . 202	7	537	205 . 371	12	783	1726 .2267	77
221	18 . 29		331	25 . 191	6	579	130 . 213	7	937	10 . 33	
222	2971 .5861	196	338	339 .1363	43	615	192 . 613	19	955	898 .1047	37
229	155 . 188	7	389	20 . 29		697	67 . 86	3	37023	520 . 587	21

37057	191 . 386	12	39679	113 . 230	7	42765	31 . 104	3	45638	187 . 299	9
063	45 . 59	2	681	4030 .9197	277	777	55 . 137	4	847	120 . 241	7
"	121 . 222	7	718	1389 .4285	127	796	19 . 33		871	1637 .4916	139
219	17 . 100	3	735	838 .1037	35	"	109 .3183	91	974	121 . 241	7
247	61 . 92	3	748	89 . 199	6	866	81 . 169	5	46011	1 . 215	6
252	95 . 193	6	758	229 . 257	9	883	2 . 35		019	49 . 104	3
259	26 . 27		871	41 . 63	2	894	67 . 95	3	063	97 . 246	7
369	120 . 223	7	878	227 . 259	9	902	3 . 35		094	185 . 301	9
394	93 . 157	5	891	11 . 205	6	939	4 . 35		163	19 . 34	
429	1874 .3473	109	914	87 . 163	5	43081	104 . 239	7	167	1091 .1285	42
522	1331 .2781	86	969	59 . 133	4	091	6 . 35		185	28 . 107	3
577	25 . 28		995	379 . 381	14	155	976 .1079	37	209	1088 .4609	129
681	134 . 635	19	997	4150 .5579	183	238	199 . 287	9	215	694 .1181	35
"	17 . 32		40033	9 . 34		273	163 . 413	12	252	3827 .9601	273
791	20 . 31		"	16 . 33		290	191 . 199	7	303	85 . 86	3
801	160 . 207	7	166	223 . 263	9	* "	529 .2591	74	341	28 . 29	
837	6 . 235	7	307	56 . 97	3	367	52 . 101	3	417	151 . 425	12
849	187 . 389	12	329	6623 .6820	247	387	8 . 35		* "	83 . 88	3
870	13 . 67	2	334	221 . 265	9	489	103 . 240	7	423	35 . 69	2
954	2395 .3027	103	393	111 . 232	7	491	5 . 211	6	513	96 . 247	7
999	118 . 225	7	417	175 . 401	12	533	7205 .9379	302	531	82 . 89	3
38052	31 . 65	2	439	22 . 31		598	197 . 289	9	566	61 . 101	3
087	2433 .3008	103	476	5 . 103	3	604	9 . 35		655	174 .1331	37
134	13 . 33		518	73 . 89	3	615	24 . 31		"	528 . 557	19
172	7 . 101	3	573	14 . 103	3	777	161 . 415	12	657	1 . 36	
186	91 . 159	5	644	29 . 67	2	903	102 . 241	7	692	25 . 71	2
213	24 . 29		657	272 . 709	21	927	37 . 67	2	702	1221 .1237	43
257	185 . 391	12	718	217 . 269	9	937	20 . 33		781	5 . 36	
311	16 . 101	3	759	110 . 233	7	946	55 . 57	2	873	80 . 91	3
* "	43 . 61	2	850	17 . 33		"	79 . 171	5	900	383 . 417	14
323	117 . 226	7	873	173 . 403	12	"	6219 .7499	247	909	3827 .4173	140
369	59 . 94	3	922	23 . 103	3	974	65 . 97	3	969	95 . 248	7
419	173 . 200	7	934	215 . 271	9	44161	11 . 106	3	* "	149 . 427	12
653	116 . 227	7	*41021	55 . 98	3	206	11 . 35		971	1727 .9289	258
673	61 . 131	4	131	109 . 234	7	225	333 . 392	13	"	16 . 35	
681	14 . 33		166	71 . 91	3	282	53 . 59	2	999	7 . 36	
682	653 .1189	37	230	11 . 69	2	289	53 . 139	4	47054	181 . 305	9
738	25 . 101	3	401	4492 .7487	231	323	101 . 242	7	101	79 . 92	3
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884	91 . 197	6	509	108 . 235	7	505	103 . 242	7	377	189 . 460	13
981	58 . 95	3	"	2521 .4338	133	548	79 . 209	6	431	94 . 249	7
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39026	89 . 161	5	598	1457 .1615	56	749	100 . 243	7	474	73 . 177	5
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305	1 . 34		834	83 . 167	5	954	51 . 61	2	812	73 . 215	6
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374	241 . 245	9	"	3529 .6668	201	101	50 . 103	3	959	872 .1297	39
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429	5 . 34		965	26 . 29		207	5600 .9469	283	987	11 . 36	
438	79 . 83	3	42029	21 . 32		212	1477 .1727	57	48078	59 . 103	3
493	100 . 441	13	254	205 . 281	9	337	155 . 421	12	097	145 . 431	12
494	235 . 251	9	267	7 . 209	6	353	17 . 71	2	146	17 . 109	3
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566	233 . 253	9	289	167 . 409	12	372	1 . 107	3	373	92 . 251	7
615	34 . 101	3	453	6247 .7904	259	409	10 . 107	3	393	25 . 32	
627	19 . 32		551	53 . 100	3	586	1413 .1531	52	565	21 . 34	
647	7 . 34		566	203 . 283	9	604	77 . 211	6	583	831 .1726	49
654	77 . 85	3	679	15 . 34		626	19 . 107	3	614	175 . 311	9

	48615	26 . 109	3	52166	163 . 323	9	55601	9 . 38		59332	65 . 233	6
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	670	7 . 73	2	297	131 . 445	12	783	29 . 75	2	383	4 . 39	
	673	143 . 433	12	351	31 . 73	2	805	161 . 199	6	444	5 . 39	
	707	18 . 35		381	12 . 37		811	40 . 113	3	445	49 . 71	2
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	861	2884 . 4325	129	612	65 . 223	6	205	53 . 67	2	687	37 . 116	3
	964	71 . 217	6	801	67 . 104	3	209	119 . 457	12	689	109 . 467	12
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	166	173 . 313	9	927	3656 . 3905	127	378	11 . 115	3	831	8 . 39	
	231	33 . 71	2	929	43 . 149	4	420	79 . 101	3	914	21 . 37	
omit	247	198 . 209	7	53009	13 . 75	2	483	3782 . 4287	133	60174	47 . 115	3
	257	47 . 145	4	261	345 . 1423	38	485	18 . 37		319	10 . 39	
	267	73 . 98	3	354	41 . 71	2	548	59 . 229	6	326	29 . 33	
	309	7 . 110	3	383	961 . 1316	39	645	157 . 203	6	409	107 . 469	12
	339	90 . 253	7	397	14 . 37		699	24 . 35		451	26 . 35	
	474	27 . 31		417	2032 . 5599	151	773	76 . 267	7	634	4213 . 4449	139
	799	895 . 1726	49	443	82 . 261	7	791	62 . 109	3	650	11 . 39	
	831	89 . 254	7	"	4 . 113	3	"	30 . 31		667	58 . 113	3
	849	139 . 437	12	467	383 . 760	21	852	207 . 725	19	732	1159 . 1433	42
	959	1238 . 2755	77	478	53 . 109	3	57038	149 . 337	9	772	53 . 235	6
	50066	69 . 181	5	513	26 . 33		057	2582 . 3061	93	845	143 . 217	6
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	449	137 . 439	12	54005	179 . 181	6	380	77 . 103	3	542	365 . 1213	31
	* 654	1 . 37		020	89 . 91	3	386	59 . 191	5	549	68 . 275	7
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	833	87 . 256	7	217	125 . 451	12	924	19 . 77	2	62063	14 . 39	
	863	70 . 101	3	245	173 . 187	6	931	74 . 269	7	173	67 . 276	7
	869	6 . 37		283	65 . 106	3	58025	4 . 271	7	244	17 . 79	2
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	996	7 . 37		343	649 . 1104	31	205	151 . 209	6	354	53 . 197	5
	51002	43 . 69	2	529	80 . 263	7	247	15 . 38		405	139 . 221	6
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	142	955 . 4103	111	605	169 . 191	6	311	925 . 2342	61	461	2914 . 6009	157
	343	86 . 257	7	749	16 . 37		357	2164 . 2325	73	558	27 . 35	
	364	67 . 221	6	845	167 . 193	6	361	38 . 115	3	565	16 . 119	3
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	534	55 . 107	3	980	83 . 97	3	653	20 . 37		833	56 . 115	3
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	673	133 . 443	12	"	64 . 107	3	** 977	37 . 155	4	245	137 . 223	6
	743	28 . 31		204	61 . 227	6	987	27 . 34		378	1255 . 1763	49
	847	1559 . 1798	57	335	953 . 1342	39	994	57 . 193	5	379	297 . 742	19
	859	85 . 258	7	445	163 . 197	6	59066	37 . 75	2	399	88 . 101	3
	52039	5 . 112	3	537	121 . 455	12	113	72 . 271	7	415	16 . 39	
	117	68 . 103	3	559	1765 . 6172	163	327	2 . 39		439	65 . 278	7

63564	5243.8059	219	66619	237.275	8	70243	60.121	3	74089	1.42	
566	133.353	9	678	3253.3497	105	245	41.79	2	109	71.118	3
583	25.79	2	709	60.283	7	252	11.41		213	5.42	
748	49.239	6	757	1139.1354	39	372	3.289	7	214	37.125	3
973	55.116	3	827	32.121	3	579	215.297	8	284	101.155	4
64001	1.40		859	235.277	8	649	12.41		387	226.793	19
027	3.40		908	5813.6635	194	665	11.124	3	438	3267.3277	98
029	86.103	3	927	80.109	3	811	141.278	7	461	47.124	3
081	64.279	7	988	117.139	4	820	61.119	3	473	3221.7418	181
092	3567.5599	151	67005	43.77	2	828	107.149	4	522	29.83	2
097	5987.6388	195	039	23.38		831	54.289	7	555	27.38	
106	51.199	5	123	233.279	8	859	19.40		583	7048.7691	221
129	97.479	12	"	1492.5367	133	907	1973.2158	63	611	201.311	8
133	21.38		214	125.361	9	71045	29.36		737	71.505	12
343	7.40		249	89.487	12	083	213.299	8	753	18.41	
407	85.104	3	319	20.39		329	79.497	12	75042	29.37	
454	131.355	9	335	13.122	3	337	74.115	3	078	6247.6875	189
477	24.37		381	59.284	7	406	121.239	6	169	48.295	7
580	67.113	3	394	433.1741	43	486	23.39		172	35.253	6
729	9.40		411	231.281	8	489	5.83	2	221	1417.2342	61
790	387.493	[illegible]	557	79.110	3	505	377.3028	73	241	33.34	
897	95.481	12	564	115.141	4	519	29.124	3	278	109.377	9
65007	184.515	13	621	52.119	3	539	53.290	7	283	199.313	8
045	133.227	6	723	229.283	8	594	43.207	5	* 419	11.42	
284	47.241	6	754	47.203	5	611	49.122	3	591	134.547	13
289	83.106	3	951	23.81	2	"	211.301	8	609	23.169	4
331	11.40		68045	127.233	6	665	14.41		"	991.1520	39
358	43.119	3	049	29.163	4	748	13.83	2	626	39.211	5
383	62.281	7	059	58.285	7	884	105.151	4	628	99.157	4
"	176.257	7	"	227.285	8	962	97.101	3	643	32.35	
539	255.257	8	166	41.121	3	72134	115.371	9	780	19.41	
548	127.129	4	236	113.143	4	163	209.303	8	823	518.725	19
563	253.259	8	357	31.122	3	169	77.499	12	885	8.127	3
585	2398.2647	79	419	225.287	8	219	73.116	3	913	47.296	7
611	251.261	8	452	43.245	6	226	95.103	3	943	46.125	3
618	5.121	3	705	32.33		253	52.291	7	979	197.315	8
644	125.131	4	743	57.286	7	289	4786.5541	157	76015	389.531	14
653	3851.5528	151	803	223.289	8	338	1.125	3	121	212.255	7
673	31.161	4	857	85.491	12	377	215.246	7	186	85.113	3
683	249.263	8	913	17.40		407	445.742	19	265	2161.2214	65
709	211.236	7	922	1.41		605	28.37		285	13.42	
715	14.121	3	929	2.41		"	119.241	6	294	511.1551	37
779	247.265	8	948	3.41		631	21.83	2	358	107.379	9
793	82.107	3	69004	111.145	4	666	31.35		445	113.247	6
807	411.422	13	046	5.41		739	207.305	8	447	31.36	
836	123.133	4	095	31.34		802	5539.5853	172	489	67.509	12
899	245.267	8	134	121.365	9	973	51.292	7	517	26.127	3
906	49.101	5	137	6.41		73017	25.167	4	663	46.297	7
66005	131.229	6	211	221.291	8	** "	16.41		677	2885.8428	201
043	61.282	7	433	8.41		036	103.153	4	699	195.317	8
"	243.269	8	578	1303.1433	42	151	28.125	3	706	5165.5793	163
* 122	33.79	2	643	219.293	8	166	113.373	9	769	3.85	2
124	121.135	4	650	9.41		220	59.121	3	879	1185.1367	38
178	19.39		673	83.493	12	261	21.40		921	20.41	
185	1219.1221	38	699	76.113	3	282	91.107	3	932	11.85	2
197	13.40		868	109.147	4	339	205.307	8	77068	97.159	4
211	241.271	8	921	10.41		444	37.251	6	289	68.121	3
278	127.359	9	967	22.39		586	41.209	5	377	65.511	12
"	25.37		70084	42.247	6	699	50.293	7	419	45.298	7
367	53.118	3	099	217.295	8	834	17.41		443	193.319	8
403	239.273	8	118	119.367	9	873	73.503	12	454	35.127	3
457	91.485	12	129	55.288	7	** 963	203.309	8	506	83.115	3
473	7.81	2	135	137.278	7	74053	4407.6172	163	623	19.85	2
508	119.137	4	154	31.81	2	074	89.109	3	653	30.37	

77663	1033 . 1066	31	81523	183 . 329	8	85309	5 . 44		89649	13 . 179	4
685	7 . 128	3	821	23 . 130	3	339	18 . 43		713	25 . 42	
714	37 . 213	5	937	55 . 521	12	527	7 . 44		836	83 . 173	4
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78181	44 . 299	7	998	97 . 389	9	633	40 . 131	3	* 957	34 . 37	
182	21 . 41		82034	33 . 217	5	708	87 . 169	4	999	85 . 122	3
211	191 . 321	8	047	64 . 125	3	715	953 . 1812	43	90083	53 . 83	2
251	25 . 128	3	081	39 . 304	7	721	91 . 116	3	097	17 . 44	
393	67 . 122	3	133	103 . 104	3	729	47 . 529	12	148	19 . 269	6
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604	95 . 161	4	251	14 . 43		913	9 . 44		331	29 . 314	7
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639	599 . 1777	42	313	1 . 87	2	86079	61 . 128	3	459	5407 . 7289	182
659	67 . 69	2	409	100 . 107	3	131	34 . 309	7	489	58 . 131	3
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724	31 . 257	6	537	80 . 827	19	212	23 . 265	6	649	37 . 539	12
812	33 . 35		693	5245 . 5403	154	"	2311 . 9505	217	783	1984 . 2339	61
949	43 . 300	7	739	59 . 77	2	255	23 . 42		785	664 . 691	19
79001	17 . 42		745	24 . 41		366	19 . 43		809	33 . 38	
003	189 . 323	8	801	859 . 1812	43	497	26 . 41		873	28 . 41	
021	44 . 127	3	823	93 . 109	3	546	29 . 221	5	980	47 . 133	3
067	65 . 71	2	873	53 . 523	12	590	33 . 37		991	37 . 134	3
109	491 . 1660	39	879	38 . 305	7	746	73 . 125	3	91126	1 . 45	
177	61 . 515	12	882	15 . 43		798	89 . 397	9	133	2 . 45	
205	109 . 251	6	927	17 . 87	2	959	33 . 310	7	134	233 . 265	7
226	27 . 85	2	956	6731 . 6769	195	963	89 . 118	3	189	4 . 45	
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345	2896 . 8439	199	174	95 . 391	9	211	171 . 341	8	289	178 . 295	7
506	1265 . 2521	61	265	4 . 131	3	227	55 . 81	2	468	7 . 45	
508	1 . 43		323	179 . 333	8	236	155 . 297	7	483	163 . 349	8
534	3 . 43		421	32 . 37		255	1261 . 2894	67	637	8 . 45	
571	4 . 43		582	4147 . 4317	122	381	13 . 44		657	35 . 541	12
604	3237 . 8135	193	603	16 . 43		431	20 . 133	3	674	23 . 43	
694	101 . 385	9	645	103 . 257	6	620	49 . 131	3	683	248 . 253	7
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819	187 . 325	8	683	37 . 306	7	673	43 . 533	12	793	83 . 124	3
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80019	8 . 43		788	89 . 167	4	88038	29 . 133	3	92001	3970 . 5129	129
089	59 . 517	12	789	95 . 112	3	147	213 . 274	7	044	81 . 175	4
219	298 . 532	13	817	17 . 175	4	164	7 . 89	2	053	27 . 316	7
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"	9 . 43		951	477 . 776	19	389	29 . 40		105	2349 . 2651	70
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645	107 . 253	6	* "	31 . 219	5	657	41 . 535	12	93158	79 . 407	9
659	185 . 327	8	292	25 . 263	6	874	27 . 223	5	249	9257 . 9871	266
674	79 . 119	3	357	2 . 307	7	939	991 . 993	28	310	29 . 41	
731	1062 . 1313	35	366	31 . 131	3	977	59 . 130	3	318	215 . 217	6
738	443 . 871	21	420	17 . 43		89110	31 . 39		322	13 . 45	
756	2983 . 3329	91	474	2063 . 2311	63	115	1 . 134	3	331	24 . 43	
759	1654 . 9883	229	493	36 . 307	7	122	71 . 127	3	347	51 . 85	2
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947	19 . 42		663	31 . 38		294	85 . 401	9	599	150 . 587	13
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107	61 . 75	2	769	49 . 527	12	369	19 . 134	3	606	209 . 223	6
235	12 . 43		779	57 . 79	2	434	447 . 475	13	674	23 . 227	5
271	28 . 39		841	410 . 1883	43	479	30 . 313	7	697	31 . 545	12
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341	3074 . 3183	91	211	3 . 44		557	3437 . 3452	97	791	31 . 40	
380	53 . 127	3	219	175 . 337	8	642	23 . 89	2	799	25 . 318	7

93869	14. 45		95478	197. 235	6	97245	29. 91	2	98611	151. 361	8
942	1171 .1583	39	569	23. 320	7	261	4031 .6617	154	623	34. 39	
94038	205. 227	6	814	25. 137	3	306	65. 133	3	666	19. 231	5
212	5. 91	2	867	1040 .1551	37	309	36. 37		667	11. 46	
326	203. 229	6	921	30: 41		337	1. 46		679	53. 136	3
348	79. 177	4	933	79. 128	3	351	23. 44		937	7. 185	4
393	3046 .4215	103	995	2241 .8279	182	363	.153. 359	8	945	4772 .9363	211
402	67. 131	3	96038	17. 45		"	3. 46		99078	185. 247	6
445	21. 44		139	485. 531	14	461	5. 46		099	3257 .6544	147
478	77. 409	9	"	155. 357	8	622	2341 .2409	65	181	19. 324	7
519	1308 .1741	43	146	21. 229	5	678	4591 .9363	211	190	27. 43	
681	24. 319	7	292	13. 275	6	679	7. 46		244	75. 181	4
705	2552 .4733	109	463	22. 321	7	* 747	35. 38		281	6985 .7198	193
729	29. 547	12	486	193. 239	6	873	23. 553	12	521	76. 131	3
753	35. 136	3	691	34. 137	3	874	5543 .8439	199	533	13. 46	
829	80. 127	3	748	77. 179	4	98065	9. 46		569	14. 139	3
939	157. 355	8	759	859 .1622	37	180	43. 137	3	883	149. 363	8
95046	199. 233	6	817	25. 551	12	269	20. 323	7	* "	151. 600	13
132	25. 43		97019	49. 87	2	279	77. 130	3	918	23. 139	3
193	379 .4154	91	062	191. 241	6	358	187. 245	6	937	33. 40	
221	16. 45		083	26. 43		404	11. 277	6			
277	55. 134	3	166	73. 413	9	477	29. 42				
387	16. 137	3	236	1429 .5579	122	534	71. 415	9			

NOTE.

In the preceding Table, wherever asterisks are prefixed to any of the numbers, it signifies that there are more pairs of cubes, though generally of a high denomination, equal to the same number.

A SUPPLEMENT,

CONTAINING ROOTS OVER FOUR PLACES OF FIGURES AND WHICH APPROACH AN EQUALITY.

Numbers.	Numerators of the Roots.		Denominators of the Roots.	Numbers.	Numerators of the Roots.		Denominators of the Roots.
19	1325880 .	1502783	670397	1020	34417 .	39023	4614
37	241757 .	333667	111492	23	285409 .	303839	36876
61	238141 .	249859	78140	78	117487 .	144467	16263
65	45976304 .	55187791	15980559	1122	39167 .	41617	4902
139	54560 .	54943	13317	1206	11891 .	12685	1456
161	1832672 .	1993963	443905	1300	151867 .	164033	18261
163	66113 .	74512	16275	78	164267 .	170587	18963
182	21293 .	22933	4923	1450	163061 .	189289	19719
203	190261 .	204371	42354	1540	183707 .	190513	20421
236	209827 .	248957	47106	1603	16655 .	17736	1853
335	260243 .	390997	61362	1925	16341 .	18034	1745
379	40335 .	44849	7436	81	22945 .	25068	2413
403	2105444 .	2253931	372225	95	68221 .	75419	7206
429	14149 .	16739	2598	2074	222731 .	281251	25227
506	1999997 .	2048003	320060	2898	10081 .	16001	1209
615	590399 .	605161	88578	2914	27043 .	27101	2388
930	32399 .	34561	4326	3036	13067 .	14257	1191

3243	114263	.	119233	9942	24379	20038	.	24179	969
3450	119017	.	129383	10374	25004	10027	.	14977	559
3675	129599	.	135001	10606	469	11213	.	14256	553
3900	16847	.	18253	1407	650	12097	.	13753	553
90	17009	.	18901	1893	26353	12583	.	13770	559
4570	22127	.	23953	1752	663	16916	.	17365	723
4740	4549681	.	4664879	345618	27323	12312	.	15011	577
5425	19351	.	24049	1574	397	11340	.	16057	569
60	22807	.	26333	1767	29302	13751	.	15551	601
89	79247	.	92628	6175	30241	14364	.	15877	613
6090	26459	.	28351	1893	31337	13589	.	17748	637
6347	35216	.	36955	2457	32419	15193	.	17226	643
6630	25793	.	33877	2037	858	16865	.	20667	746
7076	26909	.	29699	1862	33150	146249	.	152101	5853
7257	2004931	.	2175101	136236	731	15660	.	16071	661
71	130840	.	137439	8729	34099	112141	.	160651	5462
7657	167848177	.	185887823	11335260	36251	17441	.	18810	691
7812	276767	.	285697	17862	36051	10171	.	13245	446
8001	2286143	.	2322433	145164	311	24034	.	25223	921
"	2259793	.	2422895	147668	40609	14549	.	18219	608
8174	249283	.	273853	16396	41021	19530	.	21491	751
90	2321857	.	2395583	147468	147	1006264	.	1379665	44603
8385	2396159	.	2433601	149772	43238	11879	.	12455	437
8580	304127	.	313633	19014	44659	22176	.	22483	793
8749	144080	.	172977	9773	48887	11465	.	11692	399
9081	2628927	.	2890945	172732	51330	227069	.	234901	7833
9380	35837	.	39203	2246	51212	24301	.	26911	871
10710	46817	.	49573	2757	53874	14389	.	16361	515
738	58221	.	59779	3370	56153	12941	.	16184	485
11167	77788	.	77947	4389	57533	35509	.	38462	1209
342	1365173	.	1390933	77283	58463	28960	.	29483	949
628	49411	.	55241	2919	501	12662	.	15049	453
990	1443419	.	1470151	80199	786	1812527	.	1949777	61036
12498	267888	.	289487	15155	61453	38056	.	40955	1263
654	55403	.	58483	3081	65212	1939409	.	2234159	65644
14245	54757	.	59203	2966	66122	36831	.	38737	1178
820	64979	.	68401	3423	67383	10547	.	11914	349
16245	372031	.	387764	18907	453	20422	.	22453	665
17220	75599	.	79381	3783	791	11916	.	12473	377
499	14833	.	17999	804	68510	32761	.	35839	1057
845	68797	.	73963	3446	69697	33300	.	36397	1069
19691	11335	:	13982	597	73017	2409877	.	2847347	79782
866	87317	.	91477	4161	74853	12350	.	12601	373
21079	10279	.	10800	481	76531	47903	.	50494	1461
252	91279	.	99989	4359	78372	13099	.	16757	446
847	13964	.	14135	633	80081	39131	.	40950	1171
22213	11091	.	13298	551	84266	3142429	.	3483315	97984
22770	100187	.	104743	4557	89319	11341	.	15659	390
23921	11934	.	11987	523	90729	13843	.	16400	427
24031	12851	.	14613	602	94001	46494	.	47507	1303
"	20344	.	30309	1147	95071	9718	.	10055	273
803	673439	.	678180	29197					

TABLE II.

EXHIBITING THE NUMBERS OF THE NATURAL SERIES, THEIR CUBES, AND THE FIRST ORDER OF DIFFERENCES OF THE CUBES.

Nos.	Cubes.	Diff.	Nos.	Cubes.	Diff.	Nos.	Cubes.	Diff.	Nos.	Cubes.	Diff.
1	1		28	21952	2269	55	166375	8911	82	551368	19927
2	8	7	29	24389	2437	56	175616	9241	83	571787	20419
3	27	19	30	27000	2611	57	185193	9577	84	592704	20917
4	64	37	31	29791	2791	58	195112	9919	85	614125	21421
5	125	61	32	32768	2977	59	205379	10267	86	636056	21931
6	216	91	33	35937	3169	60	216000	10621	87	658503	22447
7	343	127	34	39304	3367	61	226981	10981	88	681472	22969
8	512	169	35	42875	3571	62	238328	11347	89	704969	23497
9	729	217	36	46656	3781	63	250047	11719	90	729000	24031
10	1000	271	37	50653	3997	64	262144	12097	91	753571	24571
11	1331	331	38	54872	4219	65	274625	12481	92	778688	25117
12	1728	397	39	59319	4447	66	287496	12871	93	804357	25669
13	2197	469	40	64000	4681	67	300763	13267	94	830584	26227
14	2744	547	41	68921	4921	68	314432	13669	95	857375	26791
15	3375	631	42	74088	5167	69	328509	14077	96	884736	27361
16	4096	721	43	79507	5419	70	343000	14491	97	912673	27937
17	4913	817	44	85184	5677	71	357911	14911	98	941192	28519
18	5832	919	45	91125	5941	72	373248	15337	99	970299	29107
19	6859	1027	46	97336	6211	73	389017	15769	100	1000000	29701
20	8000	1141	47	103823	6487	74	405224	16207	101	1030301	30301
21	9261	1261	48	110592	6769	75	421875	16651	102	1061208	30907
22	10648	1387	49	117649	7057	76	438976	17101	103	1092727	31519
23	12167	1519	50	125000	7351	77	456533	17557	104	1124864	32137
24	13824	1657	51	132651	7651	78	474552	18019	105	1157625	32761
25	15625	1801	52	140608	7957	79	493039	18487	106	1191016	33391
26	17576	1951	53	148877	8269	80	512000	18961	107	1225043	34027
27	19683	2107	54	157464	8587	81	531441	19441	108	1259712	34669

TABLE III.

WHICH EXHIBITS A SERIES OF ODD NUMBERS, THEIR CUBES, AND THE FIRST ORDER OF DIFFERENCES OF THE CUBES.

Nos.	Cubes.	Diff.	Nos.	Cubes.	Diff.	Nos.	Cubes.	Diff.	Nos.	Cubes.	Diff.
1	1		29	24389	4706	57	185193	18818	85	614125	42338
3	27	26	31	29791	5402	59	205379	20186	87	658503	44378
5	125	98	33	35937	6146	61	226981	21602	89	704969	46466
7	343	218	35	42875	6938	63	250047	23066	91	753571	48602
9	729	386	37	50653	7778	65	274625	24578	93	804357	50786
11	1331	602	39	59319	8666	67	300763	26138	95	857375	53018
13	2197	866	41	68921	9602	69	328509	27746	97	912673	55298
15	3375	1178	43	79507	10586	71	357911	29402	99	970299	57626
17	4913	1538	45	91125	11618	73	389017	31106	101	1030301	60002
19	6859	1946	47	103823	12698	75	421875	32858	103	1092727	62426
21	9261	2402	49	117649	13826	77	456533	34658	105	1157625	64898
23	12167	2906	51	132651	15002	79	493039	36506	107	1225043	67418
25	15625	3458	53	148877	16226	81	531441	38402	109	1295029	69986
27	19683	4058	55	166375	17498	83	571787	40346	111	1367631	72602

TABLE IV.

SHOWING A SERIES OF NUMBERS, NOT DIVISIBLE BY 3, THEIR CUBES, AND THE DIFFERENCES OF THE CUBES.

Nos.	Cubes.	Diff.	Nos.	Cubes.	Diff.	Nos.	Cubes.	Diff.	Nos.	Cubes.	Diff.
1	1		29	24389	2437	58	195112	19496	86	636056	21931
2	8	7	31	29791	5402	59	205379	10267	88	681472	45416
4	64	56	32	32768	2977	61	226981	21602	89	704969	23497
5	125	61	34	39304	6536	62	238328	11347	91	753571	48602
7	343	218	35	42875	3571	64	262144	23816	92	778688	25117
8	512	169	37	50653	7778	65	274625	12481	94	830584	51896
10	1000	488	38	54872	4219	67	300763	26138	95	857375	26791
11	1331	331	40	64000	9128	68	314432	13669	97	912673	55298
13	2197	866	41	68921	4921	70	343000	28568	98	941192	28519
14	2744	547	43	79507	10586	71	357911	14911	100	1000000	58808
16	4096	1352	44	85184	5677	73	389017	31106	101	1030301	30301
17	4913	817	46	97336	12152	74	405224	16207	103	1092727	62426
19	6859	1946	47	103823	6487	76	438976	33752	104	1124864	32137
20	8000	1141	49	117649	13826	77	456533	17557	106	1191016	66152
22	10648	2648	50	125000	7351	79	493039	36506	107	1225043	34027
23	12167	1519	52	140608	15608	80	512000	18961	109	1295029	69986
25	15625	3458	53	148877	8269	82	551368	39368	110	1331000	35971
26	17576	1951	55	166375	17498	83	571787	20419	112	1404928	73928
28	21952	4376	56	175616	9241	85	614125	42338	113	1442897	37969.

TABLE V.

CONTAINING A SERIES OF NUMBERS, NEITHER DIVISIBLE BY 2 NOR BY 3, THEIR CUBES, AND THE DIFFERENCES OF THE CUBES.

Nos.	Cubes.	Diff.	Nos.	Cubes.	Diff.	Nos.	Cubes.	Diff.	Nos.	Cubes.	Diff.
1	1		37	50653	7778	73	389017	31106	109	1295029	69986
5	125	124	41	68921	18268	77	456533	67516	113	1442897	147868
7	343	218	43	79507	10586	79	493039	36506	115	1520875	77978
11	1331	988	47	103823	24316	83	571787	78748	119	1685159	164284
13	2197	866	49	117649	13826	85	614125	42338	121	1771561	86402
17	4913	2716	53	148877	31228	89	704969	90844	125	1953125	181564
19	6859	1946	55	166375	17498	91	753571	48602	127	2048383	95258
23	12167	5308	59	205379	39004	95	857375	103804	131	2248091	199708
25	15625	3458	61	226981	21602	97	912673	55298	133	2352637	104546
29	24389	8764	65	274625	47644	101	1030301	117628	137	2571353	218716
31	29791	5402	67	300763	26138	103	1092727	62426	139	2685619	114266
35	42875	13084	71	357911	57148	107	1225043	132316	143	2924207	238588

NOTE.

The Tables II., III., IV., and V., might have been extended much further, but the generality of the calculations in which the columns of differences are useful, and to which the Tables are otherwise applicable, did not appear to require it: we therefore deemed it proper to make them thus brief. We might also have added to the number of them—that is—we might have calculated and arranged other Tables, exhibiting numbers prime to, or not divisible by, the prime numbers, 5, 7, 11, &c. respectively, or any multiple of these primes, their cubes and differences, and thus have had a special Table for each particular case or operation; but as that would have been a tedious and an almost endless task, we have preferred to substitute in the place thereof the annexed observation, containing such information and explanations, as, we hope, will lead to a full comprehension of the various operations.

OBSERVATION.

(1.) In order to show the utility of Tables II., III., IV., and V., and also the facility with which many troublesome calculations, relative to cubes, may be made by the aid of the columns of differences, &c., we shall propose, and, in as brief a manner as possible, work out several examples.

(2.) Suppose it were required to divide a given number—say 6—into 3 cubes. Then, according to Rule I. p. 122, Mathematical Miscellany, vol. I., we are to multiply 6 by a cube, and from the product deduct a series of cubes prime to the multiple cube, &c., &c. Now, to make this deduction in the ordinary way, when the number to be divided and, especially, the multiple cube, are large, would be exceedingly troublesome, and we should be greatly liable to error, whereas, by the use of the columns of differences, in the respective Tables, we shall be enabled to effect the deduction with great ease, and without a possibility of committing an error, for ever and anon a check will arise, as will appear evident from the operation. Let, for instance, the multiple cube be $(7)^3 = 343$; we shall then have $6 \times (7)^3 = 2058$, from which to deduct a series of cubes prime to $(7)^3$: and in this case, as 7 is a prime number, we use the column of differences in Table II., proceeding as follows: Look in Table II. for the cube next less than 2058, viz: $(12)^3 = 1728$, which being deducted, leaves the remainder 330. Now, to this remainder add the number in the column of differences opposite to 12, which is 397, and to this sum add 331, the next difference above, and to this 271 the next, and so on to the top or commencement of the column of differences, when the last sum must evidently be one less than the original product $6 \times (7)^3 = 2058$, else a mistake has been committed. The operation in figures will stand thus:

$$
\begin{array}{rl}
6 \times (7)^3 = & 2058, \\
(12)^3 = & \underline{1728,} \\
 & 330_{397} \\
(11)^3 \ldots & 727_{331} \\
(10)^3 \ldots & 1058_{271} \\
(9)^3 \ldots & 1329_{217} \\
(8)^3 \ldots & 1546_{169}
\end{array}
\qquad\qquad
\begin{array}{rl}
\text{check } (7)^3 \ldots\ldots & 1715 \\
(6)^3 \ldots\ldots & 1842^{127} \\
(5)^3 \ldots\ldots & 1933^{91} \\
(4)^3 \ldots\ldots & 1994^{61} \\
(3)^3 \ldots\ldots & 2031^{37} \\
(2)^3 \ldots\ldots & 2050^{19} \\
(1)^3 \ldots\ldots & 2057^{7}
\end{array}
$$

Now, this series of numbers are the remainders of a series of cubes from $(12)^3$ down to $(1)^3$ deducted from 2058, which are to be respectively looked for in Table I., and as many as are there found to correspond, so many divisions shall we be enabled to effect. On examination we find that 330, 1842, and 2050, are in the Table, consequently three different divisions may be effected; but, as the first and last of these numbers produce large roots, we shall only use 1842, which is equal to $(\frac{19}{3})^3 + (\frac{35}{3})^3$, and thence have the following equation, viz: $6 \times (7)^3 - (6)^3 = 1842 = (\frac{19}{3})^3 + (\frac{35}{3})^3$ from which we get $6 = (\frac{18}{21})^3 + (\frac{19}{21})^3 + (\frac{35}{21})^3$.

(3.) Suppose the multiple cube to be $(8)^3 = 512$; we shall then have $6 \times (8)^3 = 3072$, from which to deduct a series of cubes prime to $(8)^3$. Now, as 8 is of the form 2^n, and therefore prime to all odd numbers, we use, in this case, the differences in Table III. and proceed in the calculation as in the preceding example: Hence

$$
\begin{array}{rl}
6 \times (8)^3 = & 3072 \\
(13)^3 = & \underline{2197} \\
 & 875_{866} \\
(11)^3 \ldots & 1741_{602} \\
(9)^3 \ldots & 2343_{386} \\
(7)^3 \ldots & 2729_{218} \\
(5)^3 \ldots & 2947_{98} \\
(3)^3 \ldots & 3045^{26} \\
(1)^3 \ldots & 3071^{26}
\end{array}
$$

Now, this series of numbers are the remainders of a series of cubes prime to $(8)^3$, from $(13)^3$ down to $(1)^3$, deducted from 3072; and comparing them with Table I. we find that $875 = 7 \times (5)^3 = (\frac{20}{3})^3 + (\frac{25}{3})^3$ and $2947 = (\frac{22}{3})^3 + (\frac{41}{3})^3$. Consequently we shall have the two equations

$$6 \times (8)^3 - (13)^3 = (\tfrac{20}{3})^3 + (\tfrac{25}{3})^3 \text{ and } 6 \times (8)^3 - (5)^3 = (\tfrac{22}{3})^3 + (\tfrac{41}{3})^3,$$

and thence

$$6 = (\tfrac{20}{24})^3 + (\tfrac{25}{24})^3 + (\tfrac{39}{24})^3 \text{ or } (\tfrac{15}{24})^3 + (\tfrac{22}{24})^3 + (\tfrac{41}{24})^3.$$

(4.) Let the multiple cube $(12)^3 = 1728$ be assumed. Then, a series of cubes prime to $(12)^3$ are to be deducted from $6 \times (12)^3 = 10368$; and, in this case, as 12 is divisible by 2 and by 3, we use the differences in Table V., which has been arranged expressly for numbers or cubes prime to 2 and to 3, and which will therefore furnish the deduct-

ing cubes prime to $(12)^3$. The operation will be thus: The cube in Table V., next below 10368, is $(19)^3 = 6859$, we therefore have

$$
\begin{array}{rcl}
6 \times (12)^3 & = & 10368 \\
(19)^3 & = & 6859 \\
\hline
 & & 3509 \\
(17)^3 & \ldots & 5455^{1946} \\
(13)^3 & \ldots & 8171^{2716} \\
(11)^3 & \ldots & 9037^{866} \\
(7)^3 & \ldots & 10025^{988} \\
(5)^3 & \ldots & 10243^{218} \\
(1)^3 & \ldots & 10367^{124}
\end{array}
$$

Now, in this series of remainders we find 9037 which is also in Table I., and is equal to $(\frac{15}{2})^3 + (\frac{41}{2})^3$. Hence the equation $6 \times (12)^3 - (11)^3 = (\frac{15}{2})^3 + (\frac{41}{2})^3$ and $6 = (\frac{15}{24})^3 + (\frac{22}{24})^3 + (\frac{41}{24})^3$ as before.

(5.) Having no special Table arranged for any prime number greater than 3, it follows that when the root of a multiple cube is a prime number greater than 3, or a multiple of a prime greater than three, some of the deducting cubes in the Tables arranged and applicable to the case, will not be prime to the multiple cube. Hence such deducting cubes, and also the corresponding remainders, will be divisible by the multiple cube if its root be a prime number greater than 3, or by the cubes of the prime factors of the multiple cube whether it be or be not divisible by 3, and therefore, in the course of the calculation or operation, will serve as a check upon the correctness of our deductions, as well as that at the termination of each calculation, (see rem. and illust. (2.)) Thus, in the calculation of the first example, opposite to $(7)^3$ is the remainder 1715 which, to be correct, according to what has been said, must be divisible by $(7)^3$, which it is, for the quotient is 5. As a further illustration, let us suppose 3 to be the number to be divided into 3 cubes, and the root of the multiple cube to be $15 = 3 \times 5$: we shall then have $3 \times (15)^3 = 10125$, from which to deduct a series of cubes prime to 3 and to 5. Now, in this case, we use the differences in Table IV., because it is arranged for numbers or cubes not divisible by 3 or prime to 3; and the deducting cubes contained therein not prime to the multiple cube, and the corresponding remainders in the calculation will be divisible by $(5)^3$: and hence a check as mentioned in (2.) We shall now have the following calculation, viz:

$$
\begin{array}{lrcll}
 & 3 \times (15)^3 & = & 10125 & \\
 & (20)^3 & = & 8000 & \\
\hline
\text{check} \ldots & & \ldots & 2125 & \ldots \text{div. by } (5)^3 \\
 & (19)^3 & \ldots & 3266^{1141} & \\
 & (17)^3 & \ldots & 5212^{1946} & \\
 & (16)^3 & \ldots & 6029^{817} & \\
 & (14)^3 & \ldots & 7381^{1352} & \\
 & (13)^3 & \ldots & 7928^{547} & \\
 & (11)^3 & \ldots & 8794^{866} & \\
\text{check} \ldots & (10)^3 & \ldots & 9125^{331} & \ldots \text{div. by } (5)^3 \\
 & (8)^3 & \ldots & 9613^{488} & \\
 & (7)^3 & \ldots & 9782^{169} & \\
\text{check} \ldots & (5)^3 & \ldots & 10000^{218} & \ldots \text{div. by } (5)^3 \\
 & (4)^3 & \ldots & 10061^{61} & \\
 & (2)^3 & \ldots & 10117^{56} & \\
\text{check} \ldots & (1)^3 & \ldots & 10124^{7} & \ldots \text{being one less than 10125.}
\end{array}
$$

Comparing this series of remainders (excepting the checks divisible by $(5)^3$) with the numbers in Table I., we find that 9613 is equal to $(\frac{53}{9})^3 + (\frac{190}{9})^3$ and $10117 = (\frac{17}{5})^3 + (\frac{108}{5})^3$. Consequently, proceeding as before, there will result

$$3 = (\tfrac{53}{135})^3 + (\tfrac{72}{135})^3 + (\tfrac{190}{135})^3 \text{ or } (\tfrac{10}{75})^3 + (\tfrac{17}{75})^3 + (\tfrac{108}{75})^3$$

And in the same manner divisions of other numbers into three cubes may be readily and easily effected.

(6.) In conclusion, we hope we have fully succeeded in giving, in this observation, such information and explanations as regards the numerical operations of dividing a given number into 3 cubes, as may be satisfactory and easily comprehended; and we would further observe that the columns of differences in these Tables are highly important, and are applicable to and render many other calculations relative to cubes, which would otherwise be extremely difficult as well as tedious and troublesome, incredibly easy and simple.

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ARTICLE I.

HINTS TO YOUNG STUDENTS.—No. VI.

30. I WOULD again remind the student that the object of Algebra is not alone that of enabling him to solve difficult questions, which are often, in themselves, of little value; but to instruct him in the language of symbols—a language which he will find to be the key to the higher branches of science, and, ultimately, to the profoundest mysteries of Nature. How little this is attended to in the usual course of study, may be seen in the little power which even advanced students generally have of comprehending or interpreting their symbolical results. The most simple kind of interpretation, that of transforming Algebraical into Arithmetical results should be constantly practised throughout the course. Of this kind a very good exercise is to take formulas, such as those in Equations (6) . . . . (11), which are true for all numbers, and *verify* them in particular cases. The proof that his work is correct will be in the indentity of the values he finds for the two members of the equation. Thus, in equation (9), if $a = \frac{1}{2}$, $b = \frac{1}{3}$,

$$(a+b)(a-b) = (\tfrac{1}{2}+\tfrac{1}{3})(\tfrac{1}{2}-\tfrac{1}{3}) = \tfrac{5}{6}\cdot\tfrac{1}{6} = \tfrac{5}{36};$$
$$a^2 - b^2 = (\tfrac{1}{2})^2 - (\tfrac{1}{3})^2 = \tfrac{1}{4} - \tfrac{1}{9} = \tfrac{5}{36}.$$

He should continue his substitutions by putting, for a and b, such numbers as the following, in each of the equations:

1 and 0, 0 and -1, 100 and 1, 17 and 7, &c.;

$\frac{2}{3}$ and $\frac{1}{3}$, $\frac{4}{5}$ and $\frac{5}{6}$, $\frac{3}{4}$ and $\frac{7}{8}$, $\frac{4}{5}$ and $\frac{5}{4}$, &c.;

$1\frac{1}{3}$ and 1, $\dfrac{2\frac{1}{2}}{5}$ and -1, $3\frac{1}{8}$ and $-\dfrac{2\frac{1}{4}}{1\frac{1}{2}}$, -3 and $-2\frac{1}{3}$, &c.;

2.01 and ,1 . ,67 and ,7, -3, 1 and ,01, ,5 and ,05, &c.;

$\sqrt{2}$ and 1, $\sqrt{3}$ and $\sqrt{2}$, $\sqrt{\frac{1}{2}}$ and $\sqrt{\frac{1}{3}}$, $\sqrt[3]{5}$ and $\sqrt{3}$, &c.;

$$\sqrt{2}+1 \text{ and } \sqrt{2}-1,\ \sqrt{3}+\sqrt{2}, \text{ and } \sqrt{3}-\sqrt{2},\ \sqrt[3]{3}-\sqrt{2} \text{ and } \sqrt[3]{3}+\sqrt{5}, \text{ \&c.};$$
$$2^{\frac{1}{2}} \text{ and } 3^{\frac{1}{3}},\ \tfrac{1}{2}.2^{\frac{2}{3}} \text{ and } \tfrac{1}{3}.3^{\frac{1}{2}},\ \tfrac{1}{7}.(\tfrac{3}{4})^{\frac{3}{4}} \text{ and } \tfrac{5}{7}(\tfrac{4}{3})^{\frac{1}{4}}, \text{ \&c.};$$
$$c+d \text{ and } c-d,\ cx+d \text{ and } cx-d,\ cx+d \text{ and } ex+f, \text{ \&c.};$$
$$\frac{c}{d} \text{ and } \frac{c-d}{c+d},\ \frac{cx-dy}{cx+dy} \text{ and } \frac{c-d}{c+d},\ \frac{1+x}{x} \text{ and } \frac{x}{1-x}, \text{ \&c.};$$
$$a+\sqrt{b} \text{ and } b-\sqrt{a},\ \sqrt{a}+\sqrt{b} \text{ and } \sqrt{a}-\sqrt{b},\ \sqrt{\tfrac{1}{2}(n+1)} \text{ and } \sqrt{\tfrac{1}{2}(n-1)}, \text{ \&c.};$$
$$\frac{x}{\sqrt{x}+\sqrt{y}} \text{ and } \frac{y}{\sqrt{x}-\sqrt{y}},\ \sqrt[3]{\frac{x}{y}}, \text{ and } \sqrt[3]{\frac{y}{x}},\ \sqrt{1-y} \text{ and } y\sqrt{1-x}, \text{ \&c.};$$
&c., &c.

In these particular cases, it will be seen which member of the equation is the more easily calculated, and thus in any future case when a result is presented in one form, it can be seen, from the nature of the substitution, to what form it must be reduced so as to admit of the shortest calculation. The Algebraical substitutions will again form so many general formulas, which may be again submitted to verification, and which will often be worthy of special attention. Thus, for example, it will be seen that

$$\{\sqrt{\tfrac{1}{2}(n+1)}+\sqrt{\tfrac{1}{2}(n-1)}\}\{\sqrt{\tfrac{1}{2}(n+1)}-\sqrt{\tfrac{1}{2}(n-1)}\}=1:$$

then $\sqrt{\tfrac{1}{2}(n+1)}+\sqrt{\tfrac{1}{2}(n-1)}$ and $\sqrt{\tfrac{1}{2}(n+1)}-\sqrt{\tfrac{1}{2}(n-1)}$, is one of the general forms for all numbers which are reciprocals of each other, and may be compared with similar forms, such as

$$\frac{p}{q} \text{ and } \frac{q}{p},$$

which possesses the same property; thus, if we put

$$\sqrt{\tfrac{1}{2}(n+1)}+\sqrt{\tfrac{1}{2}(n-1)}=\frac{p}{q},$$

then solve this equation for n, and substitute the result, we shall find

$$\sqrt{\tfrac{1}{2}(n+1)}-\sqrt{\tfrac{1}{2}(n-1)}=\frac{q}{p}.$$

It is familiarity with such properties, and facility in applying them, that enables the analyst to push his researches into fields of inquiry, which would otherwise be beyond his reach; since his symbols would become so complex that he could neither operate upon, nor interpret them. A single instance, in a simple case, will exhibit this. Equations of the form

$$\sqrt{x^2+ax+b}+\sqrt{x^2-ax+b}=c,$$

are frequently met with in Geometry. The usual process for freeing the equation of radicals, would be very complicated; but the analyst seizes the truth, which the form of the equation presents to him,

$$\{\sqrt{x^2+ax+b}+\sqrt{x^2-ax+b}\}\{\sqrt{x^2+ax+b}-\sqrt{x^2-ax+b}\}=2ax;$$

then $\sqrt{x^2+ax+b}-\sqrt{x^2-ax+b}=\frac{2ax}{c}$,

$$\sqrt{x^2+ax+b}=\frac{ax}{c}+\tfrac{1}{2}c,$$

$$x^2+ax+b=\frac{a^2x^2}{c^2}+ax+\tfrac{1}{4}c^2,$$

$$\left(1-\frac{a^2}{c^2}\right)x^2=\tfrac{1}{4}c^2-b,$$

$$x=\pm\frac{c}{2}\sqrt{\frac{c^2-4b}{c^2-a^2}}.$$

31. I have placed here a number of other identical formulas which are well adapted for verification and transformation, and they can be multiplied at pleasure.

$$(x+a)(x+b)=x^2+(a+b)x+ab,$$

$$(x+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(ab+ac+bc)x+abc,$$

$$(ax-by)^2+(ay+bx)^2=(a^2+b^2)(x^2+y^2,)$$

$$(p^2-aq^2)^2+a(2pq)^2=(p^2+aq^2)^2,$$

$$(p^2-aq^2)^2+b(p^2-aq^2)(2pq+bq^2)+a(2pq+bq^2)=(p^2+bpq+aq^2)^2,$$

$$(p^3+apq^2)^2+a(pq^2+aq^3)^2=(p^2+aq^2)^3,$$

$$(p^4-6ap^2q^2+a^2q^4)^2+a(4p^3q-4apq^3)^2=(p^2+aq^2)^4,$$

$$(x+y)(xy-z^2)+(x+z)(xz-y^2)+(y+z)(yz-x^2)=0,$$

$$(x+y+z)^2+(x-y)^2+(x-z)^2+(y-z)^2=3(x^2+y^2+z^2),$$

$$\frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}+\frac{n(n+1}{1\,.\,2}+n+1=\frac{(n+1)(n+2)(n+3)}{1\,.\,2\,.\,3},$$

$$x(x-1)(x-2)+3x(x-1)+x=x^3,$$

$$x(x-1)(x-2)(x-3)+6x(x-1)(x-2)+7x(x-1)+x=x^4,$$

$$x(x-1)+4xy+y(y-1)=(x+y)(x+y-1),$$

$$x(x-1)(x-2)+3x(x-1)y+3xy(y-1)+y(y-1)(y-2)$$
$$=(x+y)(x+y-1)(x+y-2),$$

$$x(x-1)+4x+2=(x+1)(x+2)$$

$$x(x-1)(x-2)+9x(x-1)+18x+6=(x+1)(x+2)(x+3),$$

$$\frac{b}{a+b}+\frac{a-b}{b}=\frac{a^2}{ab+b^2},$$

$$\frac{a-b}{a+b}+\frac{a+b}{a-b}=\frac{2a^2+2b^2}{a^2-b^2},$$

$$\frac{x^2-2x+1}{x^2-1}=\frac{x-1}{x+1},$$

$$\frac{x^4+x^2y^2+y^4}{x^3+y^3}=x+y-\frac{xy}{x+y},$$

$$\frac{x}{a}+\frac{bx}{a^2}+\frac{b^2x}{a^3}+\frac{b^3x}{a^3(a-b)}=\frac{x}{a-b},$$

$$\frac{x}{a}-\frac{bx}{a^2}+\frac{b^2x}{a^3}-\frac{b^3x}{a^3(a+b)}=\frac{x}{a+b},$$

$$\frac{a+x}{a-x}-\frac{a-x}{a+x}-\frac{4ax}{a^2-x^2}=0,$$

$$\frac{a(a^2-bc)+b(b^2-ac)+c(c^2-ab)}{a+b+c}=a^2-bc+b^2-ac+c^2-ab,$$

$$\frac{1}{(x-y)(x-z)}+\frac{1}{(y-x)(y-z)}+\frac{1}{(z-x)(z-y)}=0,$$

$$\frac{x}{(x-y)(x-z)}+\frac{y}{(y-x)(y-z)}+\frac{z}{(z-x)(z-y)}=0,$$

$$\frac{x^2}{(x-y)(x-z)}+\frac{y^2}{(y-x)(y-z)}+\frac{z^2}{(z-x)(z-y)}=1,$$

$$\frac{(a-y)(a-z)}{(x-y)(x-z)}+\frac{(a-x)(a-z)}{(y-x)(y-z)}+\frac{(a-x)(a-y)}{(z-x)(z-y)}=1,$$

$$x\cdot\frac{(a-y)(a-z)}{(x-y)(x-z)}+y\cdot\frac{(a-x)(a-z)}{(y-x)(y-z)}+z\cdot\frac{(a-x)(a-y)}{(z-x)(z-y)}=a,$$

$$x^2\cdot\frac{(a-y)(a-z)}{(x-y)(x-z)}+y^2\cdot\frac{(a-x)(a-z)}{(y-x)(y-z)}+z^2\cdot\frac{(a-x)(a-y)}{(z-x)(z-y)}=a^2,$$

$$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=a-b,$$

$$\frac{x}{y-\sqrt{z}}=\frac{x(y+\sqrt{z})}{y^2-z},$$

$$\frac{x}{y+\sqrt{z}}=\frac{x(y-\sqrt{z})}{y^2-z},$$

$$\frac{x}{x-y}+\sqrt{\frac{x^2}{(x-y)^2}-\frac{x}{x-y}}=\frac{\sqrt{x}}{\sqrt{x}-\sqrt{y}},$$

$$\frac{x}{x-y}-\sqrt{\frac{x^2}{(x-y)^2}-\frac{x}{x-y}}=\frac{\sqrt{x}}{\sqrt{x}+\sqrt{y}},$$

$$\frac{y}{\sqrt{x^2+zy}-x}=\frac{x+\sqrt{x^2+zy}}{z}$$

$$\sqrt{\frac{x+a}{x}}+2\sqrt{\frac{a}{a+x}}=\frac{(\sqrt{a}+\sqrt{x})^2}{\sqrt{ax+x^2}}$$

$$\sqrt{xy+2x\sqrt{xy-x^2}}+\sqrt{xy-2x\sqrt{xy-y^2}}=2x,$$

$$\sqrt{x}+\sqrt{y}=\sqrt{\tfrac{1}{2}x+\tfrac{1}{2}\sqrt{x^2-y}}+\sqrt{\tfrac{1}{2}x-\tfrac{1}{2}\sqrt{x^2-y}},$$

$$\sqrt{x}-\sqrt{y}=\sqrt{\tfrac{1}{2}x+\tfrac{1}{2}\sqrt{x^2-y}}-\sqrt{\tfrac{1}{2}x-\tfrac{1}{2}\sqrt{x^2-y}},$$

$$\sqrt{x+y\sqrt{-1}}=\sqrt{\tfrac{1}{2}\sqrt{x^2+y^2}+\tfrac{1}{2}x}+\sqrt{-1}\,.\sqrt{\tfrac{1}{2}\sqrt{x^2+y^2}-\tfrac{1}{2}x},$$

$$\sqrt{x-y\sqrt{-1}}=\sqrt{\tfrac{1}{2}\sqrt{x^2+y^2}+\tfrac{1}{2}y}-\sqrt{-1}\,.\sqrt{\tfrac{1}{2}\sqrt{x^2+y^2}-\tfrac{1}{2}x},$$

$$\sqrt{x+\sqrt{y}}+\sqrt{x-\sqrt{y}}=\sqrt{2x+2\sqrt{x^2-y}},$$

$$\sqrt{x+\sqrt{y}}-\sqrt{x-\sqrt{y}}=\sqrt{2x-2\sqrt{x^2-y}},$$

$$\sqrt{x+y\sqrt{-1}}+\sqrt{x-y\sqrt{-1}}=\sqrt{2\sqrt{x^2+y^2}+2x},$$

$$\sqrt{x+y\sqrt{-1}}-\sqrt{x-y\sqrt{-1}}=\sqrt{-1}\,.\sqrt{2\sqrt{x^2+y^2}-2x},$$

$$\frac{\sqrt{\sqrt{x}+\sqrt{y}}+\sqrt{\sqrt{x}-\sqrt{y}}}{\sqrt{\sqrt{x}+\sqrt{y}}-\sqrt{\sqrt{x}-\sqrt{y}}}=\frac{x+\sqrt{x^2-y}}{\sqrt{y}},$$
$$\frac{\sqrt{\sqrt{x}+\sqrt{y}}-\sqrt{\sqrt{x}-\sqrt{y}}}{\sqrt{\sqrt{x}+\sqrt{y}}+\sqrt{\sqrt{x}+\sqrt{y}}}=\frac{x-\sqrt{x^2-y}}{\sqrt{y}};\ \&c.$$

These should first be verified generally, that is, the two members of each equation should be so operated upon as to render them identical; and afterwards for particular values, and among the numerical values of the letters, 0 and 1 must be often used; thus in the equation

$$\sqrt{x-y\sqrt{-1}}=\sqrt{\tfrac{1}{2}\sqrt{x^2+y^2}+\tfrac{1}{2}x}-\sqrt{-1}.\sqrt{\tfrac{1}{2}\sqrt{x^2+y^2}-\tfrac{1}{2}x},$$

if $x=0, y=2,\qquad \sqrt{-2\sqrt{-1}}=1-\sqrt{-1},$

which may be verified by squaring both members, when they will be found to be identical. A very useful exercise consists in taking each member of such equations, and deducing the other one from it by successive transformations, thus:

$$\begin{aligned} x(x-1)(x-2)+3x(x-1)+x &= x\{(x-1)(x-2)+3(x-1)+1\}\\ &= x\{(x-1)(x-2+3)+1\}\\ &= x\{(x-1)(x+1)+1\}\\ &= x(x^2-1+1)\\ &= x^3;\end{aligned}$$

so,
$$\begin{aligned} x^3 &= x.x.x\\ &= x(x-1+1)(x-2+2)\\ &= x\{(x-1)(x-2)+2(x-1)+x-1-1+2\}\\ &= x\{(x-1)(x-2)+3(x-1)+1\}\\ &= x(x-1)(x-2)+3x(x-1)+x.\end{aligned}$$

32. In the following equations, n is supposed to be a positive integer, although many of them are true for any values of n whatever. They should be verified for particular values of n, and several of the equations given above are among the particular cases of these,

$$(x+y)^n=x^n+\frac{n}{1}.x^{n-1}y+\frac{n.(n-1)}{1.2}.x^{n-2}y^2+\ldots\ldots\frac{n}{1}.xy^{n-1}+y^n.$$

$$2^n=1+\frac{n}{1}+\frac{n(n-1)}{1.2}+\frac{n(n-1)(n-2)}{1.2.3}+\ldots\frac{n}{1}+1.$$

$$0=1-\frac{n}{1}+\frac{n(n-1)}{1.2}-\frac{n(n-1)(n-2)}{1.2.3}+\&c.,$$

$$n^n-\frac{n}{1}(n-1)^n+\frac{n(n-1)}{1.2}(n-2)^n-\&c.\ldots=1.2.3\ldots.n,$$

$$(n-1)^{n-1}-\frac{n}{1}(n-2)^{n-1}+\frac{n(n-1)}{1.2}(n-3)^{n-1}-\&c.\ldots=1,$$

$$(n-2)^{n-1}-\frac{n}{1}(n-3)^{n-1}+\frac{n(n-1)}{1.2}(n-4)^{n-1}-\&c.\ldots=2^{n-1}-n,$$

$$(n-3)^{n-1}-\frac{n}{1}(n-4)^{n-1}+\frac{n(n-1)}{1.2}(n-5)^{n-1}-\&c.\ldots=3^{n-1}-2^{n-1}n+\frac{n(n-1)}{1.2},$$

$$(n+1)(n+2)(n+3)\ldots.(n+n)=2^n\times1.3.5.7\ldots.(2n-1)$$

$$\frac{(m+1)(m+2)(m+3)\ldots.(m+n)}{1.2.3\ldots.n}=\frac{(n+1)(n+2)(n+3)\ldots.(n+m)}{1.2.3\ldots.m}$$

$$\frac{(n+1)(n+2)\dots(n+n)}{1.2.3\dots n}=1+\left(\frac{n}{1}\right)^2+\left(\frac{n.\overline{n-1}}{1.2}\right)^2+\left(\frac{n.\overline{n-1}.\overline{n-2}}{1.2.3}\right)^2$$
$$+\dots\left(\frac{n}{1}\right)^2+1,$$

$$(x+1)(x+2)\dots(x+n)=x(x-1)\dots(x-n+1)+\frac{n^2}{1}.x(x-1)\dots(x-n+2)$$
$$+\frac{n^2(n-1)^2}{1.2}.x(x-1)\dots(x-n+3)+\&c.\dots$$
$$+\frac{n^2(n-1)^2\dots 2^2}{1.2.3\dots\overline{n-1}}.x+\frac{n^2(n-1)^2(n-2)^2\dots 1^2}{1.2.3\dots n},$$

$$(x+y)(x+y-1)\dots(x+y-n+1)=x(x-1)(x-2)\dots(x-n+1)$$
$$+\frac{n}{1}.x(x-1)\dots(x-n+2).y$$
$$+\frac{n(n-1)}{1.2}.x(x-1)\dots(x-n+3).y(y-1)+\&c.\dots$$
$$+\frac{n}{1}.x,y(y+1)\dots(y-n+2)$$
$$+y(y-1)(y-2)\dots(y-n+1),$$

$$(x+y)(x+y+1)\dots(x+y+n-1)=x(x+1)(x+2)\dots(x+n-1)$$
$$+\frac{n}{1}.x(x+1)\dots(x+n-2).y$$
$$+\frac{n(n-1)}{1.2}.x(x+1)\dots(x+n-3).y(y+1)+\&c.\dots$$
$$+\frac{n}{1}.xy.(y+1)\dots(y+n-2)$$
$$+y(y+1)(y+2)\dots(y+n-1),$$

$$(a_1+a_2+\dots a_n)^2=a_1^2+a_2^2+\dots+a_n^2+2a_1a_2+2a_1a_3+\dots+2a_2a_3+\dots,$$
$$(a_1+a_2+\dots a_n)^2+(a_1-a_2)^2+(a_1-a_3)^2+\dots+(a_2-a_3)^2+(a_2-a_4)^2+\dots,$$
$$=n(a_1^2+a_2^2+\dots a_n^2),$$

$$(a_1x_1+a_2x_2+\dots a_nx_n)^2+(a_1x_2-a_2x_1)^2+(a_1x_3-a_3x_1)^2+\dots$$
$$+(a_2x_3-a_3x_2)^2+(a_2x_4-a_4x_2)^2+\dots$$
$$+\dots\dots\dots\dots\dots\dots$$
$$=(a_1^2+a_2^2+\dots a_n^2)(x_1^2+x_2^2+\dots x_n^2),$$

$$0=\frac{1}{(x_1-x_2)(x_1-x_3)\dots(x_1-x_n)}+\frac{1}{(x_2-x_1)(x_2-x_3)\dots(x_2-x_n)}$$
$$+\&c.\dots\dots\dots\dots+\frac{1}{(x_n-x_1)(x_n-x_2)\dots(x_n-x_{n-1})},$$

$$1=\frac{x_1^{n-1}}{(x_1-x_2)(x_1-x_3)\dots(x_1-x_n)}+\frac{x_2^{n-1}}{(x_2-x_1)(x_2-x_3)\dots(x_2-x_n)}$$
$$+\&c.\dots\dots\dots\dots+\frac{x_n^{n-1}}{(x_n-x_1)(x_n-x_2)\dots(x_n-x_{n-1})}$$

$$1=\frac{(a-x_2)(a-x_3)\dots(a-x_n)}{(x_1-x_2)(x_1-x_3)\dots(x_1-x_n)}+\frac{(a-x_1)(a-x_3)\dots(a-x_n)}{(x_2-x_1)(x_2-x_3)\dots(x_2-x_n)}$$
$$+\&c.\dots\dots\dots\dots+\frac{(a-x_1)(a-x_2)\dots(a-x_{n-1})}{(x_n-x_1)(x_n-x_2)\dots(x_n-x_{n-1})},$$

$$0 = n^i - \frac{n}{1}(n-1)^i + \frac{n(n-1)}{1 \cdot 2}(n-2)^i - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}(n-3)^i + \&c. \ldots$$

$$0 = \frac{x_1^{i-1}}{(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)} + \frac{x_2^{i-1}}{(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)}$$
$$+ \&c. \ldots\ldots\ldots\ldots\ldots + \frac{x_n^{i-1}}{(x_n-x_1)(x_n-x_2)\ldots(x_n-x_{n-1})},$$

$$a = x_1^i\frac{(a-x_2)(a-x_3)\ldots(a-x_n)}{(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)} + x_2^i\frac{(a-x_1)(a-x_3)\ldots(a-x_n)}{(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)}$$
$$+ \&c. \ldots\ldots\ldots\ldots\ldots + x_n^i\frac{(a-x_1)(a-x_2)\ldots(a-x_{n-1})}{(x_n-x_1)(x_n-x_2)\ldots(x_n-x_{n-1})}.$$

The last three equations are true only so long as i is an integer less than n; x_1, x_2, x_3, &c. x_n, mean so many different numbers, and this notation is often found more convenient than that of using different letters, since the symbols may thus be made to indicate the rank of the numbers. In the advanced state of Algebra, it is essential that the student should make himself familiar with this species of notation. Exercises like the following will enable him to do so.

33. If the development of any power of a binomial be put in the form

$$(1+x)^m = A_0 + A_1x + A_2x^2 + \ldots + A_nx^n + \ldots,$$

so that A_n indicates the co-efficient of the n^{th} power of x, as, for instance, A_2 is the co-efficient of x^2, and A_{10} that of x^{10}; then it is found that

$$A_0 = 1,\ A_1 = \frac{m}{1},\ A_2 = \frac{m(m-1)}{1 \cdot 2},\ A_3 = \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3},\ \&c.$$

Now in order to deduce the expression for A_n, or the co-efficient of *any* power of n, in the development, we remark that the value of A_1 is a fraction having one factor in each member, that A_2 is a fraction having two factors iu each member, that A_3 is a fractiom having three factors in each member, &c. and we conclude that A_n is a fraction having n factors in each member. We next remark that the factors in the numerator successively decrease by an unit, the first factor being m, and therefore the last in the expression for A_n must be $m-(n-1) = m-n+1$; and the factors in the denominator successively increase by an unit, the first factor being 1, and therefore the last, in the expression for A_n, must be $1+(n-1)=n$; hence

$$A_n = \frac{m(m-1)(m-2)\ldots(m-n+1)}{1 \cdot 2 \cdot 3 \ldots n},$$

the numerator expressing the continued product of all numbers, differing by unity, from m to $m-n+1$ inclusive, and the denominator these from 1 to n inclusive; and if in this result we write, instead of n, the successive values 1, 2, 3, 4, &c. we shall get the particular values for A_1, A_2, &c.

But the analyst endeavors to express the law of continuity, among such a series of numbers, by an Algebraic relation among any two or more consecutive ones of the series. Thus, in the case of the binomial co-efficients, it is seen that

$$\frac{A^1}{A} = \frac{m}{1},\ \frac{A_2}{A_1} = \frac{m-1}{2},\ \frac{A_3}{A_2} = \frac{m-2}{3},\ \frac{A_4}{A_3} = \frac{m-3}{4},\ \&c.,$$

and it is concluded that

$$\frac{A_n}{A_{n-1}} = \frac{m-n+1}{n}, \text{ or } A_n = \frac{m-n+1}{n} \cdot A_{n-1},$$

which, together with the fact that $A_0 = 1$, makes known the whole of the co-efficients; and therefore these conclusions should be first aimed at, and the particular ones deduced from them, by the successive substitution of 1, 2, 3, &c., for n; thus, since $A_0 = 1$,

$$A_1 = \frac{m-1+1}{1} \cdot A_0 = \frac{m}{1},$$

$$A_2 = \frac{m-2+1}{2} \cdot A_1 = \frac{m}{1} \cdot \frac{m-1}{2},$$

$$A_3 = \frac{m-3+1}{3} \cdot A_2 = \frac{m}{1} \cdot \frac{m-1}{2} \cdot \frac{m-2}{3},$$

&c.

The direct mode of investigating an expression for A_n, in terms of n and given things, from such relations as these, belongs to a higher branch of analysis; but in simple cases it may be deduced by analogy in the way directed above, as well as in other ways, of which this is one:—by successive substitution,

$$A_0 = 1,$$

$$A_1 = \frac{m}{1} \cdot A_0,$$

$$A_2 = \frac{m-1}{2} \cdot A_2,$$

$$A_3 = \frac{m-2}{3} \cdot A_2,$$

$$A_4 = \frac{m-3}{4} \cdot A_4,$$

$$\begin{matrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{matrix}$$

$$A_n = \frac{m-n+1}{n} \cdot A_{n-1};$$

the continued product of the two members of these equations, gives

$$A_0 A_1 A_2 A_3 \ldots A_{n-1} A_n = \frac{m}{1} \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \ldots \frac{m-n+1}{n} \cdot A_0 A_1 A_2 \ldots A_{n-1},$$

and dividing both members by $A_0 A_1 A_2 \ldots A_{n-1}$,

$$A_n = \frac{m}{1} \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \ldots \frac{m-n+1}{n}.$$

Many remarkable properties of these quantities may be deduced from these relations among them; thus, 1°. if $s \cdot A_m$ represent their sum, by making $x = 1$,

$$(1+1)^m = A_0 + A_1 + A_2 + \ldots + A_m,$$

or $$s \cdot A^m = 2^m.$$

2°. If m be an odd integer, or $m = 2m' + 1$, by making $n = m' + 1$,

$$A_{m'+1} = \frac{2m' + 1 - m' - 1 + 1}{m' + 1} A_{m'} = A^{m'},$$

or the co-efficients of the two middle terms of such a development are equal.

3°. If m be any integer, the co-efficient standing the nth from the last or A_m, may be represented by A_{m-n}, and writing $m - n$ instead of n, in the expression for A_n,

$$\begin{aligned} A_{m-n} &= \frac{m}{1}.\frac{m-1}{2}.\frac{m-2}{3}\dots\frac{m+1}{m-n} \\ &= \frac{m}{1}.\frac{m-1}{2}.\frac{m-2}{3}\dots\frac{m-n+1}{n}\times\frac{m-n}{n+1}.\frac{m-n-1}{n+2}\dots\frac{n+1}{m-n} \\ &= \frac{m}{1}.\frac{m-1}{2}.\frac{m-2}{3}\dots\frac{m-n+1}{n} \\ &= A_n, \end{aligned}$$

or the co-efficients of the terms, equally distant in rank from the extreme terms, are the same. The student will do well to investigate in this manner, the following examples, and others of the same kind,

$$A_n - A_{n-1} = d, \qquad \text{and } A_0 = a;$$

$$A_n = rA_{n-1}, \qquad \text{and } A_0 = a;$$

$$nA_n = (n + m)A_{n-1}, \text{ and } A_0 = 1 = \frac{1.2.3\dots m}{1.2.3\dots m};$$

$$A_n - A_{n-1} = (m-2)n+1, \text{ and } A_0 = 1 = \frac{1.2}{1.2};$$

$$A_n - 2A_{n-2} + A_{n-2} = 2d^2, \qquad \text{and } A_0 = a^2, A_1 = (a + d)^2$$

$$(n + 1)A_n + nA_{n-1} = 0, A_0 = a;$$

$$n(n + 1)A_n = aA_{n-1}, A_0 = 1;$$

$$n(n-1)A_n = aA_{n-2}, A_0 = 1, A_1 = 0;$$

$$(2n + a)A_n = (2n + a - 1)A_{n-1}, A_0 = 1;$$

$$2nA_n = (2n-3)(2n-1)A_{n-2}, A_0 = 1, A_1 = \frac{1}{2^2}$$

$$2n(2n + a - 1)A_n = bA_{n-1}, A_0 = 1.$$

It will be seen that the first equation expresses the property of a series of numbers A_0, A_1, &c., in which any two consecutive terms differ by a constant quantity, d, and these numbers will therefore form a progression by differences; similarly, the numbers found from the second equation will be a progression by quotients; those from the third, the figurate numbers, of the order m; those from the fourth, the polygonal numbers, of the order m, &c.

34. In solving equations, different processes should be used, and the results should be verified, by substitution; and this will often be a more useful exercise than the solution of the equation. In the equation

$$x - \frac{ax}{a+b} + c = \frac{ac}{a-b} - \frac{b^2x}{a^2-b^2};$$

the fractions could be made to disappear and the terms collected in the usual manner; or, since,

$$x-\frac{ax}{a+b}=x\left(1-\frac{a}{a+b}\right)=x.\frac{b}{a+b},$$

and $$\frac{ac}{a-b}-c=c\left(\frac{a}{a-b}-1\right)=c.\frac{b}{a-b},$$

the equation becomes, after dividing by b,

$$\frac{x}{a+b}=\frac{c}{a-b}-\frac{bx}{a^2-b^2},$$

$$\frac{x}{a+b}+\frac{bx}{a^2-b^2}=\frac{c}{a-b},$$

$$\frac{ax}{a^2-b^2}=\frac{c}{a-b},$$

$$x=\frac{c(a+b)}{a}=c\left(1+\frac{b}{a}\right).$$

To verify this result,

$$x-\frac{ax}{a+b}+c=c\left(1+\frac{b}{a}\right)-c+c=c\left(1+\frac{b}{a}\right);$$

and $$\frac{ac}{a-b}-\frac{b^2x}{a^2-b^2}=\frac{ac}{a-b}-\frac{b^2}{a^2-b^2}.\frac{c(a+b)}{a}$$

$$=\frac{ac}{a-b}-\frac{b^2c}{a(a-b)}$$

$$=\frac{(a^2-b^2)c}{a(a-b)}=\frac{(a+b)c}{a}=c\left(1+\frac{b}{a}\right).$$

The equation should also be solved by the most direct process, and the root afterwards reduced to its most simple form; thus, by collecting the terms, as if the co-efficients were simple,

$$x\left(1-\frac{a}{a+b}+\frac{b^2}{a^2-b^2}\right)=c\left(\frac{a}{a-b}-1\right),$$

$$x=c.\frac{\frac{a}{a-b}-1}{1-\frac{a}{a+b}+\frac{b^2}{a^2-b^2}}$$

$$=c.\frac{a^2-b^2}{a^2-b^2}.\frac{\frac{a}{a-b}-1}{1-\frac{a}{a+b}+\frac{b^2}{a^2-b^2}}$$

$$=c.\frac{a(a+b)-(a^2-b^2)}{a^2-b^2-a(a-b)+b^2}$$

$$=c.\frac{a^2+ab-a^2+b^2}{a^2-b^2-a^2+ab+b^2}$$

$$=c.\frac{ab+b^2}{ab}$$

$$=c\left(1+\frac{b}{a}\right).$$

ARTICLE II.

SOLUTIONS TO THE QUESTIONS PROPOSED IN NUMBER VI.

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(31). QUESTION I. *By* ——.

Given $a = ,280796$, $b = 1,528307$, $a = 3$, $d = ,087648$, $e = ,002879$; to calculate the numerical value of the expression

$$x = \sqrt{(ab + c)}\,\frac{d}{e},$$

true to five places of decimals; and exhibit the work, without using logarithms.

SOLUTION. *By Mr. E. H. Delafield, St. Paul's College.*

```
      ,2807960                          ,087648 | ,002879
       703825,1                            8637 |30,443904 = d/e
      2807960                             12780
      1403980                             11516
        56159                              12640
        22463                              11516
          842                               11240
           20                                8637
ab =  ,4291424                               2603
 c = 3,                                      2591
                                               12
ab + c = 3,4291424 | 1,851794̇37 = √(ab + c) .  11

        1
   28 |242
      |224                                     30,443904
  365 |1891                                   73497158,1
      |1825                                    30443904
 3701 |6642                                    24355123
      |3701                                     1522195
37027 |294140                                     30444
      |259189                                     21311
 37034 |34951                                      2740
  ...  |33331                                       122
         1620                                         9
         1481                                         2
          139          d/e √(ab+c) = 56,375850 = x.
          111
           28
           26
```

(32). Question II. *By* ——.

Express the number 1006005 in a system of notation whose scale of relation is 6.

Solution. *By Mr. W. B. Benedict, Upperville, Va.*

To transform a number N, from the denary to any other system of notation, in which the scale of relation is R, it is only necessary to determine the digits or co-efficients, a, b, c, ... f, g, h, in the general expression

$$N = aR^n + bR^{n-1} + cR^{n-2} + \ldots\ldots + fR^2 + gR + h.$$

If we divide this equation by R, and denote by N′ the quotient of N by R, and by r the remainder, we get

$$N' + \frac{r}{R} = aR^{n-1} + bR^{n-2} + cR^{n-3} + \ldots\ldots + fR + g + \frac{h}{R};$$

where the fractions and whole numbers in the two members must be identical, then

$$h = r,$$

$$N' = aR^{n-1} + bR^{n-2} + cR^{n-3} + \ldots\ldots + fR + g.$$

In the same manner g is the remainder after dividing N′ by R, f the remainder after dividing the last quotient by R, &c. In the present case R = 6, N = 1006005; hence

$$\begin{array}{r|l}
6 & 1006005 \\
6 & 167667 + \frac{3}{6}, \text{ or } h = 3, \\
6 & 27944 + \frac{3}{6}, \text{ or } g = 3, \\
6 & 4657 + \frac{2}{6}, \text{ or } f = 2, \\
6 & 776 + \frac{1}{6}, \text{ or } e = 1, \\
6 & 129 + \frac{2}{6}, \text{ or } d = 2, \\
6 & 21 + \frac{3}{6}, \text{ or } c = 3, \\
6 & 3 + \frac{3}{6}, \text{ or } b = 3, \\
 & 0 + \frac{3}{6}, \text{ or } a = 3;
\end{array}$$

and the number in the senary scale, is

33321233.

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(33). Question III. *By* ——.

Given, to find x and y, the two equations

$$x + \frac{x^2}{y} + y = a,$$

$$x^2 + \frac{x^4}{y^2} + y^2 = b.$$

FIRST SOLUTION. *By Mr. D. D. Hughes, Syracuse Academy.*

Squaring the first equation, we get

$$x^2+\frac{x^4}{y^2}+y^2+\frac{2x^3}{y}+2xy+2x^2=a^2\,;$$

Subtract the second, $$\frac{2x^3}{y}+2xy+2x^2=a^2-b\,;$$

Divide this by the first equation, member by member,

$$2x=\frac{a^2-b}{a},\text{ or } x=\frac{a^2-b}{2a}.$$

By clearing the first equation of the fractions, and reducing,

$$y^2-(a-x)y=-x^2,$$

and $$y=\tfrac{1}{2}(a-x)\pm\tfrac{1}{2}\sqrt{(a-x)^2-4x^2}$$

$$=\frac{a^2+b}{4a}\pm\tfrac{1}{2}\sqrt{\frac{(a^2+b)^2}{4a^2}-\frac{(a^2-b)^2}{a^2}}.$$

SECOND SOLUTION. *By Mr. J. V. Campbell, St. Paul's College.*

Divide the second equation by the first, then

$$-x+\frac{x^2}{y}+y=\frac{b}{a}\quad\ldots\ldots\ldots\quad(3),$$

subtract (3) from the first, then

$$2x=a-\frac{b}{a},$$

or $$x=\frac{a^2-b}{2a},$$

and $$x^2=\frac{(a^2-b)^2}{4a^2}\quad\ldots\ldots\ldots\quad(4).$$

Add (4) to the second equation, then

$$\frac{x^4}{y^2}+2x^2+y^2=b+\frac{(a^2-b)^2}{4a^2}=\frac{(a^2+b)^2}{4a^2},$$

and $$y+\frac{x^2}{y}=\frac{a^2+b}{2a}\quad\ldots\ldots\ldots\quad(5).$$

Multiply (4) by (3), and subtract from the second equation,

$$\frac{x^4}{y^2}-2x^2+y^2=b-\frac{3(a^2-b)^2}{4a^2}=\frac{4a^2b-3(a^2-b)^2}{4a^2},$$

and $$y-\frac{x^2}{y}=\pm\frac{1}{2a}\sqrt{4a^2b-3(a^2-b)^2}\quad\ldots\quad(6),$$

Add (5) and (6) $$y=\frac{1}{4a}\left(a^2+b\pm\sqrt{4a^2b-3(a^2-b)^2}\right).$$

(24). QUESTION. *By the Editor.*

Given that

$$2(a+b)^2+ab=(2a+b)(a+2b).$$

It is required to divide the number

$$2x+y$$

into two factors.

SOLUTION. *By Mr. J. Blickensderfer, jun., Canal Dover, Ohio.*

If we make $(a+b)^2 = x$, and $ab = y$;
so that $(a-b)^2 = x - 4y$, and therefore

$$a + b = \sqrt{x}, \quad \text{and } a - b = \sqrt{x - 4y};$$

or $$a = \tfrac{1}{2}\sqrt{x} + \tfrac{1}{2}\sqrt{x - 4y}, \text{ and } b = \tfrac{1}{2}\sqrt{x} - \tfrac{1}{2}\sqrt{x - 4y};$$

the two factors will be

$$2a + b = \tfrac{3}{2}\sqrt{x} + \tfrac{1}{2}\sqrt{x - 4y}, \text{ and } a + 2b = \tfrac{3}{2}\sqrt{x} - \tfrac{1}{2}\sqrt{x - 4y};$$

therefore $$2x + y = \tfrac{1}{2}(3\sqrt{x} + \sqrt{x - 4y}) \times \tfrac{1}{2}(3\sqrt{x} - \sqrt{x - 4y}.$$

—— *Cor.* If we put $2x + y = n$, so that $y = n - 2x$, we have

$$n = \tfrac{1}{2}(3\sqrt{x} + \sqrt{9x - 4n}) \times \tfrac{1}{2}(3\sqrt{x} - \sqrt{9x - 4n});$$

where x may be any number whatever, and this enables us to divide any number n, an infinite number of ways, into two irrational factors; perhaps a neater form for them is

$$n = (x + \sqrt{x^2 - n})(x - \sqrt{x^2 - n}).$$

(35.) QUESTION V. *From Peirce's Algebra.*

A, B, C, D, E play together on this condition, that he who loses shall give to all the rest as much as they already have. First A loses, then B, then C, then D, and at last also E. All lose in turn, and yet at the end of the fifth game they all have the same sum, viz., each \$32. How much had each when they began to play?

SOLUTION. *By Omicron, jun., Chapel Hill, N. C.*

Let the original sum presented by A, B, C, D, E, be respectively represented by u, v, x, y, z; then, as they will have the same sum, in all, at first as at last,

$$u + v + x + y + z = 5 \times 32 = 160.$$

Now, since the person who loses pays to each of the rest as much as they already have, he must pay them all \$160 — the money he has himself, and when his losses are paid he will have left twice what he had before—160, while each winner's money is doubled; hence at the end of the several successive games the money each man has may be represented thus:

	A has	B has	C has	D has	E has
1st game,	$2u - 160$	$2v$	$2x$	$2y$	$2z$,
2nd do.	$4u - 320$	$4v - 160$	$4x$	$4y$	$4z$,
3rd do.	$8u - 640$	$8v - 320$	$8x - 160$	$8y$	$8z$,
4th do.	$16u - 1280$	$16v - 640$	$16x - 320$	$16y - 160$	$16z$,
5th do.	$32u - 2560$	$32v - 1280$	$32x - 640$	$32y - 320$	$32z - 160$;

Hence,
$$\begin{aligned} 32u - 2560 &= 32, & \text{and } u &= 80 + 1 = 81, \\ 32v - 1280 &= 32, & v &= 40 + 1 = 41, \\ 32x - 640 &= 32, & x &= 20 + 1 = 21, \\ 32y - 320 &= 32, & y &= 10 + 1 = 11, \\ 32z - 160 &= 32, & z &= 5 + 1 = 6. \end{aligned}$$

—— Omicron, deduces from this a general solution, which will be reserved for Question (100) of the Senior Department. Prof. Peirce states that this question is taken from a much older book.

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(36). Question VI. *By Mr. Geo. W. Coaklay, Peekskill Academy, N. Y.*

Find what relation must exist among the co-efficients of the equation

$$x^4 + \mathrm{A}x^3 + \mathrm{B}x^2 + \mathrm{C}x + \mathrm{D} = 0,$$

so that it may be put in either of the forms

$$(x^2 + ax)^2 + b(x^2 + ax) + c = 0,$$

or

$$(x^2 + a'x + b')^2 + c' = 0.$$

Solution. *By Mr. B. Birdsall, New-Hartford, N. Y.*

By expanding the second equation, it becomes

$$x^4 + 2ax^3 + (a^2 + b)x^2 + abx + c = 0;$$

and if the first equation can be put in this form, it must be the case that

$$2a = \mathrm{A},\ a^2 + b = \mathrm{B},\ ab = \mathrm{C},\ c = \mathrm{D};$$

then $\quad a = \frac{1}{2}\mathrm{A}, \qquad b = \mathrm{B} - a^2 = \mathrm{B} - \frac{1}{4}\mathrm{A}^2,$

and, substituting these in the equation $ab = \mathrm{C}$,

$$\mathrm{AB} - \tfrac{1}{4}\mathrm{A}^3 = 2\mathrm{C};$$

which is the relation required. Similarly the third equation gives

$$x^4 + 2a'x^3 + (a'^2 + 2b')x^2 + 2a'b'x + b'^2 + c' = 0;$$

and if the first equation can be put in this form,

$$2a' = \mathrm{A},\ a'^2 + 2b' = \mathrm{B},\ 2a'b' = \mathrm{C},\ b'^2 + c' = \mathrm{D};$$

then $\quad a' = \frac{1}{2}\mathrm{A},\ b' = \frac{1}{2}\mathrm{B} - \frac{1}{2}a'^2 = \frac{1}{2}\mathrm{B} - \frac{1}{8}\mathrm{A}^2,$

and the third becomes

$$\mathrm{AB} - \tfrac{1}{4}\mathrm{A}^3 = 2\mathrm{C},$$

which is the same relation as before; and therefore if an equation of the fourth degree can be put in one of these forms, it can be put in the other one, and the roots in either form can be found by the usual process for equations of the second degree.

—— Mr. Coaklay, the proposer, solved question (24) by this method, thus: In that question, $\mathrm{A} = -3$, $\mathrm{B} = -8\frac{3}{4}$, $\mathrm{C} = 16\frac{1}{2}$, $\mathrm{D} = -2075\frac{3}{4}$, which have the necessary relation; therefore, $a' = -\frac{3}{2}$, $b' = -\frac{11}{2}$, $c' = \mathrm{D} - b'^2 = -2106$, and the equation may be put in the form

$$(x^2 - \tfrac{3}{2}x - \tfrac{11}{2})^2 - 2106 = 0, \text{ \&c.}$$

(37.) Question VII. *By β.*

Prove that, if θ be any angle,

$$\tan^2\theta - \tan^2\tfrac{1}{2}\theta = \frac{8\sin^2\frac{1}{2}\theta\cos\frac{3}{2}\theta}{\cos^2\theta}.$$

First Solution. *By Mr. Geo. W. Coaklay.*

$$\tan^2\theta - \tan^2\tfrac{1}{2}\theta = (\tan\theta + \tan\tfrac{1}{2}\theta)(\tan\theta - \tan\tfrac{1}{2}\theta).$$

But, $\quad \tan\theta + \tan\frac{1}{2}\theta = \dfrac{\sin\theta}{\cos\theta} + \dfrac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}$

$$=\frac{\sin\theta\cos\frac{1}{3}\theta+\cos\theta\sin\frac{1}{3}\theta}{\cos\theta\cos\frac{1}{3}\theta}=\frac{\sin(\theta+\frac{1}{3}\theta)}{\cos\theta\cos\frac{1}{3}\theta}$$

$$=\frac{\sin\frac{4}{3}\theta}{\cos\theta\cos\frac{1}{3}\theta}=\frac{4\sin\frac{1}{3}\theta\cos\frac{1}{3}\theta\cos\frac{2}{3}\theta}{\cos\theta\cos\frac{1}{3}\theta}$$

$$=\frac{4\sin\frac{1}{3}\theta\cos\frac{2}{3}\theta}{\cos\theta},$$

and $$\tan\theta-\tan\tfrac{1}{3}\theta=\frac{\sin\theta}{\cos\theta}-\frac{\sin\frac{1}{3}\theta}{\cos\frac{1}{3}\theta}=\frac{\sin\frac{2}{3}\theta}{\cos\theta\cos\frac{1}{3}\theta}=\frac{2\sin\frac{1}{3}\theta}{\cos\theta};$$

therefore $$\tan^2\theta-\tan^2\tfrac{1}{3}=\frac{8\sin^2\frac{1}{3}\theta\cos\frac{2}{3}\theta}{\cos^2\theta}.$$

SECOND SOLUTION. *By Mr. R. S. Howland, St. Paul's College.*

$$\cos\tfrac{1}{3}\theta+\cos\theta=2\cos\tfrac{1}{2}(\theta+\tfrac{1}{3}\theta)\cos\tfrac{1}{2}(\theta-\tfrac{1}{3}\theta)$$
$$=2\cos\tfrac{2}{3}\theta\cos\tfrac{1}{3}\theta,$$

and $$\cos\tfrac{1}{3}\theta-\cos\theta=2\sin\tfrac{2}{3}\theta\sin\tfrac{1}{3}\theta$$
$$=4\sin^2\tfrac{1}{3}\theta\cos\tfrac{1}{3}\theta;$$

and multiplying these equations, member by member,

$$\cos^2\tfrac{1}{3}\theta-\cos^2\theta=8\sin^2\tfrac{1}{3}\theta\cos\tfrac{2}{3}\theta\cos^2\tfrac{1}{3}\theta;$$

dividing by $\cos^2\frac{1}{3}\theta\cos^2\theta$,

$$\sec^2\theta-\sec^2\tfrac{1}{3}\theta=\frac{8\sin^2\frac{1}{3}\theta\cos\frac{2}{3}\theta}{\cos^2\theta},$$
$$=\tan^2\theta-\tan^2\tfrac{1}{3}\theta.$$

THIRD SOLUTION. *By Mr. J. Blickensderfer, jun., Canal Dover, Ohio.*

In the known formula

$$\tan^2a-\tan^2b=\frac{\sin(a+b)\sin(a-b)}{\cos^2a\cos^2b},$$

take $a=\theta$, $b=\frac{1}{3}\theta$; then we have

$$\tan^2\theta-\tan^2\tfrac{1}{3}\theta=\frac{\sin\frac{4}{3}\theta\sin\frac{2}{3}\theta}{\cos^2\theta\cos^2\frac{1}{3}\theta}$$
$$=\frac{4\sin\frac{1}{3}\theta\cos\frac{1}{3}\theta\cos\frac{2}{3}\theta\,.\,2\sin\frac{1}{3}\theta\cos\frac{1}{3}\theta}{\cos^2\theta\cos^2\frac{1}{3}\theta}$$
$$=\frac{8\sin^2\frac{1}{3}\theta\cos\frac{2}{3}\theta}{\cos^2\theta}.$$

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(38). QUESTION VIII. *By the Editor.*

In a plane triangle, given that

$$b=a\sin C,\quad c=a\cos B.$$

Find its angles.

FIRST SOLUTION. *By Mr. Warren Colburn, St. Paul's College.*

By a known relation among the sides and angles of a plane triangle,

$$c=a\cos B+b\cos A;$$

but in the triangle in question,

$$c = a\cos B;$$

therefore, $0 = b\cos A$, or $\cos A = 0$, and $A = \frac{1}{2}\pi$.

Similarly, $b = a\cos C + c\cos A$

$= a\cos C$, since $A = \frac{1}{2}\pi$,

$= a\sin C$, by the question;

then $\sin C = \cos C$, and $C = \frac{1}{4}\pi$.

Finally $B = \pi - (A + C) = \pi - \frac{3}{4}\pi = \frac{1}{4}\pi$,

and the triangle is right-angled and isosceles.

SECOND SOLUTION. *By Mr. B. Birdsall.*

Multiply the second given equation by $2c$, then

$$2c^2 = 2ac\cos B,$$

but, in all triangles, $b^2 = a^2 + c^2 - 2ac\cos B$;

hence, by addition, $b^2 + c^2 = a^2$,

and therefore the triangle is right-angled, a being the hypothenuse; then $A=90°$, and $B + C = 90°$; consequently

$$\sin C = \cos B,$$

or $\dfrac{b}{a} = \dfrac{c}{a}$, and $b = c$;

or the triangle is isosceles, and $B = C = 45°$.

THIRD SOLUTION. *By Omicron, jun., Chapel Hill, N. C.*

Here $1 : \sin C :: a : b :: \sin A : \sin B$,

or $\sin B = \sin A \sin C$ . . . . (1),

also, $1 : \cos B :: a : c :: \sin A : \sin C$,

or $\sin C = \sin A \cos B$ . . . . (2).

But $\sin C = \sin(A + B) = \sin A\cos B + \cos A \sin B$,

and $0 = \cos A \sin B$;

therefore, either $\sin B = 0$, and $B = 0$, or $180°$, which cannot be;

or $\cos A = 0$, and $A = 90°$.

Then (1) becomes $\sin B = \sin C$,

or $B = C = 45°$.

—— Mr. Coaklay favored us with two solutions to this question, similar to the second and third of the above; the solution of L. was similar to the second, and that of Mr. Campbell to the third.

(39). QUESTION IX. *By* ——.

Let $a_1, a_2, a_3, \ldots\ldots$, be the sides of any plane polygon, and $\varphi_1, \varphi_2, \varphi_3, \ldots\ldots$, the angles they severally make with any straight line in the same plane, all counted in the same direction; prove that

$$a_1\sin\varphi_1 + a_2\sin\varphi_2 + a_3\sin\varphi_3 + \ldots\ldots = 0,$$
$$a_1\cos\varphi_1 + a_2\cos\varphi_2 + a_3\cos\varphi_3 + \ldots\ldots = 0.$$

FIRST SOLUTION. *By Mr. D. D. Hughes.*

It is evident that $a_1 \sin\varphi_1 + a_2\sin\varphi_2 + \ldots\ldots$, is the sum of the projections of all the sides of the polygon on a given line;—it is also clear that the sum of the projections of the sides of one-half of the po-

lygon is equal to the sum of the projections of the sides of the other half, while they have contrary signs, and therefore their sum = 0, or

$$a_1 \sin\varphi_1 + a_2 \sin\varphi_2 + a_3 \sin\varphi_3 + \ldots\ldots = 0,$$

Similarly, $a_1 \cos\varphi_1 + a_2 \cos\varphi_2 + \ldots\ldots$ is the sum of the projections of all the sides on a line perpendicular to the first one, and therefore

$$a_1 \cos\varphi_1 + a_2 \cos\varphi_2 + a_3 \cos\varphi_3 + \ldots\ldots = 0.$$

SECOND SOLUTION. *By Mr. B. Birdsall.*

Let a parallel to the given line be drawn through the vertex from which a_1 is counted, and the sides will make the same angles with this line as with the given one; then

$a_1 \sin\varphi_1$, $a_1 \sin\varphi_1 + a_2 \sin\varphi_2$, $a_1 \sin\varphi_1 + a_2 \sin\varphi_2 + a_3 \sin\varphi_3$, &c. will represent the distances of the successive vertices of the polygon from this line; but when we have reckoned entirely round the polygon and arrived at the starting point, its distance from the line passing through it = 0, and therefore

$$a_1 \sin\varphi_1 + a_2 \sin\varphi_2 + a_3 \sin\varphi_3 + \ldots\ldots = 0.$$

In the same manner, $a_1 \cos\varphi_1 + a_2 \cos\varphi_2 + \ldots\ldots$, will express the distance of the same vertex to a line passing through it, perpendicular to the given line, and therefore it must = 0.

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(40). QUESTION X. *By* ——.

Having given the sum of the sides that include the right angle of a spherical triangle, and the difference of their opposite angles; to determine the sides and angles of the triangle.

FIRST SOLUTION. *By Mr. R. S. Howland.*

$$\cos(B - A) = \cos B \cos A + \sin B \sin A \quad \ldots\ldots \quad (1),$$

$$\cos(a + b) = \cos a \cos b - \sin a \sin b \quad \ldots\ldots \quad (2).$$

But $\cos a \cos b = \cos h$, h being the hypothenuse,

and $\sin a = \sin h \sin A$, $\sin b = \sin h \sin B$ $\quad \ldots\ldots \quad (3)$,

therefore $\sin a \sin b = \sin^2 h \sin A \sin B$, and (2) becomes

$$\cos(a + b) = \cos h - \sin^2 h \sin A \sin B,$$

or $$\cos h - \cos(a + b) = \sin^2 h \sin A \sin B.$$

Divide (1) by this equation, recollecting that $\cot A \cot B = \cos h$, then

$$\frac{\cos(B - A)}{\cos h - \cos(a + b)} = \frac{\cos h + 1}{\sin^2 h} = \frac{1}{1 - \cos h};$$

whence $$1 - \cos h = \frac{1 - \cos(a + b)}{1 + \cos(B - A)},$$

or $$2\sin^2 \tfrac{1}{2}h = \frac{\sin^2 \frac{1}{2}(a + b)}{\cos^2 \frac{1}{2}(B - A)},$$

and $$\sin\tfrac{1}{2}h = \frac{\sin\frac{1}{2}(a + b)}{\cos\frac{1}{2}(B - A)} \cdot \sqrt{\tfrac{1}{2}},$$

whence h is found. Again

$$\cos(a + b) + \cos(b - a) = 2\cos a \cos b = 2\cos h,$$

therefore $\sin^2 \frac{1}{2}(a + b) + \sin^2 \frac{1}{2}(b - a) = 2 \sin^2 \frac{1}{2}h = \dfrac{\sin^2 \frac{1}{2}(a + b)}{\cos^2 \frac{1}{2}(B - A)}$,

and $\sin^2 \frac{1}{2}(b-a) = \dfrac{\sin^2 \frac{1}{2}(a+b)}{\cos^2 \frac{1}{2}(B-A)} - \sin^2 \frac{1}{2}(a+b) = \sin^2 \frac{1}{2}(a+b) \tan^2 \frac{1}{2}(B-A)$

or $\quad \sin \frac{1}{2}(b-a) = \sin \frac{1}{2}(a+b) \tan \frac{1}{2}(B-A)$,

whence $b-a$ is found, and consequently a and b. Moreover

$$\cos h = \cot A \cot B = \frac{\cos A \cos B}{\sin A \sin B} = \frac{\cos(B-A)+\cos(B+A)}{\cos(B-A)-\cos(B+A)},$$

and $$1-\cos h = \frac{\sin^2 \frac{1}{2}(a+b)}{\cos^2 \frac{1}{2}(B-A)} = \frac{-2\cos(B+A)}{\cos(B-A)-\cos(B+A)};$$

whence $$\cos(A+B) = \frac{\cos(B-A)\sin^2 \frac{1}{2}(a-b)}{\sin^2 \frac{1}{2}(a+b) - 2\cos^2 \frac{1}{2}(B-A)},$$

from which $B+A$ is found, and thence A and B.

SECOND SOLUTION. *By L., Murray Co., Geo.*

Let C be the right angle, then we shall have

$$\sin c \cos A = \cos a \sin b, \qquad \sin c \sin A = \sin a,$$
$$\sin c \cos B = \cos b \sin a, \qquad \sin c \sin B = \sin b,$$

$\therefore \sin c(\cos A + \cos B) = \sin(a+b)$, $\quad \therefore \sin c(\sin A - \sin B) = \sin a - \sin b$;

by division, $$\frac{\sin A - \sin B}{\cos A + \cos B} = \frac{\sin a - \sin b}{\sin(a+b)};$$

But $$\frac{\sin A - \sin B}{\cos A + \cos B} = \tan \tfrac{1}{2}(A-B),$$

also, $$\sin a - \sin b = 2 \sin \tfrac{1}{2}(a-b) \cos \tfrac{1}{2}(a+b),$$

and $$\sin(a+b) = 2\sin \tfrac{1}{2}(a+b) \cos \tfrac{1}{2}(a+b);$$

hence $$\tan \tfrac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)},$$

or $$\sin \tfrac{1}{2}(a-b) = \sin \tfrac{1}{2}(a+b) \tan \tfrac{1}{2}(A-B).$$

From this equation $a-b$ can be determined, and thence a and b; the side c and the angles A and B, are then found from the usual formulas.

——— Most of our correspondents solved this question by means of Napier's Analogies.

(41). QUESTION XI. *By θ.*

Given the equation

$$y^2 - yx + 1 = 0,$$

to express y, by the method of Indeterminate Co-efficients, in a series of monomials arranged 1°. according to the ascending powers of x, 2°. according to the descending powers of x.

SOLUTION. *By Δ.*

Make $y = Ax^a + Bx^b + Cx^c + \&c.$; then, by substitution,

$$\begin{aligned} 0 &= y^2 - yx + 1 \\ &= A^2x^{2a} + 2ABx^{a+b} + B^2x^{2b} + 2ACx^{a+c} + 2BCx^{b+c} + C^2x^{2c} + \&c. \\ &\quad - Ax^{a+1} - Bx^{b+1} - Cx^{c+1} - Dx^{d+1} - \&c. \\ &\quad + x^0; \end{aligned}$$

and among these exponents, we can make

1°. $2a=0$, or $a=0$; $a+b=a+1$, or $b=1$, $a+c=b+1$, or $c=2$, &c.;
hence $\quad y = A + Bx + Cx^2 + Dx^3 + \&c.$

$$0 = y^2 - yx + 1$$

$$= \begin{array}{l|l|l|l|l} A^2 & +2AB & x + B^2 & x^2 + 2AD & x^3 + C^2 \\ +1 & -A & +2AC & +2BC & +2AE \\ & & -B & -C & +2BD \\ & & & & -D \end{array}\; x^4 + \&c.$$

Then
$$\begin{array}{lll} A^2+1=0, & \text{or} & A=\pm\sqrt{-1}, \\ 2AB-A=0, & \text{or} & B=\tfrac{1}{2}, \\ B^2+2AC-B=0, & \text{or} & C=\dfrac{1}{2.4A}, \\ 2AD+2BC-C=0, & \text{or} & D=0, \\ 2AE+2BD-D+C^2=0, & \text{or} & E=\dfrac{1}{2.4.4^2.A}, \\ \&c. & & \&c. \end{array}$$

and $\quad y = \pm\sqrt{-1} + \frac{1}{2}x \mp \dfrac{\sqrt{-1}}{2\,.\,4}x^2 \mp \dfrac{\sqrt{-1}}{2.4.4^2}x^4 \mp \dfrac{1\,.\,3\,.\sqrt{-1}}{2\,.\,4\,.\,6\,.\,4^3}x^6 \;\ldots\ldots$

2°. $2a=a+1$, or $a=1$, $a+b=b+1=0$, or $b=-1$, $c+1=2b$, $c=-3$, &c.;

$$y = Ax + Bx^{-1} + Cx^{-3} + Dx^{-5} + \&c.$$

$$0 = y^2 - yx + 1$$

$$= \begin{array}{l|l|l|l} A^2 & x^2 + 2AB & x^0 + 2AC & x^{-2} + 2AD \\ -A & -B & +B^2 & +2BC \\ & +1 & -C & -D \end{array}\; x^{-4} + \&c.$$

Then
$$\begin{array}{ll} A^2-A=0, \text{ and } 2A-1=\pm 1, & A=\tfrac{1}{2}(1\pm 1), \\ 2AB-B+1=0, & B=\mp 1, \\ 2AC+B^2-C=0, & C=\mp 1=\mp\dfrac{2}{2}, \\ 2AD+2BC-D=0, & D=\mp 2=\mp\dfrac{2\,.\,6}{2\,.\,3}, \\ 2AE+2BD+C^2-E=0, & E=\mp 5=\mp\dfrac{2.6.10}{2.3.4}, \\ 2AF+2BE+2CD-F=0, & F=\mp 14=\mp\dfrac{2.6.10.14}{2.3.4.5}, \\ \&c. & \&c. \end{array}$$

and $\quad y = \frac{1}{2}(1\pm 1)x \mp x^{-1} \mp \dfrac{2}{2}x^{-3} \mp \dfrac{2\,.\,6}{2\,.\,3}x^{-5} \mp \dfrac{2.6.10}{2.3.4}x^{-7} \;\ldots\ldots$

Cor. $\dfrac{2\,.\,6\,.\,10\ldots\ldots 2(2n-3)}{1\,.\,2\,.\,3\ldots\ldots\ldots\ldots n} = \dfrac{\overline{n+1}\,.\,\overline{n+2}\ldots\ldots\overline{2n-2}}{1\,.\,2\,.\,3\ldots\ldots\ldots\overline{n-1}} =$ a whole number, for all values of n.

QUESTION XII. *By* ——.

The equation of a plane is

$$Ax + By + Cz + D = 0;$$

prove that the area of the triangle intercepted on the plane by the three rectangular co-ordinate planes, is

$$\frac{D^2}{2ABC}\,.\,\sqrt{A^2+B^2+C^2},$$

First Solution. *By L., Murray Co., Geo.*

The distances from the origin to the intersections of the plane with the axes of x, of y, and of z, are

$$x' = -\frac{D}{A},\ y' = -\frac{D}{B},\ z' = -\frac{D}{C};$$

the trace (a) on the plane of xy is

$$a = \sqrt{x'^2 + y'^2} = \frac{D}{AB}\sqrt{A^2 + B^2};$$

the perpendicular (p) from the origin upon the trace (a), is

$$p = \frac{x'y'}{a} = \frac{D}{\sqrt{A^2 + B^2}};$$

and the perpendicular (P) from the intersection of the plane with the axis of z upon the trace (a), is

$$P = \sqrt{p^2 + z'^2} = \frac{D}{C}.\sqrt{\frac{A^2 + B^2 + C^2}{A^2 + B^2}};$$

hence the area of the triangle is

$$\tfrac{1}{2}aP = \frac{D^2}{2ABC}\sqrt{A^2 + B^2 + C^2}.$$

Second Solution. *By Mr. Geo. W. Coakley.*

From the given equation we have, for the intersections of the plane with the axes $x = -\frac{D}{A}$, $y = -\frac{D}{B}$, $z = -\frac{D}{C}$. Put a, b, c for the sides of the triangle, then, evidently

$$a^2 = x^2 + y^2 = \frac{D^2}{A^2} + \frac{D^2}{B^2},\ b^2 = y^2 + z^2 = \frac{D^2}{B^2} + \frac{D^2}{C^2},\ c^2 = x^2 + z^2 = \frac{D^2}{A^2} + \frac{D^2}{C^2}.$$

But, if s be the area of the triangle, it is easy to prove that

$$4s^2 = a^2b^2 - \tfrac{1}{4}(a^2 + b^2 - c^2)^2;$$

now $$a^2b^2 = \frac{D^4}{B^4} + \frac{D^4}{A^2B^2} + \frac{D^4}{A^2C^2} + \frac{D^4}{B^2C^2},$$

and $$a^2 + b^2 - c^2 = \frac{2D^2}{B^2};$$

hence, $$4s^2 = \frac{D^4}{A^2B^2} + \frac{D^4}{A^2C^2} + \frac{D^4}{B^2C^2}$$

$$= \frac{D^4}{A^2B^2C^2}(A^2 + B^2 + C^2),$$

and $$2s = \frac{D^2}{ABC}\sqrt{A^2 + B^2 + C^2}.$$

—— *Cor.* If α, β, γ, be the angles which a perpendicular (δ) from the origin upon the plane, makes with the three axes respectively, its equation will be

$$x\cos\alpha + y\cos\beta + z\cos\gamma = \delta,$$

the area becomes $s = \tfrac{1}{2}\delta^2 \sec\alpha \sec\beta \sec\gamma$,

and the tretraedron whose surfaces are the given plane and the three co-oridinate planes has for its volume

$$\tfrac{1}{6}\delta^3 \sec\alpha \sec\beta \sec\gamma.$$

List of Contributors to the Junior Department, and of Questions answered by each. The figures refer to the number of the questions, as marked in Number VI, Article XIV., page 359.

W. B. BENEDICT, Upperville, Fauquier Co., Va., answered 2, 3, 4.
B. BIRDSALL, New-Hartford, N. Y., ans. all the questions.
J. BLICKENSDERFER, jun., Canal Dover, Ohio, ans. 3, 4, 7, 8.
J. V. CAMPBELL, Sophomore Class, St. Paul's College, ans. 1, 3, 8.
GEO. W. COAKLAY, Peekskill Academy, N. Y., ans. all the questions.
WARREN COLBURN, Sophomore Class, St. Paul's College, ans. 3, 4, 8.
E. H. DELAFIELD, Freshman Class, St. Paul's College, ans. 1, 3, 4, 5, 8.
DELTA, ans. 11.
R. S. HOWLAND, Junior Class, St. Paul's College, ans. 7, 8, 10, 12.
D. D. HUGHES, Syracuse Academy, N. Y., 3, 5, 6, 7, 9, 12.
L., Murray County, Geo., ans. all the questions.
OMICRON, jun., Chapel Hill, N. C., ans. all but 11.

ARTICLE III.

QUESTIONS TO BE ANSWERED IN NUMBER VIII.

Their solutions must arrive before August 1st, 1839.

(43). QUESTION I. *By* ——.

Transform the numbers 25 and 389 into a system of notation whose base, or scale of relation is 3; multiply them in that state, exhibiting the process, and transform the result to the decimal scale.

(44). QUESTION II. *By* ——.

Prove that

$$(n+1)(n+2)\ .\ .\ .\ .\ (n+n)=2^n\times 1\ .\ 3\ .\ 5\ .\ .\ .\ .\ (2n-1).$$

(45). QUESTION. III. *By L., Murray Co., Geo.*

To find x, y, z, there are given the three equations

$$(1),\quad xz=y^2,$$
$$(2),\quad 2xz-x-y-z=1,$$
$$(3),\quad x^2+y^2+z^2=3x+3y+3z.$$

(46). QUESTION IV. *By Mr. E. H. Delafield.*

Find what relation must have place among the co-efficients of the equation

$$x^4+\text{A}x^3+\text{B}x^2+\text{C}x+\text{D}=0,$$

so that the process for taking away its second term, may also take away its fourth term at the same time.

(47). QUESTION V. *By* ——.

Let $\quad \text{x}_0 = 1, 2\text{x}_1 = x + \frac{1}{x}, 2\text{x}_2 = x^2 + \frac{1}{x^2}, 2\text{x}_3 = x_3 + \frac{1}{x^3}$, &c.;

prove that

$$\text{x}_{n+1} + \text{x}_{n-1} = 2\text{x}_n\text{x}_1.$$

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(48). QUESTION VI. *By* ——.

Adapt the relations of the sides and angles of a plane triangle to the case where the sides are in arithmetical progression, and find the area of the triangle.

(49.) QUESTION VII. *By β.*

Prove that, θ being any angle,

$$\operatorname{cosec} \theta - 2 \operatorname{cosec} 3\theta = \cot 3\theta \sec \theta.$$

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(50). QUESTION VIII. *By* ——.

In Navigation, find the bearing and distance from a given place on the earth's surface to another one, differing from the former 10° in latitude and 10° in longitude.

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(51). QUESTION IX. *By* ——.

The earth being supposed a perfect sphere, draw a great circle arc between any two points on the surface which differ from each other 10° in latitude and 10° in longitude; find its length, and the angle it makes with the meridian of either place.

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(52). QUESTION X. *By* ——.

Find the points of intersection of the two ellipses

$$7y^2 + 4x^2 = 28,$$
$$6y^2 + 5x^2 = 30,$$

related to the same axes of co-ordinates, and determine the angles they make with each other at these points.

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(53). QUESTION XI. *By Mr. H. Clay, Eng.*

Find, when $\varphi = 0$, the value of the expression

$$\frac{1}{\varphi^2} - \frac{1}{\tan^2 \varphi}.$$

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(54). QUESTION XII. *By* ——.

AB is the diameter of a given circle, and C any point in the circumference; from C let fall CD perpendicular to AB, and upon it take CP = AD; find the curve in which the point P is always found.

SENIOR DEPARTMENT.

ARTICLE I.

SOLUTIONS TO THE QUESTIONS PROPOSED IN NUMBER V.

(82). QUESTION I. *By an Engineer.*

The following is an extract from my Note-Book:

No.	Bearing.	Distance.	Elevation.
1	N. 10° 15′ E.	27,64 ch.	+ 17° 54′
2	N. 28° 40′ W.	100,00	+ 20° 19′
3	N. 20° 00′ W.	15,00	+ 7° 43′
4	N. 20° 00′ W.	37,26	— 5° 26′
5	N. 30° 17′ E.	68,75	— 11° 13′

It is required to find the Bearing, Distance, and Elevation of a line drawn from the beginning of the first to the end of the fifth line, by a method applicable to all practical cases of the kind. The Bearing is considered as the inclination of a vertical plane through the two places with the plane of the meridian.

FIRST SOLUTION. *By Mr. R. S. Howland.*

We may suppose these to be six points in space, referred to three rectangular axes of x, y, z; the plane of xz being the plane of the meridian, that of yz the vertical east and west plane, and that of xy the horizontal plane through the first point. The formulas for calculation will then be the common ones for transformation from polar to rectangular co-ordinates: thus

$$\begin{aligned} \text{Altitude} &= \text{Distance} \times \text{sin. Elevation}, \\ \text{Diff. lat.} &= \text{Distance} \times \text{cos. Elevation} \times \text{cos. Bearing}, \\ \text{Departure} &= \text{Distance} \times \text{cos. Elevation} \times \text{sin. Bearing}. \end{aligned}$$

Then the altitude of the last point above the first is the sum of the several altitudes taken with their proper signs; the difference of latitude and departure, are the sums of the difference of latitudes and departures of the several distances; and their bearing, elevation, and distance are calculated from the converse formulas

$$\text{tan Bearing} = \frac{\text{Departure}}{\text{Diff. Lat.}},$$

$$\text{tan. Elevation} = \frac{\text{Altitude} \times \text{cos. Bearing}}{\text{Diff. Lat.}},$$

$$\text{Distance} = \frac{\text{Altitude}}{\text{sin.Elevation}}.$$

No.	Bearing.	Distance.	Elevation.	Altitude.	Diff. lat. N.	Departure W.
1	N. 10° 15′ E.	27,54	17° 54′	8,46	25,79	— 4,66
2	N. 28° 40′ W.	100,00	20° 19′	34,72	82,28	44,99
3	N. 20° 00′ W.	15,00	7° 43′	2,01	13,97	5,08
4	N. 20° 00′ W.	37,26	−5° 26′	— 3,53	34,70	12,63
5	N. 36° 17′ E.	68,75	—11° 13′	—13,37	54,36	—39,91
Total	N. 4° 54½′ W.	213,74	7° 36½′	28,29	211,10	18,13

(83). QUESTION II. *By Mr. J. F. Macully, Esq., N. Y.*

It is required to draw a chord through the focus of a given ellipse, which shall divide the area in a given ratio.

FIRST SOLUTION. *By the Proposer.*

The equation of the ellipse, the focus being the pole, is

$$r = \frac{\mathrm{B}^2}{\mathrm{A} - c\cos\varphi}:$$

and since the area, cut off by a focal chord making the angle θ with the angular axis, is to be in a given ratio with the whole area,

$$n\mathrm{AB}\pi = \tfrac{1}{2}\int_\theta^{\pi+\theta} r^2 d\varphi = \tfrac{1}{2}\mathrm{B}^4\int_\theta^{\pi+\theta}\frac{d\varphi}{(\mathrm{A}-c\cos\varphi)^2}$$

$$= \mathrm{AB}\cot^{-1}\left(\frac{\mathrm{C}}{\mathrm{B}}\sin\theta\right) - \frac{\mathrm{AB}^2 c\sin\theta}{\mathrm{A}^2 - c^2\cos^2\theta}.$$

Let, then, ψ be an angle such that $\cot\frac{1}{2}\psi = \frac{\mathrm{C}}{\mathrm{B}}\sin\theta$, then

$$\psi - \sin\psi = 2n\pi.$$

If ψ_1 be an approximate root of this equation, the correction to be applied in order to obtain a nearer value, will be

$$\delta\psi = \frac{2n\pi - (\psi_1 - \sin\psi_1}{2\sin^2\frac{1}{2}\psi_1)}.$$

Having found ψ from this equation, the angle θ is determined by

$$\sin\theta = \frac{\mathrm{B}}{\mathrm{C}}\cot\tfrac{1}{2}\psi.$$

SECOND SOLUTION. *By Mr. B. Birdsall.*

The equation of the ellipse, referred to the transverse axis and a diameter making the angle a with it as axes of co-ordinates, is

$$(\mathrm{A}^2\sin^2 a + \mathrm{B}^2\cos^2 a)y^2 + 2\mathrm{B}^2\cos a\, xy + \mathrm{B}^2x^2 = \mathrm{A}^2\mathrm{B}^2,$$

and if we put $\mathrm{A}^2\sin^2 a + \mathrm{B}^2\cos^2 a = k^2\sin^2 a$, we shall have

$$y\sin a = -\frac{\mathrm{B}^2}{k^2}\cot ax \pm \frac{\mathrm{AB}}{k^2}\sqrt{k^2 - x_2};$$

that is, if y' and y'' are the segments into which a chord parallel to the axis of y is divided by the axis of x,

$$y' \sin a = -\frac{\text{B}^2}{k^2} \cot a \,.\, x + \frac{\text{AB}}{k^2}\sqrt{k^2 - x^2},$$

$$y'' \sin a = +\frac{\text{B}^2}{k^2} \cot a \,.\, x + \frac{\text{AB}}{k^2}\sqrt{k^2 - x^2};$$

and if m represent the given ratio, the area cut off by a chord through the focus, parallel to the axis of y, is

$$m\text{AB}\pi = \int_{\text{A}e}^{\text{A}} (y'dx \sin a + y''dx \sin a)$$

$$= \int_{\text{A}e}^{\text{A}} \frac{2\text{AB}}{k^2} \,.\, dx\sqrt{k^2 - x^2},$$

$$= \text{AB} \tan^{-1} \frac{\text{A}\sqrt{k^2 - \text{A}^2 e^2} - \text{A}e\sqrt{k^2 - \text{A}^2}}{\text{A}^2 e + \sqrt{(k^2 - \text{A}^2)(k^2 - \text{A}^2 e^2)}} + \frac{\text{A}^2 \text{B}}{k^2}\sqrt{k^2 - \text{A}^2}$$

$$+ \frac{\text{A}^2 \text{B}}{k^2}\sqrt{k^2 - \text{A}^2} - \frac{\text{A}^2 \text{B}e}{k^2}\sqrt{k^2 - \text{A}^2 e^2},$$

and restoring the value of $k^2 = \text{A}^2 + \text{B}^2 \cot^2 a$,

$$m\pi = \tan^{-1} \frac{\text{AB}(1 - e \cos a)}{\text{A}^2 e \sin a + \text{B}^2 \cot a} + \frac{\text{AB}(\cot a - e \operatorname{cosec} a)}{\text{A}^2 + \text{B}^2 \cot^2 a},$$

from which the angle a may be found.

THIRD SOLUTION. *By Mr. O. Root, Syracuse Academy.*

Let a, b be the semiaxes of the ellipse, c the distance from the centre to the focus. If the circle, radius a, whose projection is the ellipse, be divided by a chord into two parts having the given ratio, the projected areas will have the same ratio; this will be the case when

$$\psi - \tfrac{1}{2} \sin 2\psi : \pi - \psi + \tfrac{1}{2} \sin 2\psi = m : n$$

or
$$2\psi - \sin 2\psi = \frac{2m\pi}{m+n},$$

ψ being the angle included by the greater segment, and $m : n$ the given ratio. Having found ψ by this equation, let θ' and θ be the angles the dividing line and its projection make with the axis, we have

$$c \sin \theta' = a \cos \psi, \text{ or } \sin \theta' = \frac{a}{c} \cos \psi,$$

and
$$\tan \theta = \frac{b}{a} \tan \theta'.$$

The question is impossible if $\cos \psi > \frac{c}{a}$, or $\sin \psi < \frac{b}{a}$,

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(84). QUESTION III. *By Investigator.*

Find the *polar* equation of a straight line on a plane; and bring it to the form best adapted to general use. Apply it to finding the equation of a tangent to the ellipse at any point, the pole being at the focus and the angular axis the line of the foci.

SOLUTION. *By Prof. C. Avery, Hamilton College.*

Let p be the perpendicular from the origin on the line, α the angle made by p with the angular axis, r the radius vector, ω the angle r makes with the axis; then

$$r\cos(\omega-\alpha)=p, \quad \text{or } r=p\sec(\omega-\alpha) \quad . \quad . \quad . \quad . \quad (1).$$

is the polar equation of the straight line. If b be the intercept of the angular axis, between the origin and the line, and β the angle the line makes with the axis, we shall have

$$\alpha+\beta=\tfrac{1}{2}\pi, \quad p=b\sin\beta=b\cos\alpha \quad . \quad . \quad . \quad . \quad . \quad (2),$$

which may be used instead of p and α, if more covenient.

If $r'\omega'$, $r''\omega''$ be two points in any curve, we shall have for the secant passing through them

$$r'=p\sec(\omega'-\alpha), \quad r''=p\sec(\omega''-\alpha)$$

and $$\frac{r'-r''}{r'}=\frac{\sec(\omega'-\alpha)-\sec(\omega''-\alpha)}{\sec(\omega'-\alpha)}=\frac{\cos(\omega''-\alpha)-\cos(\omega'-\alpha)}{\cos(\omega''-\alpha)}$$

$$=\frac{2\sin\tfrac{1}{2}(\omega'-\omega'')\sin\tfrac{1}{2}(\omega'+\omega''-2\alpha)}{\cos(\omega''-\alpha)},$$

and, ultimately, when this secant becomes a tangent,

$$\frac{dr'}{r'}=d\omega'\tan(\omega'-\alpha),$$

$$\text{or,} \quad \alpha=\omega'-\tan^{-1}\frac{dr'}{r'd\omega'}, \text{ and } p=r'\cos(\omega'-\alpha) \quad . \quad . \quad . \quad (3);$$

the equation of the tangent is then

$$r\cos(\omega-\alpha)=r'\cos(\omega'-\alpha).$$

But $$\omega-\alpha=(\omega-\omega')+(\omega'-\alpha),$$

and $$\cos(\omega-\alpha)=\cos(\omega-\omega')\cos(\omega'-\alpha)-\sin(\omega-\omega')\sin(\omega'-\alpha),$$

$$\therefore\; r\{\cos(\omega-\omega')-\sin(\omega-\omega')\tan(\omega'-\alpha)\}=r',$$

$$\text{or} \quad r\left\{\cos(\omega-\omega')-\sin(\omega-\omega')\frac{dr'}{r'd\omega'}\right\}=r' \quad . \quad . \quad . \quad . \quad . \quad (4)$$

is the equation of the tangent to a given curve at the point $r'\omega'$.

For the ellipse, $$\frac{\text{A}(1-e^2)}{r'}=1+e\cos\omega',$$

and $$\frac{dr'}{r'd\omega'}=\frac{e\sin\omega'}{1+e\cos\omega}.$$

Hence the equation of a tangent to the ellipse is

$$r\{\cos(\omega-\omega')+e\cos\omega\}=\text{A}(1-e^2) \quad . \quad . \quad . \quad . \quad (5).$$

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(85). QUESTION IV. *By Mr. P. Barton, jun.*

The co-ordinates of the vertex of a cone of revolution are

$$x=-4, \quad y=3, \quad z=-2;$$

the equations of its axis are

$$x=\tfrac{1}{2}z-3, \quad y=-\tfrac{1}{2}z+2\tfrac{1}{2};$$

and its vertical angle is 90°. It is required to find where its surface is intersected by the line whose equations are

$$x=z+6, \quad y=-z-5.$$

FIRST SOLUTION. *By Prof. M. Catlin, Hamilton College.*

Let the vertex of the cone be xyz, a point in the given line at the distance D from the vertex, be $x''y''z''$, and a point in the axis at the distance D′ from the first, be $x'y'z'$; then, from the equations of the line,

$$\begin{aligned} D^2 &= (x''-x)^2+(y''-y)^2+(z''-z)^2 \\ &= (z''+10)^2+(z''+8)^2+(z''+8)^2 \\ &= 3z''^2+40z''+162 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1). \end{aligned}$$

Also,

$$\begin{aligned} D'^2 &= (x'-x'')^2+(y'-y'')^2+(z'-z'')^2 \\ &= (z''-\tfrac{1}{2}z'+9)^2+(z''-\tfrac{1}{3}z'+7\tfrac{1}{3})^2+(z'-z'')^2 \\ &= \tfrac{49}{36}z'^2-\tfrac{11}{3}z'z''+3z''^2-\tfrac{125}{9}z'+\tfrac{98}{3}z''+134\tfrac{7}{9} \quad (2). \end{aligned}$$

If D′ be perpendicular to the axis, we shall have

$$\frac{dD'}{dz'}=\tfrac{49}{18}z'-\tfrac{11}{3}z''-\tfrac{125}{9}=0, \text{ or } z'=\frac{66z''+250}{49} \quad (3),$$

and

$$D'^2=\frac{26z''^2+684z''+4868}{49} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Now in order that the point $(x''y''z'')$ may be in the surface of the cone whose vertical angle is 90°, we must have $D^2=2D'^2$, or

$$49(3z''^2+40z''+168)=2(26z''^2+684z''+4868),$$
$$95z''^2+592z''=1504;$$

therefore

$$\begin{aligned} z'' &= -8{,}16947, &\text{ or } z'' &= 1{,}93789, \\ x'' &= -2{,}16947, &\text{ or } x'' &= 7{,}93789, \\ y'' &= 3{,}16947, &\text{ or } y'' &= -6{,}93789, \end{aligned}$$

are the co-ordinates of the two points of intersection.

SECOND SOLUTION. *By Mr. Geo. R. Perkins, Clinton Liberal Institute.*

The equation of any line passing through the vertex of the cone is

$$x+4=a(z+2),\ y-3=b(z+2);$$

and if this be a linear element of the cone whose vertical angle $= 90°$,

$$\cos.45°=\sqrt{\tfrac{1}{2}}=\frac{1+aa'+bb'}{\sqrt{1+a^2+b^2}.\sqrt{1+a'^2+b'^2}},$$

or

$$(1+a^2+b^2)(1+a'^2+b'^2)=2(1+aa'+bb')^2;$$

and substituting the values

$$a=\frac{x+4}{z+2},\ b=\frac{y-3}{z+2},\ a'=\tfrac{1}{2},\ b'=-\tfrac{1}{3},$$

we get the equation of the surface of the cone

$$23z^2-41y^2-31x^2-48yz+72xz-24xy+524z+54y-32x+379=0.$$

Combining this with the equations of the given straight line, we have, for the points of intersection,

$$\begin{aligned} x &= 7{,}9378, &\text{ or } x &= -2{,}1694, \\ y &= -6{,}9378, &\text{ or } y &= 3{,}1694, \\ z &= 1{,}9378, &\text{ or } z &= 8{,}1694. \end{aligned}$$

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(86). QUESTION X. *By* ψ.

The circumference of a circle is divided into n equal parts, and from the points of division perpendiculars are drawn upon a given diameter of the circle. If lines be drawn from any given point in the plane of the circle to the points where these perpendiculars intersect the diameter, it is required to find the sum of the squares of these lines.

SOLUTION. *By Mr. O. Root.*

Let r = radius of the circle, a = distance of the given point from the centre, φ = angle between a and the fixed diameter, $r\theta$ the arc between the first point and the extremity of the fixed diameter, and $r\theta + r \cdot \frac{2m\pi}{n}$ will be the arc between any other one and that extremity; then a general expression for the square of any of the required lines is

$$a^2 - 2ar\cos\varphi\cos\left(\theta+\frac{2m\pi}{n}\right)+r^2\cos^2\left(\theta+\frac{2m\pi}{n}\right)$$

$$= a^2+\tfrac{1}{2}r^2 - 2ar\cos\varphi\cos\left(\theta+\frac{2m\pi}{n}\right)+\tfrac{1}{2}r^2\cos\left(2\theta+\frac{4m\pi}{n}\right),$$

m having all the integral values from 0 to $n-1$, or from 1 to n inclusive; hence the sum required is

$$n(a+\tfrac{1}{2}r^2) - 2ar\cos\varphi\left\{\cos\theta+\cos\left(\theta+\frac{2\pi}{n}\right)+\ldots\cos\left(\theta+\frac{2(n-1)\pi}{n}\right)\right\}$$
$$+\tfrac{1}{2}r^2\left\{\cos 2\theta+\cos\left(2\theta+\frac{4\pi}{n}\right)+\ldots\cos\left(2\theta+\frac{4(n-1)\pi}{n}\right)\right\}.$$

But $\cos\theta+\cos\left(\theta+\frac{2\pi}{n}\right)+\cos\left(\theta+\frac{4\pi}{n}\right)+\ldots\cos\left(\theta+\frac{2(n-1)\pi}{n}\right)=0,$

and $\cos 2\theta+\cos\left(2\theta+\frac{4\pi}{n}\right)+\cos\left(2\theta+\frac{8\pi}{n}\right)+\ldots\cos\left(2\theta+\frac{4(n-1)\pi}{n}\right)=0:$

hence the sum is $\quad n(a^2+\tfrac{1}{2}r^2)$;

except when $n=2$, and then,

$$\cos 2\theta+\cos\left(2\theta+\frac{4\pi}{n}\right)+\ldots\cos\left(2\theta+\frac{4(n-1)\pi}{n}\right)=2\cos 2\theta,$$

and the sum is $\quad 2(a^2+r^2\cos^2\theta)$.

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(87). QUESTION VI. *By* ——.

Def. In the ellipse or hyperbola, the parameter of any diameter is that chord of the system it bisects which is a third proportional to that diameter and its conjugate.

It is required

1°. To find what diameters may properly be said to have parameters.

2°. To find the locus of the middle points of all the parameters of the same curve.

3°. Having given a parameter, to find if possible, another one perpendicular to it.

SOLUTION. *By Mr. Geo. R. Perkins.*

Let a and b be the semi-axes of the ellipse or hyperbola, a' and b' any two semi-conjugate diameters, making the angles w and w' with a, r the the distance of the parameter from the centre, measured on a' and $2p$ the parameter. Then

1°. $\qquad a'^2p^2 \pm b'^2r^2 = \pm a'^2b'^2,$

or, since $\qquad a'^2p^2=b'^4,\ r^2=a'^2 \mp b'^2 \quad . \quad . \quad . \quad . \quad (1),$

so that, in the ellipse, when the upper signs are used, the diameter has

a parameter, only when $a'>b'$, or when it is included between the equal conjugates. In the hyperbola it is limited by the circumstance that p is only real when a' is real, and therefore those diameters only, included between the assympotes have parameters.

2°. $$a'^2 = \frac{\pm a^2 b^2}{a^2 \sin^2 w \pm b^2 \cos^2 w},\ b'^2 = \frac{a^2 b^2}{a^2 \sin^2 w' \pm b^2 \cos^2 w'};$$

but $$\tan w \tan w' = \mp \frac{b^2}{a^2};$$

therefore $$\sin^2 w' = \frac{b^4 \cos^2 w}{a^4 \sin^2 w + b^4 \cos^2 w},\ \cos^2 w' = \frac{a^4 \sin^2 w}{a^4 \sin^2 w + b^4 \cos^2 w},$$

and $$b'^2 = \pm \frac{a^4 \sin^2 w + b^4 \cos^2 w}{a^2 \sin^2 w \pm b^2 \cos^2 w};$$

$$\therefore \quad r^2 = (a^2 \mp b^2) . \frac{-a^2 \sin^2 w \pm b^2 \cos^2 w}{a^2 \sin^2 w \pm b^2 \cos^2 w} \quad . \quad . \quad . \quad . \quad . \quad (2),$$

which is the polar equation of the locus. Its rectangular equation is
$$(x^2+y^2)(a^2y^2 \pm b^2x^2) = (a^2 \mp b^2)(-a^2y^2 \pm b^2x^2) \ . \ . \ . \ . \ . \ (3).$$
The curve passes through the foci; for the ellipse it is of the form of the *Lemniscata*, the centre being the multiple point; for the hyperbola, the assymptotes are also those of the curve.

3°. If there are two parameters at right angles to each other, the diameters to which they are parallel are so. Now if β be the angle the equal conjugates of the ellipse, or the assymptotes of the hyperbola make with the transverse axis, so that $\tan \beta = \frac{b}{a}$, we have seen that the limits of w are $-\beta$ and β, and the limits of w' are β and $\pi-\beta$, then, if there are parameters perpendicular to each other, they must be parallel to diameters within the limits $w'=\beta$ and $w'=\frac{1}{2}\pi-\beta$, or $w'=\frac{1}{2}\pi+\beta$ and $w'=\pi-\beta$; and if the angle w' of the given parameter be within these limits, we shall have the angle of its perpendicular $=\frac{1}{2}\pi+w'$, and for corresponding diameter

$$\tan w = \mp \frac{b^2}{a^2} \cot (\tfrac{1}{2}\pi + w') = \pm \frac{b^2}{a^2} \tan w'$$

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(88). QUESTION VII. *By* ——.

The theorem of M. Sturm, published in the "Memoirs préséntes par des Savans Estrangers," for 1835, may be stated thus:

Let $\mathrm{x} = 0$, be any algebraical equation whose co-efficients are real, and whose roots are unequal, and let $\mathrm{x}_1 = \frac{d\mathrm{x}}{dx}$. Apply to the two polynomials x, x_1 the process for finding their greatest common measure, the several remainders having all their signs changed from + to —, and from — to +, before they are used as new divisors, and in that state let them be represented by $\mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4 \ldots\ldots \mathrm{x}_m$. In the series of polynomials

$$\mathrm{x}, \mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \ldots\ldots \mathrm{x}_m,$$

which are of continually decreasing dimensions in x, x_m being independent of x, let any two numbers p and q be successively substituted for x, noting the signs of the two series of results. *Then the difference between*

the number of variations of the first series of signs, and that of the second, expresses exactly the number of real roots of the given equation, which are comprised between the two numbers p and q.

It required to apply this theorem to the general equation

$$x^4 + ax^2 + bx + c = 0,$$

in order to determine the number and nature of its real roots.

SOLUTION. *By Prof. Peirce, Cambridge University.*

We have

$$\begin{aligned} \mathrm{X} &= x^4 + ax^2 + bx + c, \\ \mathrm{X}_1 &= 4x^3 + 2ax + b, \\ \mathrm{X}_2 &= -2ax^2 - 3bx - 4c, \\ \mathrm{X}_3 &= -(2a^3 + 9b^2 - 8ac)x - (a^2 b + 12bc) = -a'x - b', \\ \mathrm{X}_4 &= 2ab'^2 - 3a'bb' + 4a'^2 c = a''. \end{aligned}$$

By substituting for x $\qquad x = +\infty,$

the values become, when none are deficient,

$$\mathrm{X} = +\infty, \mathrm{X}_1 = +\infty. \mathrm{X}_2 = -a\infty, \mathrm{X}_3 = -a'\infty, \mathrm{X}_4 = a'';$$

substituting for x $\qquad x = -\infty,$

they become $\mathrm{X} = \infty, \mathrm{X}_1 = -\infty, \mathrm{X}_2 = -a\infty, \mathrm{X}_3 = a'\infty, \mathrm{X}_4 = a''$;

substituting for $\qquad x = 0,$

they become $\mathrm{X} = c$, $\mathrm{X}_1 = b$, $\mathrm{X}_2 = -4c$, $\mathrm{X}_3 = -b'$, $\mathrm{X}_4 = a''$.

In order that all the roots may be real, we must then have

$$a < 0, \quad a' < 0, \quad a'' > 0;$$

and if moreover, we have $\qquad c > 0,$

two of the roots are positive and two are negative.

But if $c < 0$ we must obviously have $bb' > 0$ in order that $a'' > 0$, and therefore, $\qquad (a^2 + 12c)b^2 > 0$, or $c > -\frac{1}{12}a^2$.

If, then, $b > 0$, one of the roots is positive and three are negative; and if $b < 0$, three roots are positive and one is negative.

In all other cases but the preceding, the equation has two real roots and no more when a'' is negative, and none when a'' is positive.

If c is also negative, one of the roots is positive and the other negative.

If c is positive the two real roots must both have the opposite sign to b.

Particular Cases.

1. When $b = 0$ the roots are all real, when $c > 0$ and $< \frac{1}{4}a^2$, two being positive and two negative; but if $c > \frac{1}{4}a^2$ they are all imaginary. If $c < 0$, two of the roots are real, the one being positive, the other negative.
2. When $c = -\frac{1}{12}a^2$, there are two real roots, one of which is positive, the other negative.
3. When $c = 0$, it is reduced to a cubic equation.
4. When $a'' = 0$, the equation has a pair of equal roots each of which is
$$x = -\frac{b'}{a'},$$
and the other roots are easily found.
5. When $a' = 0$, and a positive, the equation has no real roots, but if a is negative it has two real roots, which, when c is negative are, the

one positive, the other negative; but when c is positive, they both have the same sign as $-b$. If a is zero, the equation is reduced to $x^4+c=0$,

6. When $a=0$, there are no real roots if b and $256c^3 - 27b^4$ are both positive or both negative. But if b is positive and $256c^3-27b^4$ negative, one of the two real roots is positive and the other negative; and if b is negative while $256c^3-27b^4$ is positive, both the real roots are positive. If $256c^3-27b^4=0$, the equation has two real roots each equal to $-\dfrac{4c}{3b}$.

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(89). QUESTION VIII. *By Prof. B. Peirce, Harvard University.*

Prove that if all the roots of the equation

$$x^n - \text{A}x^{n-2} + \text{B}x^{n-3} - \&c. = 0,$$

are real that we shall have

$$n(n-1)(3\text{B})^2 < (n-2)^2(2\text{A})^3.$$

FIRST SOLUTION. *By the Proposer.*

Put

$$x = x' + a$$

in the given equation, and it becomes

$$x'^n + nax'^{n-1} + \left[\frac{n(n-1)}{2}a^2 - \text{A}\right]x'^{n-2} + \left[\frac{n(n-1)(n-2)}{2.3}a^3 - (n-2)\text{A}a + \text{B}\right] x'^{n-3} + \&c. = 0;$$

and if we take for a such a value as to satisfy the equation

$$\frac{n(n-1)}{2}a^2 - \text{A} = 0,$$

we reduce it to

$$x'^n + nax'^{n-1} + [\text{B} - \tfrac{2}{3}.(n-2)\text{A}a]x'^{n-3} + \&c. = 0,$$

in which, since a term is wanting, the co-efficients preceding and following it must not have the same sign, or all the roots would not be real as they are in the given equation, and as they must be in this equation also, since A must be positive, and therefore a is real. The quotient of these co-efficients must then be negative, or

$$\frac{\text{B}}{a} < \tfrac{2}{3}(n-2)\text{A},$$

$$\text{or} \qquad 3\text{B} < 2(n-2)\text{A}a.$$

$$\text{Squaring} \quad (3\text{B})^2 < (n-2)^2 a^2 (2\text{A})^2,$$

$$\text{and by substitution} \quad (3\text{B})^2 < \frac{(n-2)^2(2\text{A})^3}{n(n-1)},$$

$$\text{or} \qquad n(n-1)(3\text{B})^2 < (n-2)^2(2\text{A})^3;$$

which, for the cubic equation, is

$$3\,.\,2\,(3\text{B})^2 < (2\text{A})^3, \text{ or } (\tfrac{1}{2}\text{B})^2 < (\tfrac{1}{3}\text{A})^3;$$

for the biquadratic,

$$4\,.\,3\,(3\text{B})^2 < 4(2\text{A})^3, \text{ or } \text{B}^2 < (\tfrac{2}{3}\text{A})^3;$$

for the 5th degree,

$$5\,.\,4\,(3\text{B})^2 < 3^2(2\text{A})^3, \text{ or } 5\text{B}^2 < 2\text{A}^3,$$

and for n very large $(3\text{B})^2 < (2\text{A})^3$

SECOND SOLUTION. *By Prof. M. Catlin, Hamilton College.*

Let a, b, c, e, &c., be the roots of the given equation; then, since the second term is wanting, we shall have the following relations:

$$a + b + c + e + \&c. = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$
$$a^2 + b^2 + c^2 + e^2 + \&c. = 2\text{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$
$$a^3 + b^3 + c^3 + e^3 + \&c. = 3\text{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

(See Hutton's Math. Vol. 2. p. 262, Cor. 5.) Therefore the given inequality becomes

$$(n-2)^2(a^2+b^2+c^2+e^2+\&c.)^3 > n(n-1)(a^3+b^3+c^3+e^3+\&c.)^2 \quad (4),$$

or

$$(n-2)^2(a^2+b^2+c^2+e^2+\&c.)^3 - n(n-1)(a^3+b^3+c^3+e^3+\&c.)^2 = +\text{P} \; (5)$$

P being essentially positive. To prove (5) we will find the minimum value of P. Differentiate (5), eliminate da by means of (1), and equate the co-efficients of db, dc, &c., separately with zero, and we easily get

$$\left.\begin{array}{l} (n-2)^2(2\text{A})^2(a-b) - n(n-1)(3\text{B})(a^2-b^2) = 0 \\ (n-2)^2(2\text{A})^2(a-c) - n(n-1)(3\text{B})(a^2-c^2) = 0 \\ (n-2)^2(2\text{A})^2(a-e) - n(n-1)(3\text{B})(a^2-e^2) = 0 \\ \quad\&c. \qquad\qquad\qquad\qquad \&c. \end{array}\right\} \quad . \quad . \quad (6).$$

Hence, b, c, e, &c., are obviously equal to each other, and therefore any one of them must by (1)

$$= -\frac{a}{n-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

By virtue of (7) equation (5) becomes

$$\text{P} = (n-2)^2\left(a^2 + \frac{a^2}{n-1}\right)^3 - n(n-1)\left(a^3 - \frac{a^3}{(n-1)^2}\right)^2 = 0 \; . \; (8).$$

Therefore (5) becomes

$$(n-2)^2(2\text{A})^3 = n(n-1)(3\text{B})^2 \quad . \quad . \quad . \quad . \quad (9),$$

when $n-1$ of the roots of the given equation are equal to each other. In all other cases we shall have $\text{P} > 0$, and consequently

$$(n-2)^2(2\text{A})^3 > n(n-1)(3\text{B})^2 . \quad . \quad . \quad . \quad . \quad (10).$$

Cor. Whenever we have the equation $(n-2)^2(2\text{A})^3 = n(n-1)(3\text{B})^2$, we may infer that $n-1$ of the roots are equal to each other.

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(90). QUESTION IX. *By Prof. F. N. Benedict, University of Vt.*

To determine the locus of the intersection of two tangents or normals to the common parabola which include an angle whose tangent varies as a given function of the co-ordinates of the point of intersection.

SOLUTION. *By Prof. Avery.*

I. Let the equation of the parabola be $y^2 = 2mx$; the equations of the tangents at the points $y'x'$, $y''x''$, are

$$yy' = m(x+x') = mx + \tfrac{1}{2}y'^2, \; yy'' = m(x+x'') = mx + \tfrac{1}{2}y''^2 \quad . \quad (1),$$

hence, at their point of intersection, xy,

$$y(y'-y'') = \tfrac{1}{2}(y'^2 - y''^2), \text{ or } y' + y'' = 2y \quad . \quad . \quad (2);$$

and $\quad y(y'+y'') = 2mx + \tfrac{1}{2}(y'+y'')^2 - y'y'', \text{ or } y'y'' = 2mx \; (3).$

Since $\frac{m}{y'}$ and $\frac{m}{y''}$ are the tangents of their inclinations with the axis of x, the tangent of their mutual inclination is

$$\frac{\frac{m}{y''}-\frac{m}{y'}}{1+\frac{m^2}{y'y''}}=\frac{m(y'-y'')}{y'y''+m^2}=\frac{2\sqrt{y^2-2mx}}{2x+m}=f(x,y),$$

and $$2\sqrt{y^2-2mx}=(2x+m)f(x,y) \quad . \quad . \quad . \quad . \quad (4),$$
is the equation of the locus required.

II. The equations of the normals are

$$y-y'=-\frac{y'}{m}(x-x') \text{ or } m^2y=m(m-x)y'+\tfrac{1}{2}y'^3 \quad . \quad . \quad (5),$$

and $$y-y''=-\frac{y''}{m}(x-x''), \text{ or } m^2y=m(m-x)y''+\tfrac{1}{2}y''^3 \quad . \quad . \quad (6);$$

and the tangent of their angle of intersection is

$$\frac{m(y''-y')}{y'y''+m^2}=f'(x,y) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

And if y', y'' be eliminated among equations (5), (6), (7), the result will be the equation of the curve.

Equations (5) and (6) show that, when

$$27y^2 < 2m(m-x)^3,$$

there are three real values of y', and three of y'', or that from any point within the evolute of the parabola, three normals can be drawn to the parabola.

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(91). QUESTION X. *By Wm. Lenhart, Esq., York. Penn.*

Having given a series of whole numbers whose third order of differences are constant, and of which a given term is divisible by a given prime number m; it is required to find that term in the series which is divisible by m^n, n being a given whole number.

FIRST SOLUTION. *By the Proposer.*

Let $a_0, a_1, a_2 \ldots a_{n'}$ represent the given series; $A+Bn'+Cn'^2+Dn'^3$, a general expression for any term of the series, A, B, C and D being constants to be determined from the first four terms of the series, and $a_{n'}$ a term divisible by m. Then

$$a_{n'}=A+Bn'+Cn'^2+Dn'^3=A_1m \quad . \quad . \quad . \quad . \quad . \quad (1).$$

In (1) write $n'+n''m$ for n', and

$$a_{n'+n''m}=A+(n'+n'm)B+(n'+n''m)^2C+(n'+n''m)^3D;$$

from which

$$a_{n'+n''m}=\left\{\begin{array}{l} A+n'B+n''mB \\ \quad n'^2C+2n'n''mC+n''^2m^2C \\ \quad n'^3D+3n'^2n''mD+3n'n''^2m^2D+{}''^3m^3D \end{array}\right\}$$

$$=\begin{cases} A_1 m + n''m(B + n'(2C + 3n'D)) \\ \quad + n''^2 m^2 (C + 3n'D) \\ \quad + n''^3 m^3 D; \end{cases}$$

Or putting $B + n'(2C + 3n'D) = s$, and $C + 3n'D = s'$, and dividing by m

$$\frac{1}{m} \cdot a_{n'+n''m} = A_1 + n''s + n''^2 ms' + n''^3 m^2 D = A_2 m \quad . \quad . \quad (2).$$

Now find a value of n'' as directed at the end of the solution, that (2) may divide by m, and make the division; then

$$\frac{1}{m^2} \cdot a_{n'+n''m} = A_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

We may here remark that since A_1, in (2), is prime to m, s must be so also; for otherwise, (2) could not be made divisible by m, and in that case the problem would be impossible.

Again, for n'', in (2), write $n'' + n'''m$, and it will become

$$\frac{1}{m} \cdot a_{n'+n''m+n'''m^2} = A_1 + (n'' + n'''m)s + m(n'' + n'''m)^2 s' + m^2(n'' + n'''m)^3 D;$$

from which, by a development as above, we shall have

$$\frac{1}{m} \cdot a_{n'+n''m+n'''m^2} = \begin{cases} A_1 + n''s + n'''ms \\ \quad n''^2 ms' + 2n''n'''m^2 s' + n'''^2 m^3 s' \\ \quad n''^3 m^2 D + 3n''^2 n''' m^3 D + 3n''n'''^2 m^4 D + n'''^3 m^5 D \end{cases}$$

$$= \begin{cases} A_2 m + n'''m(s + n''m(2s' + 3n''mD)) \\ \quad + n'''^2 m^3 (s' + 3n''mD) \\ \quad + n'''^3 m^5 D; \end{cases}$$

Or, restoring the values of s and s', and dividing by m,

$$\frac{1}{m^2} \cdot a_{n'+n''m+n'''m^2} = \begin{Bmatrix} A_2 + n'''[B + n'(2C + 3n'D) + n''m(2C + 6n'D + 3n''mD)] \\ + m^2 n'''^2 (C + 3n'D + 3n''mD) \\ + n'''^3 m^4 D \end{Bmatrix}$$

$$= A_3 m \quad . \quad . \quad . \quad (4).$$

Or, finding n''', as directed, and dividing by m,

$$\frac{1}{m^3} \cdot a_{n'+n''m+n'''m^2} = A_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Again, by writing $n''' + n''''m$ for n''', in (4), and developing as before, we shall find

$$\frac{1}{m^3} \cdot a_{n'+n''m+n'''m^2+n''''m^3} = \begin{Bmatrix} A_3 + n''''[B + n'(2C + 3n'D) + n''m(2C + 6n'D + 3n''mD) \\ + n'''m^2(2C + 6n'D + 6n''mD + 3n'''m^2D)] \\ + n''''^2 m^3 (C + 3n'D + 3n''mD + 3n'''m^2D) + n''''^3 m^6 D \end{Bmatrix}$$

$$= A_4 m \quad . \quad . \quad (6).$$

Or, finding n'''', and dividing by m,

$$\frac{1}{m^4} \cdot a_{n'+n''m+n'''m^2+n''''m^3} = A_4 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

We shall now have generally,

$$\frac{1}{m^{n-1}} \cdot a_{n'+n''m \ldots n^{(n)}m^{n-1}} = A_{n-1} + n^{(n)}(B + 2Cp + 3Dp^2)$$
$$+ n^{(n)2} m^{n-1}(C + 3Dp) + n^{(n)3} m^{2n-2} D = A_n m \quad . \qquad (8);$$

wherein $\qquad p = (n' + n''m \ldots n^{(n-1)} m^{n-2})$.

And to find a value of $n^{(n)}$ that shall render (8) divisible by m, we prefer, among several, the following simple method, namely:—Let the remainder of $A_{n-1} \div m$ be denoted by R, and the remainder of s or $(B + n'(2C + 3n'D)) \div m$ by R' which is constant, as the nature of the process or inspection alone plainly indicates. Then from

$$R + n^{(n)} R' = mT \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

we have $n^{(n)}$, and thence dividing (8) by m, we obtain finally

$$\frac{1}{m^n} \cdot a_{n' + n''m \ldots n^{(n)} m^{n-1}} = A_n,$$

$$\text{or,} \qquad a_{n' + n''m \ldots n^{(n)} m^{n-1}} = A_n m^n = \text{term required} \quad . \quad . \quad (10).$$

We may now readily perceive the general law or system which pervades the whole subject, and consequently be enabled to resolve, not only each particular case of each example, but also the general problem itself, namely, when the n^{th} differences are constant.

Example. Given 13, 14, 17, 23, &c.

Appl. Let $a_0 = A = 13$; $a_1 = A + B + C + D = 14$; $a_2 = A + 2B + 4C + 8D = 17$; $a_3 = A + 3B + 9C + 27D = 23$; then $A = 13$, $B = \frac{1}{3}$, $C = \frac{1}{2}$, and $D = \frac{1}{6}$. Assume $m = 7$, then $n' = 1$, $A_1 = 2$, and $s = 1\frac{4}{6}$.

Remainder of $A_1 \div m = R = 2$, and remainder of $s \div m = R' = \frac{5}{3}$ constant. Then from

(9), $n'' = 4$, and (2) becomes 644: therefore $A_2 = 92$. Rem. $A_2 \div m = R = 1$,
(9), $n''' = 2$, " (4) " 7133: " $A_3 = 1019$. " $A_3 \div m = R = 4$,
(9), $n'''' = 1$, " (6) " 50771: " $A_4 = 7253$. " $A_4 \div m = R = 1$,
(9), $n''''' = 2$, " (8) &c., &c.

Now, if we assume $n = 3$, we shall have from (10)

$$a_{127} = a_3 m^3 = 1019 \times 343 = 349517 = \text{term required.}$$

SECOND SOLUTION. *By the same gentleman.*

Let the given series be denoted by $a_0, a_1, a_2 \ldots . a_n$; a general expression for any term in the series by $A + Bn' + Cn'^2 + Dn'^3$; in which A, B, C, and D are constants to be determined by the first four terms of the series, and let $a_{n'}$ be a term divisible by m. Then

$$a_{n'} = A + n'(B + n'(C + n'D)) = A_1 m \quad . \quad . \quad . \quad . \quad (1).$$

In (1), write $n' + n''m$ for n', and

$$\begin{aligned} a_{n'+n''m} &= A + (n' + n''m)(B + (n' + n''m)(C + n'D + n''mD)) \\ &= A + (n' + n''m)(B + n'(C + n'D) + n''m(C + 2n'D + n''mD)) \\ &= A_1 m + n''m\left(\frac{A_1 m - A}{n'} + (n' + n''m)(C + 2n'D + n''mD)\right), \end{aligned}$$

by comparing terms with their values deduced from (1). Or, reducing further, putting $\dfrac{A_1 m - A}{n'} + n'(C + 2n'D) = s$, and dividing by m,

$$\frac{1}{m} \cdot a_{n'+n''m} = A_1 + n'(s + n''m(C + 3n'D + n''mD)) = A_2 m \quad . \quad . \quad (2).$$

Find n'', as directed at the end of the solution, to make (2) divide by m, and effect the division: then

$$\frac{1}{m^2} \cdot a_{n'+n''m} = A_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

We may here remark that since A_1, in (2), is prime to m, s must be so too, else (2) could not be made to divide by m, and in that case the problem would be impossible.

For n'' write $n'' + n'''m$, in (2), and it will become

$$\frac{1}{m} \cdot a_{n'+n''m+n'''m^2} = A_1 + (n'' + n'''m)(s + m(n'' + n'''m)(C + 3n'D + n''mD + n'''m^2D)).$$

From which, reducing as above, and restoring the values of s and s', we shall have

$$\frac{1}{m^2} \cdot a_{n'+n''m+n'''m^2} = A_2 + n'''\left(\frac{A_2m - A_1}{n''} + m(n'' + n'''m)(C + 3n'D + 2n''mD + n'''m^2D)\right) = A_3m \quad . \quad . \quad (4).$$

Or, finding n''', as directed, and dividing by m,

$$\frac{1}{m^3} \cdot a_{n'+n''m+n'''m^2} = A_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

In the same way, writing $n''' + n''''m$ for n''', in (4), and reducing, &c., we shall find

$$\frac{1}{m^3} \cdot a_{n'+n''m+n'''m^2+n''''m^3} = A_3 + n''''\left(\frac{A_3m - A_2}{n'''} + m^2(n''' + n''''m)(C + 3n'D + 3n''mD + 2n'''m^2D + n''''m^3D)\right) = A_4m \quad . \quad (6).$$

$$\frac{1}{m_4} \cdot a_{n'+n''m+n'''m^2+n''''m^3} = A_4 \quad . \quad . \quad . \quad . \quad (7).$$

And thence generally

$$\frac{1}{m^{n-1}} \cdot a_{n'+n''m \dots n^{(n)}m^{n-1}} = A_{n-1} + n^{(n)}\left(\frac{A_{n-1} - A_{n-2}}{n^{(n-1)}} + m^{n-2}(n^{(n-1)} + n^{(n)}m)(C + 3Dp + m^{n-2}(2n^{(n-1)} + n^{(n)}m)D)\right) = A_nm \quad . \quad (8);$$

wherein $\quad p = (n' + n''m \dots \dots n^{(n-2)}m^{n-3}$.

To find $n^{(n)}$ that (8) may divide by m, we shall use this simple method, namely:—Let the remainder of $A_{n-1} \div m$ be denoted by R, and the remainder of $s \div m$ by R' which is constant, as is plainly indicated by the process. Then from

$$R + n^{(n)}R' = mT \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

we have $n^{(n)}$, and thence, dividing (8) by m, obtain finally

$$\frac{1}{m^n} \cdot a_{n'+n''m \dots n^{(n)}m^{n-1}} = A_n;$$

$$\text{or,} \qquad a_{n'+n''m \dots n^{(n)}m^{n-1}} = A_nm^n = \text{term required } (10).$$

Note. A solution after the manner of that which we have given in Speculation No. 2, p. 331, Vol. I. Miscel., will be found to be exceedingly curious and interesting. If the series be represented by

$$\begin{array}{ccccccccc} a'_0 & & a'_1 & & a'_2 & & a'_{n'} & & \\ & b'_0 & & b'_1 & & b'_2 & & b'_{n'} & \\ & & a & a+b & a+2b \ldots & a+n'b & & & \\ & & & b & b & b & b & & \end{array}$$

the general formulas, by the process of that solution, will be found to be

$a_0^{(n)} =$ first term of the n^{th} series.

$$b_0^{(n)} = b^{(n-1)}_{n^{(n-1)}} + \tfrac{1}{6}m^{n-2}(m-1)(3p+m^{n-2}(m+1)b). \quad (\text{A}),$$

$$m^{n-1}(p + m^{n-1}b) \quad \ldots \ldots \quad (\text{B}),$$

and $$m^{2n-2}b \quad \ldots \ldots \quad (\text{C}).$$

Also $a_{n^{(n)}}^{(n)} = a_0^{(n)} + n^{(n)}(b_0^{(n)} + \tfrac{1}{6}(n^{(n)}-1)m^{n-1}(3p+(n^{(n)}+1)m^{n-1}b))$ (D), which will divide by m.

$$b_{n^{(n)}}^{(n)} = b_0^{(n)} + \tfrac{1}{2}n^{(n)}m^{n-1}(2p+(n^{(n)}+1)m^{n-1}b) \quad \ldots \quad (\text{E}),$$

$$m^{n-1}(p+(n^{(n)}+1)m^{n-1}b) \quad \ldots \ldots \quad (\text{F}),$$

and $$m^{2n-2}b \quad \ldots \ldots \quad (\text{G}).$$

Wherein $p = (a - b + b(n' + n''m \ldots . n^{(n-1)}m^{n-2}))$; the remainder of $b_0^{(n)} \div m$ constant, and $n^{(n)}$ found from (9).

A, B, and C, are respectively the terms of the first, second and third order of differences corresponding to $a_0^{(n)}$, and

E, F, and G, those corresponding to $a_{n^{(n)}}^{(n)}$.

THIRD SOLUTION. *By Dr. T. Strong, New-Brunswick, N. J.*

Let $\text{A} = m\text{A}'$ denote the term which is divisible by m, and let ΔA, $\Delta^2\text{A}$, $\Delta^3\text{A}$ be the first terms of the successive orders of differences, and mx the distance of another term from A, then the term is expressed by

$$m\text{A}' + mx\Delta\text{A} + \frac{mx(mx-1)}{2}\Delta^2\text{A} + \frac{mx(mx-1)(mx-2)}{2\,.\,3}.\Delta^3\text{A} = m^n p,$$

if this term be divisible by m^n, p being an integer, or putting

$$6\text{A}' = a,\ 6\Delta\text{A} - 3\Delta^2\text{A} + 2\Delta^3\text{A} = b,\ 3(\Delta^2\text{A} - \Delta^3\text{A}) = c,\ \Delta^3\text{A} = d,$$

then $$a + bx + mcx^2 + m^2dx^3 = 6pm^{n-} \quad \ldots \quad (1),$$

Put $$a + bx = my \quad \ldots \ldots \quad (2).$$

and by solving this equation as an indeterminate one, if v be the least value of x which satisfies the equation, $x = v + mx'$ will also satisfy it, and by substituting this value, and dividing by m, we get an equation of the form

$$a' + b'x' + mc'x'^2 + m^2x'^3 = qm^{n-2} \quad \ldots \ldots \quad (3);$$

q being an integer, and the exponent of m one less than in (1); putting then $a' + b'x' = my'$, and proceeding as for (2), we can successively lessen the exponent of m, until that exponent becomes zero.

This method is the same as Legendre's, who solves such equations in their most general form, in his *Theory of Numbers*.

—— The solutions of Messrs. Avery, Catlin, and Perkins were very complete, and we regret our inability to insert them. The theorem they deduce is comprised in the general expression in the third solution.

(92). QUESTION XI. *By J. F. Macully, Esq.*

Required the value of n terms of the continued product
$$(1+2\cos\theta)(1+2\cos 3\theta)(1+2\cos 9\theta)\ldots.$$

FIRST SOLUTION. *By Dr. Strong.*

Let $P=(1+2\cos\theta)(1+2\cos 3\theta)(1+2\cos 9\theta)\ldots(1+2\cos 3^{x-1}\theta)$,

then, $\quad P+\Delta P=P(1+2\cos 3^{x}\theta,$

and, $\quad \dfrac{\Delta P}{P}=2\cos 3^{x}\theta=3-4\sin^{2}3^{x}.\tfrac{1}{2}\theta-1$

$$=\frac{\sin 3^{x+1}.\frac{1}{2}\theta}{\sin 3^{x}.\frac{1}{2}\theta}-1=\frac{\Delta\sin 3^{x}.\frac{1}{2}\theta}{\sin 3^{x}.\frac{1}{2}\theta},$$

which is satisfied by putting
$$P=\frac{\sin 3^{x}.\frac{1}{2}\theta}{\sin\frac{1}{2}\theta},$$

the product to x terms.

SECOND SOLUTION. *By Mr. B. Birdsall.*

$$\text{Sin }\tfrac{3}{2}\theta-\sin\tfrac{1}{2}\theta=2\sin\tfrac{1}{2}(\tfrac{3}{2}\theta-\tfrac{1}{2}\theta)\cos\tfrac{1}{2}(\tfrac{3}{2}\theta+\tfrac{1}{2}\theta)=2\sin\tfrac{1}{2}\theta\cos\theta;$$

therefore $\quad 1+2\cos\theta=\dfrac{\sin\frac{3}{2}\theta}{\sin\frac{1}{2}\theta},$

similarly $\quad 1+2\cos 3\theta=\dfrac{\sin\frac{9}{2}\theta}{\sin\frac{3}{2}\theta},$

&c.

$$\therefore(1+2\cos\theta)(1+2\cos 3\theta)\ldots(1+2\cos 3^{n-1}\theta)$$
$$=\frac{\sin\frac{3}{2}\theta}{\sin\frac{1}{2}\theta}\times\frac{\sin\frac{9}{2}\theta}{\sin\frac{3}{2}\theta}\times\ldots\times\frac{\sin 3^{n}.\frac{1}{2}\theta}{\sin 3^{n-1}.\frac{1}{2}\theta}$$
$$=\frac{\sin 3^{n}.\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}.$$

THIRD SOLUTION. *By Mr. Geo. R. Perkins.*

If e be the base of the Naperian logarithms, and $x=\theta\sqrt{-1}$, we have
$$1+2\cos\theta=1+e^{x}+e^{-x},$$
$$1+2\cos 3\theta=1+e^{3x}+e^{-3x},\ \&c.$$
By actually multiplying two or three of the first factors, we shall perceive that the product is
$$1+e^{x}+e^{2x}+e^{3x}+\ldots+e^{\frac{1}{2}(3^{n}-1)x}$$
$$+e^{-x}+e^{-2x}+e^{-3x}+\ldots+e^{-\frac{1}{2}(3^{n}-1)x}$$
$$=1+\frac{e^{\frac{1}{2}(3^{n}+1)x}-e^{x}}{e^{x}-1}+\frac{e^{-\frac{1}{2}(3^{n}+1)x}+e^{-x}}{e^{-x}-1}$$
$$=\frac{e^{\frac{1}{2}(3^{n}-1)x}+e^{-\frac{1}{2}(3^{n}-1)x}-e^{\frac{1}{2}(3^{n}+1)x}-e^{-\frac{1}{2}(3^{n}+1)x}}{2-e^{x}-e^{-x}}$$
$$=\frac{\cos\frac{1}{2}(3^{n}-1)\theta-\cos\frac{1}{2}(3^{n}+1)\theta}{1-\cos\theta}=\frac{\sin\frac{1}{2}.3^{n}\theta}{\sin\frac{1}{2}\theta}.$$

—— Prof. Peirce, after paying a just compliment to the inventor of this beautiful question, proceeds thus:

"In general, $$P_n = \frac{\sin \frac{1}{2} . 3^n\theta}{\sin \frac{1}{2}\theta}.$$

Case I. When	$\theta = 4m\pi$, m being an integer,	$P_n = 3^n$.
II. When	$\theta = (2m+1) . 2\pi$,	$P_n = (-1)^n$.
III. When	$\theta = (2m+1)\pi$,	$P_n = 1$.
IV. When	$3^n\theta = 2m'\pi$, but θ does not $= 2m\pi$,	$P_n = 0$.
V. When	$\theta = \frac{(4m'+1)\pi}{3^n-1}$,	$P_n = \cot \frac{1}{2}\theta$.
VI. When	$\theta = \frac{4m\pi}{3^n-1}$,	$P_n = 1$.
VII. When	$\theta = \frac{(4m+2)\pi}{3^n-1}$,	$P_n = -1$.
VIII. When	$\theta = \frac{(4m+3)\pi}{3^n-1}$,	$P_n = -\cot \frac{1}{2}\theta$.
IX. When	$\theta = \frac{4m\pi}{3^n-2}$,	$P_n = 2 \cos \frac{1}{2}\theta$.
X. When	$\theta = \frac{(4m+2)\pi}{3^n-2}$,	$P_n = -2 \cos \frac{1}{2}\theta$."

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(93). QUESTION XII. *By Prof. B. Peirce.*

To find a curve whose radius of curvature is a given function of its arc.

FIRST SOLUTION. *By the Proposer.*

Let s = the arc, ρ = the radius of curvature, φ = the angle which ρ makes with the axis of x, and let the given equation be

$$\rho = \mathrm{f}(s).$$

We have then $\quad ds = \rho d\varphi = \mathrm{f}(s)d\varphi, \quad$ and $\varphi = \int \frac{ds}{\mathrm{f}(s)}$.

Then $\quad dx = ds . \sin \varphi, \qquad x = \int ds . \sin \int \frac{ds}{\mathrm{f}(s)}$.

$$dy = ds . \cos \varphi, \qquad y = \int ds . \cos \int \frac{ds}{\mathrm{f}(s)}.$$

Example I. Let $\mathrm{f}(s)$ = constant = R; then $\int \frac{ds}{\mathrm{f}(s)} = \frac{s}{R}$,

$$x = \int ds \sin \frac{s}{R} = -R \cos \frac{s}{R}, \quad y = \int ds \cos \frac{s}{R} = R \sin \frac{s}{R},$$
$$x^2 + y^2 = R^2,$$

which is the circle.

Example II. Let $\mathrm{f}(s) = As + B$, and we have $\int \frac{ds}{\mathrm{f}(s)} = \frac{1}{A}.\log.(As+B)$.

If we put $\quad \log(As+B) = As', \quad A = \tan\beta$;

$$x = \int ds \sin s' = \int ds' e^{As'} \sin s' = \cos \beta e^{As'} \sin (s' - \beta),$$

$$y = \int ds \cos s' = \int ds' e^{As'} \cos s' = \cos \beta e^{As'} \cos (s' - \beta).$$

Using the polar co-ordinates, r and θ, counting the angle θ from an axis inclined to that of x by angle β, we have

$$\tan (\theta - \beta) = \tan (s' - \beta), \quad \theta = s';$$
$$r = \cos \beta . e^{As'} = \cos \beta . e^{A\theta};$$

which is the equation of the logarithmic spiral.

Example III. Let $f(s) = \sqrt{1 - s^2}$, then $\varphi = \int \frac{ds}{\sqrt{1 - s^2}} = \text{arc sin.}\, s, s = \sin \varphi$;

$$x = \int ds \sin \varphi = \int d\varphi \cos \varphi \sin \varphi = \tfrac{1}{4} \cos 2\varphi,$$

$$y = \int ds \cos \varphi = \int d\varphi \quad \cos^2 \quad \varphi = \tfrac{1}{4}(2\varphi - \sin 2\varphi);$$

which are the equations of the cycloid.

SECOND SOLUTION. *By Prof. M. Catlin, Hamilton College.*

Let x and y be the co-ordinates of the required curve, s its length, r the radius of curvature, and z the angle formed by dx and ds.

$$\text{Then} \qquad r = \frac{dsdx}{d^2y} \quad \ldots\ldots\ldots \quad (1);$$

Also, $\quad dx = ds \cos z, \quad$ and $dy = ds \sin z \quad \ldots\ldots \quad (2).$
Hence, if ds is constant, $\quad d^2y = dsdz \cos z \quad \ldots\ldots\ldots \quad (3),$

$$(1) \text{ becomes} \qquad r = \frac{ds}{dz}, \text{ or } ds = rdz \quad \ldots\ldots\ldots \quad (4),$$

and (2) becomes $\quad dx = r \cos zdz, \; dy = r \sin zdz \quad \ldots\ldots\ldots \quad (5).$

When the form of the function r is known, the required curve will be determined by (4) and (5).

For instance; Let $r = \sqrt{1 - s^2}$; then, by (4), $s = \sin z$, and $r = \cos z$.

$$\therefore \; dx = \cos^2 zdz, \; dy = \sin z \cos zdz \quad \ldots\ldots \quad (6),$$
$$x = \tfrac{1}{4} \sin 2z + \tfrac{1}{2}z + c, \quad y = \tfrac{1}{2} \sin^2 z + c' \quad \ldots\ldots \quad (7),$$

which give the equation required, by the elimination of z.

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(94). QUESTION XIII. *By Prof. Peirce, Cambridge University.*

Find a curve which is its own involute.

SOLUTION. *By the Proposer.*

Let $\quad s =$ the arc of the given curve,
$\qquad \rho =$ its radius of curvature,
$\qquad \varphi =$ the angle which ρ makes with a fixed axis;

and let s', ρ', φ' be the corresponding quantities for the evolute whose fixed axis makes an angle $= \alpha - 90^\circ$, with that of its involute.

We have, then,
$$d\varphi' = d\varphi, \quad \varphi' = \varphi + a,$$
$$ds' = \varrho' d\varphi' = \varrho' d\varphi = d\varrho,$$
$$\varrho' = \frac{d\varrho}{d\varphi};$$
so that if the curve is determined by the equation
$$\varrho = f(\varphi),$$
we have, for the evolute,
$$\varrho' = f(\varphi') = f(\varphi + a) = \frac{df(\varphi)}{d\varphi},$$

If, now, we suppose
$$f(\varphi) = \text{A}e^{m\varphi} + \text{A}'e^{m'\varphi} + \&\text{c}.$$
we have A, A′, &c. arbitrary, and m, m', &c. the different roots of the equation,
$$e^{ma} = m, \text{ or } e^{ma} - m = 0.$$

The first member of this equation increases with m, when its differential co-efficient $ae^{ma} - 1$, is positive; and this differential co-efficient constantly increasing from $m = -\infty$ to $m = \infty$, is equal to zero, when
$$m = -\frac{\text{hyp. log. } a}{a}.$$

Case I. When $a > 0$, this value of m is real and corresponds to a minimum of the given first member, which then assumes the value
$$\frac{1 + \text{hyp. log. A}}{a}.$$
The given equation has, therefore,

no real root when hyp. log. $a > -1$, that is, when $a > \frac{1}{e}$;

it has two real roots when hyp. log. $a < -1$, that is, when $a < \frac{1}{e}$;

it has one real root when hyp. log. $a = -1$, that is, when $a = \frac{1}{e}$;

which is
$$m = e.$$

Case II. When $a < 0$, hyp. log. a is imaginary, and the differential co-efficient is always negative, so that the given equation has but one real root which is always positive.

Case III. When $a = 0$, the only possible root is $m = 1$.

The equation
$$e^{ma} - m = 0,$$
has, however, *an infinite number of imaginary roots* when a differs from zero, as we will now proceed to demonstrate. If we represent a pair of these roots by
$$g \pm h\sqrt{-1},$$
the corresponding terms, in the value of ϱ, may be reduced to the one
$$\text{B}e^{g\varphi} \sin(h\varphi + \varepsilon),$$
in which B and ε are arbitrary constant quantities. And g and h are determined by the equation
$$g = h \cot ah,$$
$$\frac{h}{\sin ah} - e^{ah \cot ah} = 0.$$

Now if, in this last equation, $(n+\frac{1}{2})\pi$ be substituted for ah, its first member becomes

$$\frac{(n+\frac{1}{2})\pi}{a\cos n\pi}-1,$$

which, provided $\quad (n+\frac{1}{2})\pi > a,$
has opposite signs for even and odd values of n, and that whether n or a is positive or negative; there is then, in general, a root of this equation between every two values

$$\frac{(n-\frac{1}{2})\pi}{a} \text{ and } \frac{(n+\frac{1}{2})\pi}{a},$$

n being any integer positive or negative; and thence a corresponding value of g from the first of these equations.

We may here observe that if $a=(2n+\frac{1}{2})\pi$, n being any integer positive or negative, we may take $h=1$, in which case we have $g=0$, and the term of ρ becomes

$$\text{B}\sin(\varphi+\varepsilon).$$

Having thus determined the equation

$$\rho=f(\varphi),$$

which contains an infinite number of arbitrary constant quantities, we have $\quad x=-\int\rho d\varphi\,.\sin\varphi,\quad y=\int\rho d\varphi\,.\cos\varphi;$
and, on account of the infinite number of arbitrary constants, a curve which is its own involute may be found so as to pass through any points whatever.

Example 1. Suppose all the arbitrary constants, but one, to be zero, which one we will suppose to correspond to a real root of m. The curve is, in this case, the logarithmic spiral.

Example 2. Suppose $\quad a=(2n+\frac{1}{2})\pi,\ h=1,$
and suppose all the arbitrary constant quantities to be zero, but those which correspond to this value of h, we have

$$\rho=\text{B}\sin(\varphi+\varepsilon),$$

and by changing the fixed axis by an angle $=\varepsilon+90°$, we have

$$\rho=\text{B}\sin(\varphi+90°)=\text{B}\cos\varphi,$$

which corresponds to the cycloid.

Example 3. Suppose all but two of the constants to be zero, and these two to correspond to the real roots m and m'. We have then

$$\rho=\text{A}e^{m\phi}+\text{A}'e^{m'\phi}.$$

Let $\quad m=\tan\beta,\ m'=\tan\beta',$

$$x=\text{A}\cos\beta e^{m\phi}\cos(\varphi+\beta)+\text{A}'\cos\beta' e^{m'\phi}\cos(\varphi+\beta'),$$

$$y=\text{A}\cos\beta e^{m\phi}\sin(\varphi+\beta)+\text{A}'\cos\beta' e^{m'\phi}\sin(\varphi+\beta');$$

and if we put

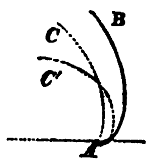
$$x_1=\text{A}\cos\beta e^{m\phi}\cos(\varphi+\beta),\ x_2=\text{A}'\cos\beta' e^{m'\phi}\cos(\varphi+\beta'),$$

$$y_1=\text{A}\cos\beta e^{m\phi}\sin(\varphi+\beta),\ y_2=\text{A}'\cos\beta' e^{m'\phi}\sin(\varphi+\beta'),$$

we have $\quad x=x_1+x_2,\quad y=y_1+y_2,$
in which x_1, y_1 are the co-ordinates of one logarithmic spiral, and x_2, y_2 those of another, so that if upon the two radius vectors of these spirals which correspond to the same value of φ, a parallelogram is formed, the diagonal drawn from the origin is the corresponding radius vector of the required curve. It appears, therefore, that when φ is very large, the

form of the curve is almost exactly like that of the spiral which corresponds to the larger value of m, and when φ is a very large negative quantity, the curve is almost identical with the other spiral. If A and A′ have opposite signs, the curve has a cusp of the first species corresponding to the value of φ

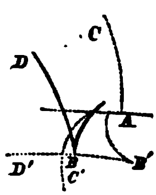
$$\varphi = \frac{\log. A' - \log. A}{(m - m') \log. e}.$$



Thus if $a = 18°$, we have

$$m = 5.3212,$$
$$m' = 1,7127,$$
$$\beta = 79° 21',$$
$$\beta' = 59° 43',$$

and if $A = 2$, $A' = 3$. The curve is AB in fig. 1, in which AC is the spiral for m, and AC′ that for m'.



But if $A = 2$, $A' = -3$, the curve is ABD, fig. 2, in which AC is the spiral for m, and AC′ for m', the cusp B corresponding to $\varphi = 5° 26'$, and AB′D′ is the evolute.

Example 4. Suppose all the arbitrary constants to be zero, but those which correspond to a pair of imaginary roots of m. The term of φ is, in this case,

$$Be^{g\phi} \sin (h\varphi + \varepsilon),$$

and if we suppose $\varepsilon = 0$, it is reduced to $Be^{g\phi} \sin h\varphi$,

and if we take $\quad \tan \gamma = \frac{h+1}{g}, \tan \gamma' = \frac{h-1}{g}, B' = \frac{B}{2g}$

we get $\quad x = B'e^{g\phi}[\cos \gamma \cos (\overline{h+1}.\varphi - \gamma) - \cos \gamma' \cos (\overline{h-1}.\varphi - \gamma')],$

$$y = B'e^{g\phi}[\cos \gamma \sin (\overline{h+1}.\varphi - \gamma) + \cos \gamma' \sin (\overline{h-1}.\varphi - \gamma')].$$

If we take the polar co-ordinates r and θ, θ being the angle which the radius vector r makes with the axis of x, we have to determine θ and r,

$$\tan (\theta + \tfrac{1}{2}\gamma - \tfrac{1}{2}\gamma' - \varphi) = \tan (\tfrac{1}{2}\gamma + \tfrac{1}{2}\gamma' - h\varphi). \cot \tfrac{1}{2}(\gamma' + \gamma) \cot \tfrac{1}{2}(\gamma - \gamma'),$$

$$r' = \frac{B'e^{g\phi} \cos \gamma \sin(2h\varphi - \gamma - \gamma')}{\sin(\overline{h-1}.\varphi - \gamma' + \theta)}$$

$$= B'e^{g\phi} \sqrt{\cos^2 \gamma + \cos^2 \gamma' - 2\cos\gamma\cos\gamma' \cos(2h\varphi - \gamma + \gamma')}.$$

When $h\varphi = n\pi$, in which n is any integer, we have

$$r = B'e^{g\phi} \sin (\gamma + \gamma'),$$

and θ such, that

$$\cos \theta = \cos n\pi \sin (\varphi - \gamma + \gamma') = \cos (\tfrac{3}{2}\pi - \gamma + \gamma' - \overline{h-1}\,.\varphi),$$
$$\sin \theta = - \cos n\pi \cos (\varphi - \gamma + \gamma') = \sin(\tfrac{3}{2}\pi - \gamma + \gamma' - \overline{h-1}\,.\varphi);$$

or $\qquad \theta = \tfrac{3}{2}\pi - \gamma + \gamma' - (h-1)\varphi = \tfrac{3}{2}\pi - \left(1 - \frac{1}{h}\right)n\pi - \gamma + \gamma'.$

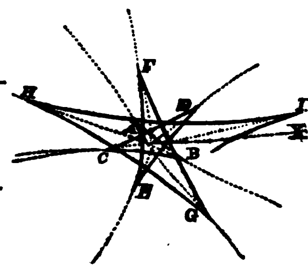
Now at all such points the radius of curvature is zero, and the curve has a cusp of the first species; and each of these cusps is upon a logarithmic spiral whose equation is

$$r = B' \sin(\gamma + \gamma') e^{\frac{g}{h-1}(\frac{3}{2}\pi - \gamma + \gamma' - \theta)}$$

the radius vectors at the cusps making angles of $\left(1 - \frac{1}{h}\right)\pi$, with each other.

Thus in the figure, which is constructed for

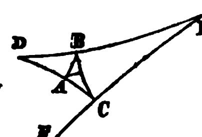
$A = 254° 21'$, $h = 6$, $g = 0{,}40417$,
$\gamma = 86° 42'$, $\gamma' = 85° 23'$, $B' = 1$;
the curve is, in its course, ABCDEFGHI, &c.



Example 5. The curve itself is parallel to its involute when $a = (2n + \frac{1}{2})\pi$ and if the co-efficient corresponding to $h = 1$ is zero, the curve is strictly its own involute without change of position in all other cases. Thus it is so for all the logarithmic spirals corresponding to the real root of the equation

$$e^{-m(2n - \frac{1}{2})\pi} - m = 0.$$

and the curve corresponding to a pair of imaginary roots would be of the form ABCDEFGHI, &c.



Example 6. A great variety of curves might be obtained by using several terms with various constants. Thus, if the real root of the equation

$$e^{-m(2n - \frac{1}{2})\pi} - m = 0$$

were combined with the root corresponding to the cycloid, a curve would be obtained which in the outset would hardly differ from the cycloid for very large negative values of φ, and it would approach very near the logarithmic spiral at the end where φ was very large.

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(95). QUESTION XIV. *By Prof. Avery.*

Suppose a rod to descend as in Question (63), Miscellany, and that a particle, whose weight is inconsiderable with respect to that of the rod, is placed on it and begins to descend by gravity, without friction, at the instant the rod commences its motion. Required the point on the rod where the particle must be placed, in order that it may arrive at the lowest extremity of the rod at the time the rod becomes horizontal.

FIRST SOLUTION. *By the Proposer.*

Let x, y be the co-ordinates of the place of the particle at any time t, from the origin of motion, a the length of the rod, φ the angle a line drawn from the origin to the centre of the rod makes with the axis of y,

vertical, r the distance of the particle from the foot of the rod; φ', r' the initial values of φ, r; then

$$x=(a-r)\sin\varphi,\quad y=r\cos\varphi \quad . \quad . \quad . \quad . \quad . \quad (1),$$

and, by the solutions to question (63), Math. Miscellany,

$$dt=\sqrt{\frac{a}{3g}}.\frac{d\varphi}{\sqrt{\cos\varphi'-\cos\varphi}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

By Dynamics,

$$\frac{d^2x}{dt^2}\delta x+\left(\frac{d^2y}{dt^2}+g\right)y=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3);$$

substitute (1) and (2) in (3), r being the only variable with regard to δ,

$$\frac{d^2r}{dt^2}+(a\sin^2\varphi-r)\frac{d\varphi^2}{dt^2}-\tfrac{3}{2}g\sin^2\varphi\cos\varphi+g\cos\varphi=0 \quad . \quad . \quad (4).$$

Now, since, from (1), φ is a function of t, r is a function of t, and by Maclaurin's Theorem,

$$r=(r)+\left(\frac{dr}{dt}\right)t+\tfrac{1}{2}\left(\frac{d^2r}{dt^2}\right)t^2+\frac{1}{2\,.\,3}\left(\frac{d^3r}{dt^3}\right)t^3+\&c. \quad . \quad (5);$$

but, at the beginning of motion, when $t=0$, $\frac{d\varphi}{dt}=0$, $\frac{dr}{dt}=0$, by hypothesis, so that from (4), we easily get,

$$(r)=r',\left(\frac{dr}{dt}\right)=0,\left(\frac{d^2r}{dt^2}\right)=\tfrac{1}{2}g\cos\varphi'(1-3\cos^2\varphi'),\left(\frac{d^3r}{dt^3}\right)=0,\ \&c.,$$

Therefore $\qquad r=r'+\tfrac{1}{4}g\cos\varphi'(1-3\cos^2\varphi')t^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$

and, by (2), when the rod is horizontal, and $r=0$,

$$r'=\tfrac{1}{4}g\cos\varphi'(3\cos^2\varphi'-1)t^2=\frac{a}{12}\cos\varphi'(3\cos^2\varphi'-1)\left\{\int_{\varphi'}^{\frac{1}{2}\pi}\frac{d\varphi}{\sqrt{\cos\varphi'-\cos\varphi}}\right\}^2 (7),$$

the integral contained in this equation being easily determined from Legendre's Tables of Elliptic Functions.

SECOND SOLUTION. *By Mr. O. Root.*

If R be the re-action of the rod, r the distance of the particle from the lowest point of the rod, and the notation otherwise as in my solution to question (63), the equations of its motion will be

$$\frac{d^2x}{dt^2}+R\cos\varphi=0,\quad \frac{d^2y}{dt^2}+R\sin\varphi+g=0 \quad . \quad . \quad . \quad (1),$$

or, eliminating R, $\qquad -\frac{d^2x}{dt^2}\sin\varphi+\left(\frac{d^2y}{dt^2}+g\right)\cos\varphi=0 \quad . \quad . \quad (2).$

But $\qquad x=(2a-r)\sin\varphi,\quad y=r\cos\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)'$

and (2) becomes, by substitution,

$$\frac{d^2r}{dt^2}+(2a\sin^2\varphi-r)\frac{d\varphi^2}{dt^2}-2a\sin\varphi\cos\varphi\,\frac{d^2\varphi}{dt^2}+g\cos\varphi=0 \quad . \quad (4),$$

and, by Maclaurin's Theorem,

$$r=r'+\left(\frac{dr'}{dt}\right)t+\tfrac{1}{2}\,.\left(\frac{d^2r'}{dt^2}\right)t^2+\&c. \quad . \quad . \quad . \quad (5),$$

therefore if T represent the whole time of motion, or

$$\mathrm{T}=\sqrt{\frac{2a}{3g}}\cdot\int_{\varphi'}^{\frac{1}{2}\pi}\frac{d\varphi}{\sqrt{\cos\varphi'-\cos\varphi}}\quad\ldots\ldots\quad(6),$$

we have, at the end of the motion, since $\left(\frac{dr'}{dt}\right)=0$,

$$r'=-\tfrac{1}{2}\left(\frac{d^2r'}{dt^2}\right)\mathrm{T}^2-\frac{1}{2\,.\,3}\left(\frac{d^3r'}{dt^3}\right)\mathrm{T}^3-\&\text{c.}\quad\ldots\quad(7),$$

the values of $\left(\frac{d^2r'}{dt^2}\right)$, $\left(\frac{d^3r'}{dt^3}\right)$, &c., being derived from (4), when $t=0$, and $\varphi=\varphi'$; and the values of $\left(\frac{d\varphi}{dt}\right)$, $\left(\frac{d^2\varphi}{dt^2}\right)$, &c., from their values at page 312, vol. I.

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(96). Question XV. *By Mr. W. S. B. Woolhouse, London.*

A crown piece being twirled any how on a perfectly smooth horizontal plane, it is required to investigate the circumstances of the motion and the velocities of its points when it acquires any given position, disregarding the thickness of the metal.

First Solution. *By Dr. Strong.*

Let the notation be as in the article on Rotary motion, in a subsequent part of this Number of the Miscellany; the axes of x and y being fixed on the horizontal plane, that of z vertical and counted downwards; X, Y, Z will be the co-ordinates of the centre of the plate; the co-ordinates x, y, z of any element dm of the plane having their origin at that point. Then, if we *fix* the axes of x', y', z' in the plate, that of z' being its axis, and those of x', y' on its plane, the angle φ of that article will represent the angle which the axis of x' makes with the tangent, at the point of contact of the plate and plane, to the path of that point; χ will be the angle made by the axis of x, with the same tangent, and θ will be the inclination of the plate to the horizon. Then if these co-ordinates refer to the point of contact, x_1, y_1 being the co-ordinates of the same point referred to the first axes, ds the element of the path of the point on the plane, R_1 the radius of the plate, and M' its mass, we shall have for this point,

$$\left.\begin{array}{l} x'=\mathrm{R}_1\sin\varphi,\ y'=\mathrm{R}_1\cos\varphi,\ z'=0;\\ x_1-\mathrm{X}=x=ax'+by'=\mathrm{R}_1\cos\theta\sin\chi,\\ y_1-\mathrm{Y}=y=a'x'+b'y'=\mathrm{R}_1\cos\theta\sin\chi,\\ \therefore\ \mathrm{Z}=\sqrt{\mathrm{R}_1{}^2-(x_1-\mathrm{X})^2-(y_1-\mathrm{Y})^2}=\mathrm{R}_1\sin\theta \end{array}\right\}\quad\ldots\quad(\mathrm{A}),$$

For a slight change of position, when x_1, y_1 become x_1+dx_1, y_1+dy_1, we have

$$(x_1-\mathrm{X})dx_1+(y_1-\mathrm{Y})dy_1=0,\ \text{or}\ \sin\chi dx_1+\cos\chi dy_1=0;$$

which is satisfied by putting $\frac{dx_1}{ds}=\cos\chi$ $\frac{dy_1}{ds}=\sin\chi$. Now let $\mathrm{M}'\mathrm{T}=$ the resistance the plate meets with in the direction of the element ds

arising from friction; $\text{M}'\text{V}$ = the resistance in the plane in a direction at right angles to ds, and $\text{M}'\text{R}''$ = the re-action of the plane in a vertical direction. Then the resultants of all the accelerating forces acting on any element, dm, in the direction of the three axes, being represented by X', Y', Z', are

$$\left.\begin{aligned} \text{X}' &= -\text{T}\frac{dx_1}{ds} + \text{V}\frac{dy_1}{ds} = -\text{T}\cos\chi - \text{V}\sin\chi, \\ \text{Y}' &= -\text{T}\frac{dy_1}{ds} - \text{V}\frac{dx_1}{ds} = \quad \text{T}\sin\chi - \text{V}\cos\chi, \end{aligned}\right\} \quad . \quad (\text{B}).$$
$$\text{Z}' = \text{R}'' - g$$

the equations of motion (b') of the centre of gravity, become

$$\frac{d^2\text{X}}{dt^2} = -\text{T}\cos\chi - \text{V}\sin\chi,\ \frac{d^2\text{Y}}{dt^2} = \text{T}\sin\chi - \text{V}\cos\chi,\ \frac{d^2\text{Z}}{dt^2} = \text{R}'' - g \quad . \quad (\text{C}).$$

$$\left.\begin{aligned} \therefore\ \text{T} &= -\frac{d^2\text{X}}{dt^2}\cos\chi + \frac{d^2\text{Y}}{dt^2}\sin\chi, \\ \text{V} &= -\frac{d^2\text{X}}{dt^2}\sin\chi - \frac{d^2\text{Y}}{dt^2}\cos\chi, \\ \text{R}'' &= \frac{d^2\text{Z}}{dt^2} + g \end{aligned}\right\} \quad . \quad . \quad (\text{D}).$$

The equations of rotation of the plate around the axes of x', y', z', are those in (z') of the article cited, which are easily adapted to this case. For since $z' = 0$, and the plate is symmetrical and homogeneous, by (k),

$$\text{A} = \text{B} = \text{S}x'^2 dm = \tfrac{1}{4}\text{R}_1{}^2\text{M}',\ \text{C} = 2\text{A} = \tfrac{1}{2}\text{R}_1{}^2\text{M}';$$
$$\text{P} = -\text{T}\cos\chi - \text{V}\sin\chi,\ \text{Q} = \text{T}\sin\chi - \text{V}\cos\chi,\ \text{R} = \text{R}'';$$

by substituting in (n) the proper values of a, b, c, &c., given in (10),

$$\text{R}' = \text{R}''\cos\theta - \text{V}\sin\theta,\ \text{Q}' = \text{T}\sin\varphi - \text{V}\cos\varphi\cos\theta - \text{R}''\sin\theta\cos\varphi,$$
$$\text{P}' = -\text{T}\cos\varphi - \text{V}\cos\theta\sin\varphi - \text{R}''\sin\theta\sin\varphi,$$ and the equations (z') become

$$\left.\begin{aligned} \frac{dp}{dt} + qr &= \frac{4\cos\varphi}{\text{R}_1}(\text{R}''\cos\theta - \text{V}\sin\theta,) \\ \frac{dq}{dt} - pr &= -\frac{4\sin\varphi}{\text{R}_1}(\text{R}''\cos\theta - \text{V}\sin\theta, \\ \frac{dr}{dt} &= \frac{2\text{T}}{\text{R}_1'} \end{aligned}\right\} \quad . \quad . \quad . \quad (\text{E}),$$

In the case where the plate rolls, without sliding, the point of contact may be regarded as momentarily at rest in consequence of the opposite motions arising from the motion of the centre of gravity, and of rotation; then

$$\frac{dx_1}{dt} = \frac{d\text{X}}{dt} + \frac{dx}{dt} = 0,\ \frac{dy_1}{dt} = \frac{d\text{Y}}{dt} + \frac{dy}{dt} = 0 \quad . \quad . \quad . \quad (\text{F})$$

But equations (f) become in this case,

$$\text{L} = -ry' = -\text{R}_1 r\cos\varphi,\ \text{M} = rx' = \text{R}_1 r\sin\varphi,\ \text{N} = py' - qx' = \text{R}_1(p\cos\varphi - q\sin\varphi),$$

and substituting these, together with the values of a, b, c, &c., from (10), in (h),

$$\left.\begin{aligned}\frac{d\text{x}}{dt}=-\frac{dx}{dt}=\text{R}_1\left(r\cos\chi+\sin\theta\sin\chi\frac{d\theta}{dt}\right),\\ \frac{d\text{y}}{dt}=-\frac{dy}{dt}=\text{R}_1\left(-r\sin\chi+\sin\theta\cos\chi\frac{d\theta}{dt}\right),\end{aligned}\right\}\quad\ldots\quad(\text{G});$$

therefore, (D) become, by substituting these

$$\text{V}=\text{R}_1\left(r\frac{d\chi}{dt}+\frac{d^2\cos\theta}{dt^2}\right),\ \text{T}=-\text{R}_1\left(\frac{dr}{dt}+\frac{d\theta}{dt}\cdot\frac{d\chi}{dt}\sin\theta\right),$$

$$\text{R}''=\text{R}_1\frac{d^2.\sin\theta}{dt^2}+g.$$

and these substituted in (E), together with the value of p, q, r, from (e), give the equations for rolling motion,

$$\left.\begin{aligned}3\frac{dr}{dt}+2\sin\theta\frac{d\theta}{dt}\cdot\frac{d\chi}{dt}=0,\\ d\left(\sin^2\theta\frac{d\chi}{dt}\right)-2rd.\cos\theta=0,\\ \frac{5}{4}\cdot\frac{d^2\theta}{dt^2}-\tfrac{1}{4}\sin\theta\left(\cos\theta\frac{d\chi}{dt}+6p\right)\frac{d\chi}{dt}+\frac{g}{\text{R}_1}\cos\theta=0,\end{aligned}\right\}\quad.\quad(\text{H}).$$

which agree with the equations found at p. 66, No. 10, of the Math. Diary.

Again, if the plane be perfectly smooth, so that there is no friction, $\text{T}=0$, $\text{V}=0$, $\text{R}''=\text{R}_1\dfrac{d^2.\sin\theta}{dt^2}+g$, and (C) become $\dfrac{d^2\text{x}}{dt^2}=0$, $\dfrac{d^2\text{y}}{dt^2}=0$, or integrating

$$\text{x}=\text{A}t+\text{B},\ \text{y}=\text{A}'t+\text{B}',$$

A, B, A′, B′ being constants, determined from the initial position and velocity of the centre, and eliminating t we get $\text{A}'\text{x}-\text{A}\text{y}=\text{A}'\text{B}-\text{A}\text{B}'$, which shows that the centre of the plate is always on the same vertical plane, having its height from the horzontal plane $\text{z}=\text{R}_1\sin\theta$. The equations (E) of rotation become by substituting for p, q, r and slightly modifying them

$$\left.\begin{aligned}-\frac{d^2\theta}{dt^2}+\frac{d\chi^2}{dt^2}\sin\theta\cos\theta+2r\sin\theta\frac{d\chi}{dt}=4\cos\theta\left(\frac{d^2\sin\theta}{dt^2}+\frac{g}{\text{R}_1}\right),\\ d\left(\sin^2\theta\frac{d\chi}{dt}\right)+2r\sin\theta d\theta=0,\\ dr=0.\end{aligned}\right\}\ .\ (\text{I}).$$

The last of these gives $r=$ const. $=$ the angular velocity of the plate about its axis, then the second is integrable and gives

$$\sin^2\theta\,.\frac{d\chi}{dt}-2r\cos\theta=\text{const.}=n\ .\ .\ .\ .\ .\ .\ .\ (\text{K}).$$

If the first be multiplied by $d\theta$, and subtracted from the second multiplied by $d\chi$, the integral of the resulting equation is

$$\left.\begin{aligned}\frac{d\theta^2}{dt^2}+4\frac{(d\sin\theta)^2}{dt^2}+\sin^2\theta\frac{d\chi^2}{dt^2}+\frac{8g}{\text{R}_1}\sin\theta=\text{const.}=m,\\ \text{or, by (K)},\ (\sin^2\theta+\sin^2 2\theta)\frac{d\theta^2}{dt^2}=m\sin^2\theta-(2r\cos\theta+n)^2-\frac{8g}{\text{R}_1}\sin^3\theta\end{aligned}\right\}(\text{L});$$

$$\left.\begin{array}{ll}\text{and, by } (e), & \dfrac{d\varphi}{dt} - \cos\theta\dfrac{d\chi}{dt} = r, \\ \text{or} & \sin^2\theta\dfrac{d\varphi}{dt} = r(1+\cos^2\theta) + n\cos\theta,\end{array}\right\} \quad \ldots\ldots \quad (\text{M}).$$

The angular motions are therefore all given in functions of θ by (K), (L), (M). If we suppose $\frac{d\chi}{dt}$ = const., (K) will give θ = const., (M) will give $\frac{d\varphi}{dt}$ = const., therefore the plate would revolve uniformly about its axis, the point of contact describing a circle with a uniform motion on the horizontal plane, when $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$; but if these are not $= 0$, the aforesaid motion continues, while the centre of the plate describes a straight line parallel to the horizontal plane. If $\frac{d\chi}{dt} = 0$, the point of contact describes a right line on the plane, parallel to the right line described by the centre, and the plate revolves uniformly about its axis in all cases, except when $r = 0$.

SECOND SOLUTION. *By Prof. C. Avery.*

Let X, Y, Z be the co-ordinates of any element dm of the crown-piece, referred to three rectangular axes, the two first on the horizontal plane on which the disc moves; t the time from any epoch; φ, χ, θ the three angles which determine the position of the crown-piece and of its principal axes at any time; R the radius of the disc, and m its mass. Then we have by the general formula of Dynamics

$$\text{S}dm\left(\frac{d^2\text{X}}{dt^2}\,\delta\text{X} + \frac{d^2\text{Y}}{dt^2}\,\delta\text{Y} + \frac{d^2\text{Z}}{dt^2}\,\delta\text{Z} + g\delta\text{Z}\right) = 0 \quad . \quad . \ (1).$$

Assume

$$\text{X} = x + ax' + by',$$
$$\text{Y} = y + a'x' + b'y',$$
$$\text{Z} = z + a''x' + b''y',$$

where x, y, z are the co-ordinates of the centre of the plate, referred to the same axes; x', y', z' the co-ordinates of dm, referred to three rectangular axes passing through the centre of gravity, those of x', y', in the disc itself, and that of z' perpendicular to it, z' being neglected on account of its minuteness; a, b, a', &c., having the values in equations (10) of Dr. Strong's Article on Rotation in this number.*

The crown-piece will be subjected to move on the horizontal plane by the equation of condition

$$\text{L} = z - \text{R}\sin\theta = 0 \quad \ldots\ldots\ldots \quad (2).$$

and if we put, to facilitate the transformation of (1),

$$2\text{T} = \text{S}dm\left(\frac{d\text{X}^2}{dt^2} + \frac{d\text{Y}^2}{dt^2} + \frac{d\text{Z}^2}{dt^2}\right),\ \text{V} = g\text{S}dmz \quad . \ . \ . \quad (3);$$

then, since

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* The Editor has taken the liberty of changing these references for the reader's convenience, so as to prevent the necessity of his consulting a number of different authors.

$$\text{S}dm = m,\ \text{S}x'dm = \text{S}y'dm = 0,\ \text{S}x'^2dm = \text{S}y'^2dm = \tfrac{1}{4}\text{R}^2m,\ \text{S}x'y'dm = 0,$$

$$2\text{T} = m\,.\,\frac{dx^2+dy^2+dz^2}{dt^2} + \tfrac{1}{4}\text{R}^2m\,.\,\frac{da^2+da'^2+da''^2+db^2+db'^2+db''^2}{dt^2},$$

$$\text{V} = gmz.$$

But, using Dr. Strong's equations of reduction, in (c') and (e),

$$da^2+da'^2+da''^2+db^2+db'^2+db''^2 = (p^2+q^2+2r^2)dt^2$$
$$= d\theta^2+(1+\cos^2\theta)d\chi^2+2d\varphi^2-4\cos\theta d\varphi d\chi$$

$$\therefore\ 2\text{T} = m.\frac{dx^2+dy^2+dz^2}{dt^2}+\frac{\text{R}^2m}{4}\Big\{\frac{d\theta^2}{dt^2}+(1+\cos^2\theta)\frac{d\chi^2}{dt^2}+2\frac{d\varphi^2}{dt^2}$$
$$-4\cos\theta\frac{d\varphi}{dt}.\frac{d\chi}{dt}\Big\}.$$

Then there will result, as La Grange has shown, from the substitution in (1), for each of the six quantities $x, y, z, \theta, \chi, \varphi$ that enter into it, an equation of the form

$$d.\frac{\delta\text{T}}{\delta dx}-\frac{\delta\text{T}}{\delta x}+\frac{\delta\text{V}}{\delta x}+\lambda.\frac{\delta\text{L}}{\delta x}=0,$$

λ being an indeterminate co-efficient. Hence the equations of motion

$$\left.\begin{array}{r}\dfrac{d^2x}{dt^2}=0,\qquad \dfrac{d^2y}{dt^2}=0,\qquad \dfrac{d^2z}{dt^2}+g+\dfrac{\lambda}{m}=0,\\ \dfrac{d^2\theta}{dt^2}+\tfrac{1}{2}\sin2\theta\dfrac{d\chi^2}{dt^2}-2\sin\theta\dfrac{d\varphi}{dt}.\dfrac{d\chi}{dt}-\dfrac{4\lambda}{\text{R}m}\cos\theta=0,\\ \dfrac{d.\{(1+\cos^2\theta)d\chi-2\cos\theta d\varphi\}}{dt^2}=0,\\ \dfrac{d.\{d\varphi-\cos\theta d\chi\}}{dt^2}=0,\end{array}\right\}\ .\ (4).$$

The two first equations give, by integration,

$$x = a_1t+b_1,\quad y = a_2t+b_2\ \ .\ .\ .\ .\ .\ .\ (5)$$

which show that the centre of the disc is confined to a vertical plane, the position of which depends on the arbitrary constants a_1, a_2, b_1, b_2, or on the initial position and impulse of that point. The third give

$$\frac{\lambda}{m} = -\frac{d^2z}{dt^2}-g = -\text{R}\frac{d^2(\sin\theta)}{dt^2}-g\ .\ \ .\ \ .\ \ .\ (6),$$

and the fifth and sixth,

$$\left.\begin{array}{r}(1+\cos^2\theta)\dfrac{d\chi}{dt}-2\cos\theta\dfrac{d\varphi}{dt}=\text{constant}=n,\\ \dfrac{d\varphi}{dt}-\cos\theta\dfrac{d\chi}{dt}=\text{constant}=r,\end{array}\right\}\ .\ \ .\ \ .\ (7),$$

$$\text{or}\qquad\left.\begin{array}{r}\sin^2\theta\,.\dfrac{d\chi}{dt}=2r\cos\theta+n,\\ \sin^2\theta\,.\dfrac{d\varphi}{dt}=(1+\cos^2\theta)r+n\cos\theta,\end{array}\right\}\ .\ \ .\ \ .\ (8).$$

For another integral, we have in this case, the equation of living forces,

$$\text{T}+\text{V}=\text{const.}$$

or combining the constants $\frac{dx}{dt}$, $\frac{dy}{dt}$ with this constant, and reducing

$$(1+4\cos^2\theta)\frac{d\theta^2}{dt^2}+(1+\cos^2\theta)\frac{d\chi^2}{dt^2}+2\frac{d\varphi^2}{dt^2}-4\cos\theta\frac{d\varphi}{dt}.\frac{d\chi}{dt}+\frac{8g}{R}\sin\theta=m,$$

or, by (8),

$$(\sin^2\theta+\sin^2 2\theta)\frac{d\theta^2}{dt^2}+(2r\cos\theta+n)^2+(2r^2-m)\sin^2\theta+\frac{8g}{R}\sin^3\theta=0 \text{ (9).}$$

The equations (8) and (9) give the angular velocities, at any time, in terms of θ, and the constants, which are to be determined from the initial impulses. Since r represents the velocity of rotation about the axis of the disc, that velocity is constant. Equation (6) shows the vertical pressure, p, of the disc on the plane to be, at the time t,

$$p=gm+Rm.\frac{d^2(\sin\theta)}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

The path of the centre is determined from the equations (2) and (5), and that of the point of contact on the plane, by making $x'=R\sin\varphi$, $y'=R\cos\varphi$; then its co-ordinates are, $Z=0$,

$$X=x+ax'+by'=x+R\cos\theta\sin\chi,$$
$$Y=y+a'x'+b'y'=y+R\cos\theta\cos\chi,$$

between which, x, y, θ, χ may be eliminated since they are all functions of t.

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(97). Question XVI. *By Mr. W. S. B. Woolhouse*

It is required to solve the preceding Question, when, instead of the circular disc, any solid of revolution is substituted, as for instance, a *spheroid*, the semi-axes of which are a and b.

First Solution. *By Dr. Strong.*

Let the centre of gravity of the *ellipsoid* be defined at any time t, by the rectangular co-ordinates X, Y, Z the first two being in the horizontal plane and the last vertical and directed upwards, and let the co-ordinates of any element dm of the body be x, y, z when referred to these axes, and x', y', z' when referred to the principal axes A′, B′, C′ of the ellipsoid as axes of co-ordinates, then the equation of the ellipsoid (when the point x', y', z' is at the surface) is

$$u=A'^2B'^2z'^2+A'^2C'^2y'^2+B'^2C'^2x'^2-A'^2B'^2C'^2=0 \quad . \quad . \quad (A),$$

but from the eq. of transformation (3), (see the article on Rotary Motion),

$$x'=a(x-X)+a'(y-Y)+a''(z-Z),$$
$$y'=b(x-X)+b'(y-Y)+b''(z-Z),$$
$$z'=c(x-X)+c'(y-Y)+c''(z-Z),$$

and if these be substituted in (A) we shall have the equation of the surface in function of the fixed co-ordinates x, y, z. Hence if we suppose two sections of the solid, passing through the point of contact of the body with the plane, and parallel to the co-ordinates planes xz, yz respectively, we shall have

$$\frac{du}{dx}dx+\frac{du}{dz}dz=0,\ \frac{du}{dy}dy+\frac{du}{dz}dz=0;$$

which will enable us to draw a tangent at any point of either section, but the horizontal plane touches each section, and for the point of contact

$$z=0, dz=0, \frac{du}{dx}=0, \frac{du}{dy}=0.$$

But by the above equations,

$$\frac{du}{dx}=\frac{du}{dx'}.\frac{dx'}{dx}+\frac{du}{dy'}.\frac{dy'}{dx}+\frac{du}{dz'}.\frac{dz'}{dx}=\text{B}'^2\text{C}'^2 a\,x'+\text{A}'^2\text{C}'^2 b\,y'+\text{A}'^2\text{B}'^2 c\,z'=0,$$

$$\frac{du}{dy}=\frac{du}{dx'}.\frac{dx'}{dy}+\frac{du}{dy'}.\frac{dy'}{dy}+\frac{du}{dz'}.\frac{dz'}{dy}=\text{B}'^2\text{C}'^2 a'x'+\text{A}'^2\text{C}'^2 b'y'+\text{A}'^2\text{B}'^2 c'z'=0;$$

and it is evident, from eq. (4) of transformation, that these equations, as well as (A,) will be satisfied by putting

$$x'=\text{A}'^2a''v, y'=\text{B}'^2b''v, z'=\text{C}'^2c''v, \text{ if } v=\pm(\text{A}'^2a''^2+\text{B}'^2b''^2+\text{C}'^2c''^2)^{-\frac{1}{2}} \text{ (B)}.$$

Also, since $z-\text{z}=a''x'+b''y'+c''z'$, for this point, where $z=0$,

$$\text{z}=-(a''x'+b''y'+c''z')=\pm\frac{1}{v}=(\text{A}'^2a''^2+\text{B}'^2b''^2+\text{C}'^2c''^2)^{\frac{1}{2}} \text{ . (C)}.$$

Let M' be the mass of the plate, and $\text{M}'\text{R}''$ the vertical reaction of the plane upon the solid; then the equations of translation of the centre are

$$\frac{d^2\text{X}}{dt^2}=0, \quad \frac{d^2\text{Y}}{dt^2}=0, \quad \frac{d^2\text{Z}}{dt^2}=\text{R}''-g \quad . \quad . \quad . \quad . \text{ (D)},$$

therefore, $\quad \text{X}=\text{A}_1t+\text{A}_2, \ \text{Y}=\text{B}_1t+\text{B}_2, \text{R}''=\dfrac{d^2\text{Z}}{dt}+g.$

$\text{A}_1, \text{A}_2, \text{B}_1, \text{B}_2$ being arbitrary constants; eliminating t, we have

$$\text{A}_1\text{Y}-\text{B}_1\text{X}=\text{A}_1\text{B}_2-\text{A}_2\text{B}_1,$$

which shows that the centre of the ellipsoid is always in the same vertical plane. The equations of rotation are those in (z') of the subsequent article on rotation, and are adopted to this case by making

$$\text{P}=0, \text{Q}=0, \text{P}'=a''\text{R}, \text{Q}'=b''\text{R}, \text{R}'=c''\text{R}, \text{S}dm\text{R}=\text{M}'\text{R}'',$$

and writing in them the values of x', y', z' found in (B), after putting $v=\dfrac{1}{\text{z}}$; they thus become

$$\left.\begin{aligned}\text{A}\frac{dp}{dt}+(\text{C}-\text{B})qr&=\frac{\text{M}'\text{R}''}{\text{z}}(\text{C}'^2-\text{B}'^2)b''c'',\\ \text{B}\frac{dq}{dt}+(\text{A}-\text{C})pr&=\frac{\text{M}'\text{R}''}{\text{z}}(\text{A}'^2-\text{C}'^2)a''c'',\\ \text{C}\frac{dr}{dt}+(\text{B}-\text{A})pq&=\frac{\text{M}'\text{R}''}{\text{z}}(\text{B}'^2-\text{A}'^2)a''b'',\end{aligned}\right\} \quad . \quad . \quad \text{(E)}.$$

If we multiply these equations severally by a'', b'', c'' and add them together, reduce the result by equations (c') of the article on Rotation, we get

$$d.[\text{A}a''p+\text{B}b''q+\text{C}c''r]=0,$$

and $\qquad \text{A}a''p+\text{B}b''q+\text{C}c''r=\text{const.}=m' \quad . \quad . \quad \text{(F)}$

Multiply (E) by pdt, qdt, rdt, severally, add the products, and reduce by the same equations and by (C), then

$$\text{A}pdp+\text{B}qdq+\text{C}rdr=-\text{M}'\text{R}''dz=-\text{M}'dz\left(\frac{d^2\text{z}}{dt^2}+g\right),$$

or, integrating,

$$\text{A}p^2 + \text{B}q^2 + \text{C}r^2 + \text{M}'\left(\frac{dz^2}{dt^2} + 2g\text{z}\right) = \text{const.} = n' \quad . \quad (\text{G}).$$

See Poisson's Traité de Mecanique, Vol. II., page 178, ed. 1811, from which the above solution, with some slight variations, has been taken.

If we put $\text{A}' = \text{B}'$, the solid becomes a spheroid, the axis of z' being the axis of revolution, and those of x', y' two rectangular diameters of its equator, then

$$\text{A} = \text{B} = \text{S}(x'^2 + z'^2)\,dm = \tfrac{1}{5}\text{M}'(\text{A}'^2 + \text{C}'^2), \quad \text{C} = \text{S}(x'^2 + y'^2)dm = \tfrac{2}{5}\text{M}'\text{A}'^2,$$

$$\text{z} = \sqrt{\text{A}'^2(a''^2 + b''^2) + \text{C}'^2c''^2} = \sqrt{\text{A}'^2 \sin^2\theta + \text{C}'^2 \cos^2\theta},$$

and put $m' = \frac{1}{5}\text{M}'l$, $n' = \frac{1}{5}\text{M}'l'$; then the last of equations (E) together with (F) and (G) become

$$\left.\begin{aligned} dr = 0, \text{ or } r = \text{constant}, \\ (\text{A}'^2+\text{C}'^2)(a''p + b''q) + 2\text{A}'^2c''r = l, \\ (\text{A}'^2+\text{C}'^2)(p^2+q^2) + 2\text{A}'^2r + 5\left(\frac{dz^2}{dt^2} + 2g\text{z}\right) = l', \end{aligned}\right\} \quad . \quad . \quad . \quad (\text{H}).$$

Write in these the values of p, q, r from equations (e), and those of a'', b'', c'' from equations (10) of the subsequent article, restoring the value of z, and putting

$$\frac{2r\text{A}'^2}{\text{A}'^2 + \text{C}'^2} = \text{E}, \quad \frac{5(\text{A}'^2 - \text{C}'^2)^2}{\text{A}'^2 + \text{C}'^2} = \text{G}^2, \quad \frac{10g}{\text{A}'^2 + \text{C}'^2} = g',$$

$$\frac{l}{\text{A}'^2 + \text{C}'^2} = -\text{F}, \quad \frac{l'}{\text{A}'^2 + \text{C}'^2} = \text{H},$$

$$\left.\begin{aligned} \frac{d\varphi}{dt} - \cos\theta\frac{d\chi}{dt} = r, \\ \sin^2\theta\frac{d\chi}{dt} - \text{E}\cos\theta = \text{F}, \\ \frac{d\theta^2}{dt^2} + \sin^2\theta\frac{d\chi^2}{dt^2} + \frac{\text{G}^2\sin^2\theta\cos^2\theta}{\text{A}'^2\sin^2\theta + \text{C}'^2\cos^2\theta}\cdot\frac{d\theta^2}{dt^2} + g'\sqrt{\text{A}'^2\sin^2\theta + \text{C}'^2\cos^2\theta} = \text{H}, \end{aligned}\right\} (\text{I})$$

the constants r, F, H being determined by any given state of the body. These equations give the three angular velocities of the body in terms of θ; and by eliminating χ from the second and third, θ can be found from the resulting equation in terms of t.

If C' be so small that it may be neglected, and $\text{A}' = \text{R}_1$, these equations become

$$\left.\begin{aligned} \frac{d\varphi}{dt} - \cos\theta\frac{d\chi}{dt} = r, \\ \sin^2\theta\frac{d\chi}{dt} - 2r\cos\theta = \text{F}, \\ \frac{d\theta^2}{dt^2} + \sin^2\theta\frac{d\chi^2}{dt^2} + 5\cos^2\theta.\frac{d\theta^2}{dt^2} + \frac{10g}{\text{R}_1}.\sin\theta = \text{H}, \end{aligned}\right\} \quad . \quad . \quad . \quad (\text{K}).$$

By comparing these with the corresponding ones (K), (L), (M) in the solution to the last question, we perceive that the two first agree with (K) and (M), but the last does not agree with (L), the quantity $4\frac{(d\sin\theta)^2}{dt^2} + \frac{8g}{\text{R}_1}\sin\theta$, in (L), corresponding to the terms $5\frac{(d\sin\theta)^2}{dt^2} + \frac{10g}{\text{R}_1}\sin\theta$

in this equation; hence the motion of a very thin spheroidal disk is not the same as that of a very thin cylindrical disk; and the reason is that a greater portion of the mass is accumulated round the centre of the one than the other, *since, however thin, a spheroidal disk cannot be considered as equally thick throughout.*

SECOND SOLUTION. *By Prof. M. Catlin, Hamilton College.*

Let x, y, and z be the rectangular co-ordinates of the centre of gravity, the plane xy being horizontal; R the reaction of the plane, and M the mass of the moving body. Then we shall have for the motion of the centre of gravity,

$$\text{M}\frac{d^2x}{dt^2}=0,\ \text{M}\frac{d^2y}{dt^2}=0,\ \text{M}\left(\frac{d^2z}{dt^2}+g\right)-\text{R}=0 \quad . \quad . \quad (1).$$

Let x', y', z' be the point of contact of the solid with the horizontal plane, referred to the principal axes passing through the centre of gravity; A, B, C the moments of inertia, and p, q, r the angular velocities about the same axes. Then, p, q, r being as in equation (e), and a'', b'', c'' as in eq. (10), of Dr. Strong's Article on Rotation, we shall have, for the rotatory motion of the body,

$$\left.\begin{aligned}\text{A}dp+(\text{C}-\text{B})qrdt&=\text{M}\left(\frac{d^2z}{dt^2}+g\right)(c''y'-b''z')dt\\ \text{B}dq+(\text{A}-\text{C})rpdt&=\text{M}\left(\frac{d^2z}{dt^2}+g\right)(a''z'-c''x')dt\\ \text{C}dr+(\text{B}-\text{A})pqdt&=\text{M}\left(\frac{d^2z}{dt^2}+g\right)(b''x'-a''y')dt\end{aligned}\right\}\quad(2).$$

Let $\text{L}=0$ be the equation of the surface of the moveable referred to the axes of x', y' and z', and put

$$\text{V}=\left[\left(\frac{d\text{L}}{dx'}\right)^2+\left(\frac{d\text{L}}{dy'}\right)^2+\left(\frac{d\text{L}}{dz'}\right)^2\right]^{-\frac{1}{2}}$$

then the above general equations must be subjected to the following conditions, which are equivalent to four independent equations, $\text{L}=0$,

$$z+a''x'+b''y'+c''z'=0,\ a''=\text{V}\frac{d\text{L}}{dx'},\ b''=\text{V}\frac{d\text{L}}{dy'},\ c''=\text{V}\frac{d\text{L}}{dz'}\ .\ .\ (3).$$

The equations above are sufficient to completely determine the motion of any solid, twirled in any manner upon a smooth horizontal plane. We shall now proceed, as required by the question, to investigate the particular case of a solid of revolution.

Let the axis of z' be the axis of the figure — then evidently we shall have $x'=y'\tan\varphi$, but $a''=b''\tan\varphi$; hence $b''x'-a''y'=0$. We have also $\text{A}=\text{B}$, consequently equations (2) become by substitution

$$\left.\begin{aligned}\text{A}dp+(\text{C}-\text{A})qrdt&=\text{M}\left(\frac{d^2z}{dt^2}+g\right)(c''y'-b''z')dt\\ \text{A}dq+(\text{A}-\text{C})rpdt&=\text{M}\left(\frac{d^2z}{dt^2}+g\right)(a''z'-c''x')dt\\ \text{C}dr&=0\end{aligned}\right\}\ .\ (4).$$

Integrating (4) by the the usual method we obtain

$$\left.\begin{array}{l}\text{A}(p^2+q^2)+\text{M}\left(\dfrac{dz^2}{dt^2}+2gz\right)=k\\ \text{A}(a''p+b''q)+\text{C}c''r=k',\ r=k''\end{array}\right\}\quad\ldots\ (5),$$

where k, k' and k'' are arbitrary constants. The first k includes $\text{C}k''$.

But $\quad a''p+b''q=-\sin^2\theta\dfrac{d\chi}{dt};\ p^2+q^2=\sin^2\theta\dfrac{d\chi^2}{dt^2}+\dfrac{d\theta^2}{dt^2};$

therefore equations (5) are reduced to

$$\left.\begin{array}{l}\dfrac{d\chi}{dt}=\dfrac{\text{C}k''\cos\theta-k'}{\text{A}\sin^2\theta},\ \dfrac{d\varphi}{dt}=\dfrac{\cos\theta(\text{C}k''\cos\theta-k')+\text{A}k''\sin^2\theta}{\text{A}\sin^2\theta},\\ \text{A}\dfrac{d\theta^2}{dt^2}+\text{M}\dfrac{dz^2}{dt^2}=\dfrac{\text{A}\sin^2\theta(k-2g\text{M}z)-(\text{C}k''\cos\theta-k')^2}{\text{A}\sin^2\theta}\end{array}\right\}\ (6)$$

When the form of the equation of $\text{L}=0$ is given, z and $\dfrac{dz}{dt}$ will be known by means of (3) in terms of θ, φ and χ, which being substituted in (6) will give the velocities $\dfrac{d\chi}{dt}$, $\dfrac{d\varphi}{dt}$ and $\dfrac{d\theta}{dt}$ for any given position of the moveable. Integrating (1)

$$\frac{dx}{dt}=l,\ \frac{dy}{dt}=l',\ x=lt+m,\ y=l't+m'\quad\ldots\quad(7).$$

Equations (7) determine the velocity of the centre of gravity, which is uniform in a direction parallel to the horizontal plane, and the projection of the locus of the centre of gravity upon the plane (xy) is a right line.

By integrating the third of (6) we may find the time t in terms of θ, which being substituted in (12), will give us the values of x, y, and z for any given values of θ, φ and χ. The locus of the point of contact will then become known by means of the equations

$$x_1=x+ax'+by'+cz',\ y_1=y+a'x'+b'y'+c'z'\ .\quad.\ (8),$$

where x_1 and y_1 are the co-ordinates of that point referred to the axes of x and y, and a, b, &c., are as in Dr. Strong's eqs. (10).

We will now proceed to examine the less general case of a spheroid of revolution. We shall now have

$$\text{L}=\beta^2z'^2+\alpha^2(x'^2+y'^2)-\alpha^2\beta^2\quad\ldots\quad(9),$$

α and β representing the semi-axes of the ellipsoid. Taking the partial differentials of (9), with respect to x', y', z', and substituting them, together with the values of a'', b'', c'' in (3), we have after reduction

$$\left.\begin{array}{l}z=\sqrt{\alpha^2\cos^2\theta+\beta^2\sin^2\theta},\ \dfrac{dz}{dt}=\dfrac{(\beta^2-\alpha^2)\sin\theta\cos\theta}{z}.\dfrac{d\theta}{dt},\\ x'=\dfrac{\beta^2\sin\theta\sin\varphi}{z},\ y'=\dfrac{\beta^2\sin\theta\cos\varphi}{z},\ z'=\dfrac{\alpha^2\cos\theta}{z}\end{array}\right\}\ .\ (10),$$

and equations (6) become, for the spheroid of revolution,

$$\left.\begin{array}{l}\dfrac{d\chi}{dt}=\dfrac{\text{C}k''\cos\theta-k'}{\text{A}\sin^2\theta},\ \dfrac{d\varphi}{dt}=\dfrac{\cos\theta(\text{C}k''\cos\theta-k')+\text{A}k''\sin^2\theta}{\text{A}\sin^2\theta},\\ \dfrac{d\theta}{dt}=\dfrac{z}{\sin\theta}\sqrt{\dfrac{\text{A}\sin^2\theta(k-2g\text{M}z)-(\text{C}k''\cos\theta-k')^2}{\text{A}^2z^2+\text{AM}(\beta^2-\alpha^2)^2\sin^2\theta\cos^2\theta}}\end{array}\right\}\ .\ (11)$$

and the velocities of its centre of gravity are

$$\frac{dx}{dt}=l, \frac{dy}{dt}=l', \frac{dz}{dt}$$

$$=(\beta^2-\alpha^2)\cos\theta\sqrt{\frac{\textsc{a}\sin^2\theta(k-2g\textsc{m}z)-(\textsc{c}k''\cos\theta-k')^2}{\textsc{a}^2z^2+\textsc{a}\,\textsc{m}(\beta^2-\alpha^2)^2\sin^2\theta\cos^2\theta}} \quad . \quad . \quad (12).$$

Hence all the velocities are given in terms of θ, and the time t may be found, by approximation, from the third of (11); and the loci of the point of contact and the centre of gravity from (8), and (7).

THIRD SOLUTION. *By Mr. Geo. R. Perkins, Clinton.*

In this solution I shall follow Mr. Poisson's notation, as given in the second volume of his Mechanics. Let the ellipsoid's equation be

$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{a^2}+\frac{\gamma^2}{b^2}-1=\textsc{l}=0 \quad . \quad . \quad . \quad . \quad (1),$$

referred to its principal axes. Call K the point of contact with the plane, M its mass, G its centre of gravity; if to the weight of the ellipsoid we add the force R of unknown magnitude, which is the resistance of the plane against the ellipsoid, we may consider its centre of gravity as a free material point; if its co-ordinates are x_1, y_1, z_1, referred to fixed axes, of which the plane of x, y coincides with the given plane, and the axis of z is counted upwards, the differential equations of its motion will be

$$\textsc{m}\frac{d^2x_1}{dt^2}=0, \textsc{m}\frac{d^2y_1}{dt^2}=0, \textsc{m}\frac{d^2z_1}{dt^2}=\textsc{r}-\textsc{m}g \quad . \quad . \quad . \quad (2).$$

At the same time the ellipsoid will turn about G as about a fixed point, in virtue of the forces R and Mg applied at the points K and G, but the force Mg can have no influence on this rotation, since it passes through the point G, considered as fixed. Now the momenta of the force R, referred to the principal axes will be

$$\textsc{r}(\alpha b''-\beta a''), \textsc{r}(\gamma a''-\alpha c''), \textsc{r}(\beta c''-\gamma b'') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

and the three equations of rotation are

$$\left.\begin{array}{l} \textsc{c}\,dr+(\textsc{b}-\textsc{a})pq\,dt=\textsc{r}(\alpha b''-\beta a'')dt, \\ \textsc{b}dq+(\textsc{a}-\textsc{c})pr\,dt=\textsc{r}(\gamma a''-\alpha c'')dt, \\ \textsc{a}dp+(\textsc{c}-\textsc{b})qr\,dt=\textsc{r}(\beta c''-\gamma b'')dt, \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

(——The symbols A, B, C; p, q, r; a'', b'', c'' have here precisely the same signification and relations with the angles θ, χ, φ, as in Dr. Strong's article on Rotation, and Mr. Perkins deduces from these equations, in the same manner as in the first solution—)

$$\textsc{a}\,a''p+\textsc{b}\,b''q+\textsc{c}c''r=l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

$$\textsc{a}\,pdp+\textsc{b}\,qdq+\textsc{c}\,rdr=\textsc{r}(\alpha da''+\beta db''+\gamma dc'') \quad . \quad . \quad (6).$$

The equation of the given plane, referred to the principal axes, is

$$z_1+a''\alpha+b''\beta+c''\gamma=0,$$

hence $\quad \alpha da''+\beta db''+\gamma dc''=-dz_1-(a''d\alpha+b''d\beta+c''d\gamma);$

and, since the normal at the point K, is parallel to the fixed axis of z_1,

$$a''=\textsc{v}.\frac{d\textsc{l}}{d\alpha}, b''=\textsc{v}.\frac{d\textsc{l}}{d\beta}, c''=\textsc{v}.\frac{d\textsc{l}}{d\gamma}; \text{ where } \textsc{v}=\left\{\left(\frac{d\textsc{l}}{d\alpha}\right)^2+\left(\frac{d\textsc{l}}{d\beta}\right)^2+\left(\frac{d\textsc{l}}{d\gamma}\right)^2\right\}^{-\frac{1}{2}}$$

$$\therefore \quad a''d\alpha+b''d\beta+c''d\gamma=\textsc{v}.\left\{\frac{d\textsc{l}}{d\alpha}d\alpha+\frac{d\textsc{l}}{d\beta}d\beta+\frac{d\textsc{l}}{d\gamma}d\gamma\right\}=\textsc{v}d\textsc{l}=0,$$

and equation (6) becomes, by substituting this, and the value of z, and integrating

$$Ap^2+Bq^2+Cr^2+M\left(\frac{dz_1^2}{dt^2}+2gz'\right)=h \quad . \quad . \quad . \quad . \quad (7);$$

also since the solid is one of revolution $A=B$, and $\alpha b''-\beta a''=0$, hence the first of equations (4) gives,

$$dr=0, \text{ and } r=\frac{d\varphi}{dt}-\cos\theta\frac{d\chi}{dt}=\text{const.}=n \quad . \quad . \quad . \quad . \quad . \quad (8).$$

By substitution, we shall also find

$$z_1=-\sqrt{a^2\sin^2\theta+b^2\cos^2\theta},\ \frac{dz_1}{dt}=\frac{(a^2-b^2)\sin\theta\cos\theta}{z_1}\cdot\frac{d\theta}{dt},$$

$$a''p+b''q=-\sin^2\theta\frac{d\chi}{dt},\ p^2+q^2=\sin^2\theta\frac{d\chi^2}{dt^2}+\frac{d\theta^2}{dt^2};$$

and equations (5) and (6) thus become

$$Cn\cos\theta-A\sin^2\theta\frac{d\chi}{dt}=l, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$A\left(\sin^2\theta\frac{d\chi^2}{dt^2}+\frac{d\theta^2}{dt^2}\right)+M\left\{\frac{(a^2-b^2)^2\sin^2\theta\cos^2\theta}{z_1^2}\cdot\frac{d\theta^2}{dt^2}+2gz_1\right\}=h\ . \ . \ (10),$$

where the constant h includes $-Cn^2$. The equations (8), (9), (10) give the angular velocities of the solid. The equations of translation show that the projection of the path of the point G on the plane is a straight line uniformly described, and its distance from the plane is $z_1=-\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$.

——Professor Peirce's solutions to the last three questions did not arrive until this part of the copy was prepared for the press. His results confirm those of the gentlemen whose solutions are published.——

List of Contributors, and of Questions answered by each. The figures refer to the number of the Questions, as marked in Number V. Art. XX.

Professor C. Avery, Hamilton College, N. Y. ans. all the questions.
Professor F. N. Benedict, University of Vermont, ans. 9.
P. Barton, Jr., Athol, Massachusetts, ans. 4.
B. Birdsall, New-Hartford, N. Y., ans. 1, 2, 3, 4, 11.
Professor M. Catlin, Hamilton College, N. Y., ans. all the questions.
R. S Howland, Junior Class, St. Paul's College, ans. 1, 3.
Investigator, ans. 3.
William Lenhart, York, Penn., ans. 10.
J. F. Macully, Teacher of Mathematics, New-York, ans. 1, 2, 11.
Professor B. Peirce, Harvard University, Mass., ans. all the questions.
George R. Perkins, Clinton Liberal Institute, ans. all the questions.
ψ, ans. 5.
O. Root, Principal of Syracuse Acad., N. Y., ans. 1, 2, 3, 4, 5, 9, 11, 12, 14.
Prof. T. Strong, LL. D., New Brunswick, N. J., ans. all the questions.

*** All communications for Number VIII. which will be published on the first day of November, 1839, must be post paid, addressed to the Editor, St. Paul's College, Flushing, L. I, and must arrive before the first of August, 1839. New Questions must be accompanied with their solutions.

☞ Mr. George R. Perkins, of Clinton, Oneida Co., N. Y. is desirous of obtaining a situation as Teacher of Mathematics. For further particulars reference may be had to the following gentlemen: Hon. John A. Dix, Albany; Hon. John H. Prentiss, Cooperstown, Otsego Co., N. Y.; Rev. George B. Miller, D. D., Principal of Hartswick Seminary, Otsego Co., N. Y.; Rev. C. B. Thummel, A. M., Georgetown, S. C.

*** We have been obliged, for want of room in this number, to defer the insertion of Mr. Lenhart's Diophantine speculations, No. III.; as well as an article on the application of Sturm's Theorem to the general equations of the 4th and 5th degrees, by Δ.

ARTICLE II.

NEW QUESTIONS TO BE ANSWERED IN NUMBER IX,

Their solutions must arrive before February 1st, 1840.

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(113). QUESTION I. *By Prof. N. May, M. D., St. Paul's College.*

What degree of temperature will be indicated by the same number on two scales, attached to the same thermometer, and graduated, the one by the centigrade, the other by Fahrenheit's, method of division.

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(114). QUESTION II. *From Peirce's Algebra,*

Solve the two equations

$$x^2y^4-8y^2x^2+16x^2=90xy+60(x-y^2)-720(y-1),$$
$$\frac{(y^2-4y+4)x}{5}=3-\frac{12}{x}.$$

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(115). QUESTION III. *By P.*

Find the value of the infinitely continued fraction

$$\sqrt[n]{a+\cfrac{b}{\sqrt[n]{a+\cfrac{b}{\sqrt[n]{a+\cfrac{b}{\sqrt[n]{a+\&c.,}}}}}}}$$

and give an example when $a=23$, $b=10$, $n=2$.

(116). QUESTION IV. *By Mr. Sam. J. Gummere, Haverford School, Pa.*

Find the value of $x^{\frac{1}{x}}$, when $x = 0$.

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(117). QUESTION V. *By J. F. Macully, Esq., New-York.*

Points are taken on the plane of a given triangle, so that the sum of the squares of the three perpendiculars drawn from any one point to the sides, shall be equal to the area of the triangle. These points will all be found in the periphery of an ellipse, whose position and magnitude are required.

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(118). QUESTION VI. *By Mr. P. Barton, jun., Athol, Mass.*

It is required to circumscribe the least isosceles triangle about two circles, touching each other, and the base of the triangle; the diameters of the circles being 16 and 20.

(119.) QUESTION VII. *By Wm. Lenhart, Esq., York, Pa.*

Find parallelograms whose sides and diagonals are integers.

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(120). QUESTION VIII. *By J. F. Macully, Esq., New-York.*

Find the sum of n terms of the series

$$\frac{1+2\sin^2\frac{1}{2}\theta}{(1-2\cos\theta)^2}+\frac{1}{9}\cdot\frac{1+2\sin^2\frac{1}{6}\theta}{(1-2\cos\frac{1}{3}\theta)^2}+\frac{1}{9^2}\cdot\frac{1+2\sin^2\frac{1}{18}\theta}{(1-2\cos\frac{1}{9}\theta)^2}+\&c.$$

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(121). QUESTION IX. *By Mr. J. S. Van de Graaff, Lexington, Ken.*

(From the Mathematical Diary.)

To find, on the surface of a given sphere, the area of the greatest triangle, whose perimeter is a semi-circumference, and whose greater angle is just double the smaller.

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(122). QUESTION X. *By Investigator.*

Two bodies of given masses, and composed of matter the particles of which attract each other with forces inversely proportional to the squares of their distances, are placed at a given distance from each other, and then projected in *opposite* directions along the same straight line. It is required to find all the circumstances of their motion until they are in contact.

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(123). QUESTION XI. *By Mr. W. S. B. Woolhouse, London.*

Sum the series

$$\frac{xy}{x+y}+\frac{x^2y^2}{(x+y)(x^2+y^2)}+\frac{x^4y^4}{(x+y)(x^2+y^2)(x^4+y^4)}+\&c.$$

(124). QUESTION XII. *By Mr. George R. Perkins, Clinton, N. Y.*

a_1	a_2	a_3	a_4
b_1	b_2	b_3	b_4
c_1	c_2	c_3	c_4
d_1	d_2	d_3	d_4

The points A, B, C, D and A′, B′, C′, D′ correspond severally with the vertices of any two regular tetraedrons inscribed within a given sphere, and the cosines of the arcs drawn from A′ to the points A, B, C, D are represented by a_1, a_2, a_3, a_4; the cosines of the arcs drawn from B′, to the same points, by b_1, b_2, b_3, b_4; and similarly for those drawn from C′ and D′ to the same points, these cosines being arranged as in the annexed square. It is required to show that the sum of the squares of the four cosines in any horizontal line, as well as in any vertical line, is equal to $\frac{4}{3}$; and that the sum of the products two and two, such as $a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$ with regard to any two horizontal lines, as well as with regard to any two vertical lines, is equal to $-\frac{4}{9}$.

(125). QUESTION XIII. *By Δ.*

If from a given point in the transverse axis of a spherical ellipse, (Vol. I, page 194,) perpendicular arcs be let fall upon the great circle tangents of the curve; the vertices of the right angles will be in a spherical curve, whose equation and properties are required.

(126). QUESTION XIV. *By Δ.*

It is required to find the equations of the surfaces described in questions (17) and (18), Vol. I. pp. 95 and 97, or to integrate the partial differential equations

$$1^\circ.\ (z-px-qy)^2(1-p-q) = A^2pq,$$
$$2^\circ.\ (z-px-qy)^4(1+p^2+q^2) = A^4(t^2-p^2-q^2)^2;$$

in which, t and A are constants, $p=\frac{dz}{dx}, q=\frac{dz}{dy}$.

(127). QUESTION XV. *By Prof. B. Peirce, Harvard University, Mass.*

Supposing a very light body, of such a substance as not to be impeded in its motions by collision with the earth, to be moving in an orbit in the plane of the ecliptic, with a time of revolution nearly half that of the earth, and an aphelion distance nearly equal to the distance of the earth from the sun, which we will suppose to be constant, and that at the instant of its passing the aphelion it is nearly in conjunction with the earth; to find the perturbations in its motion as caused by the earth.

ARTICLE III.

MOTION OF A SYSTEM OF BODIES ABOUT A FIXED AXIS.

By Prof. T. Strong, LL. D., New-Brunswick, N. J.

1. We will commence this paper by showing how to change the rectangular co-ordinates of a point when referred to one system of axes, to those referred to another system, having the same origin.

Let x, y, z denote the co-ordinates of the point referred to in the first system, and x', y', z', those of the same point referred to in the second system, the origin being the same; let L denote the right line drawn from the origin to the point; a, b, c the cosines of the angles which the axis of x makes with the axes of x', y', z'; a' b' c' the corresponding cosines for the axis of y, and a'', b'', c'' those for the axis of z.

It is evident that x=the projection of L on the axis of x,

=the sum of the projections of x', y', z' on that axis,

and similarly for y and z; hence

$$x = ax' + by' + cz',\ y = a'x' + b'y' + c'z',\ z = a''x' + b''y' + c''z' \quad . \ (1).$$

Since $$L^2 = x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2;$$

if we substitute the values of x, y, z from (1), the resulting equation will be an identical one, true for all values of x', y', z', and therefore, by the principle of indeterminate co-efficients,

$$\left.\begin{array}{lll} a^2 + a'^2 + a''^2 = 1, & b^2 + b'^2 + b''^2 = 1, & c^2 + c'^2 + c''^2 = 1; \\ ab + a'b' + a''b'' = 0, & ac + a'c' + a''c'' = 0, & bc + b'c' + b''c'' = 0. \end{array}\right\} \ (2).$$

If we now multiply the equations in (1) by a, a', a'' severally, and add the results; then successively by b, b', b'' and c, c', c'', we get

$$x' = ax + a'y + a''z,\ y' = bx + b'y + b''z,\ z' = cx + c'y + c''z \quad . \ (3);$$

from which we get, as before,

$$\left.\begin{array}{lll} a^2 + b^2 + c^2 = 1, & a'^2 + b'^2 + c'^2 = 1, & a''^2 + b''^2 + c''^2 = 1; \\ aa' + bb' + cc' = 0, & aa'' + bb'' + cc'' = 0, & a'a'' + b'b'' + c'c'' = 0. \end{array}\right\} \ . (4),$$

which are evidently equivalent to (2), and therefore only three of the nine cosines, a, b, c, a' &c. are arbitrary. Since, by (2)

$$bc + b'c' + b''c'' = 0,\ c'^2 + c''^2 = 1 - c^2,\ b'^2 + b''^2 = 1 - b^2,$$

we get
$$\begin{aligned} b^2 + c^2 &= b^2 + c^2 - 2bc(bc + b'c' + b''c'') \\ &= b^2(1-c^2) + c^2(1-b^2) - 2bc(b'c' + b''c'') \\ &= b^2(c'^2 + c''^2) + c^2(b'^2 + b''^2) - 2bc(b'c' + b''c'') \\ &= (bc' - cb')^2 + (bc'' - b''c)^2 \\ &= 1 - (b'c'' - c'b'')^2, \end{aligned}$$

since we have, identically,

$$(bc' - cb')^2 + (bc'' - b''c)^2 + (b'c'' - b''c')^2$$
$$= (b^2 + b'^2 + b''^2)(c^2 + c'^2 + c''^2) - (bc + b'c' + b''c'')^2 = 1.$$

But, by (4), $$b^2 + c^2 = 1 - a^2,$$

therefore, $a^2 = (b'c'' - c'b'')^2$, and $a = \pm (b'c'' - c'b'')$;

the sign (—) does not, however, apply, for supposing the co-ordinates x, y, z to coincide with x', y', z', we have a=1, b'=1, c''=1, b''=0, c'=0 therefore

$$a = b'c'' - b''c',$$

and in a similar manner we get the equations

$$\left.\begin{array}{lll} a = b'c'' - b''c', & b = a''c' - a'c'', & c = a'b'' - a''b', \\ a' = b''c - bc'', & b' = ac'' - a''c, & c' = a''b - ab'', \\ a'' = bc' - b'c, & b'' = ac' - a'c, & c'' = ab' - a'b, \end{array}\right\} \quad . \quad (5).$$

2. We will now show how to find values of a, b, c, a', &c., which will satisfy the equations of condition thus obtained.

Imagine (with La Place, Mec. Cel., p. 58, or Com. p. 111,) that the origin of the co-ordinates is at the centre of the earth, that x, y are in the plane of the ecliptic, x', y' in that of the equator, and that the axes of z, z' are drawn to the north poles of the ecliptic and equator; let χ, $\frac{1}{2}\pi + \chi$ denote the angles made by the axes of x and y, with the earth's radius drawn to the vernal equinox, these angles being reckoned according to the order of the signs of the zodiac; let φ, $\frac{1}{2}\pi + \varphi$ denote the angles which the axes of x' and y' make with the same radius, counted in direction of the earth's rotation about its axis; and let θ denote the obliquity of the ecliptic = the angle made by the axes of z and z'. It is evident that the sum of the projections of x, y, z on any right line = the sum of the projections of x', y', z' on that line, since they are each = the projection of L on that line; hence, if we project the two systems of co-ordinates on the line of the equinoxes, the projections of z and z' are each = 0, and we have

$$x\cos\chi - y\cos(\tfrac{1}{2}\pi + \chi) = x'\cos\varphi + y'\cos(\tfrac{1}{2}\pi + \varphi),$$

or $$x\cos\chi - y\sin\chi = x'\cos\varphi - y'\sin\varphi \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Similarly, if the two systems be projected on the line of the solstices,

$$x\sin\chi + y\cos\chi = (x'\sin\varphi + y'\cos\varphi)\cos\theta + z'\sin\theta \quad . \quad . \quad (7);$$

and if the systems be projected on the line of intersection of the plane of z, z', or solstitial colure, and that of the equator,

$$(x\sin\chi + y\cos\chi)\cos\theta - z\sin\theta = x'\sin\varphi + y'\cos\varphi \quad . \quad . \quad (8).$$

By a very obvious reduction, we get from these equations

$$\begin{array}{l} x = x'(\cos\theta\sin\chi\sin\varphi + \cos\chi\cos\varphi) + y'(\cos\theta\sin\chi\cos\varphi - \cos\chi\sin\varphi) + z'\sin\theta\sin\chi, \\ y = x'(\cos\theta\cos\chi\sin\varphi - \sin\chi\cos\varphi) + y'(\cos\theta\cos\chi\cos\varphi + \sin\chi\sin\varphi) + z'\sin\theta\cos\chi, \\ z = -x'\sin\theta\sin\varphi - y'\sin\theta\cos\varphi + z'\cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9), \end{array}$$

which agree with those of Laplace, at the place cited.

By comparing the values of x, y, z in (1) and (9), we get

$$\begin{array}{lll} a = \cos\theta\sin\chi\sin\varphi + \cos\chi\cos\varphi, & b = \cos\theta\sin\chi\cos\varphi - \cos\chi\sin\varphi, & c = \sin\theta\sin\chi; \\ a' = \cos\theta\cos\chi\sin\varphi - \sin\chi\cos\varphi, & b' = \cos\theta\cos\chi\cos\varphi + \sin\chi\sin\varphi, & c' = \sin\theta\cos\chi; \\ a'' = -\sin\theta\sin\varphi, & b'' = -\sin\theta\cos\varphi, & c'' = \cos\theta \; (10), \end{array}$$

which, on trial, will be found to fulfil the equations of condition (2), (4), (5); so that χ, φ, θ will be indeterminates, as they ought to be.

We shall now suppose that a system of bodies m, m' m'', &c. is revolving about any fixed point, and shall indefinitely denote any body of the system, by m; we shall denote by x, y, z the co-ordinates of m, when referred to any system of rectangular axes, fixed in space, at the time t from any given epoch; let also P, Q, R be the resultants of all the forces that affect an unit of m, when decomposed in the direction of x, y, z severally, these forces being supposed to tend to increase the co-ordinates; then we shall have

$$m\frac{d^2x}{dt^2} = m\text{P}, \; m\frac{d^2y}{dt^2} = m\text{Q}, \; m\frac{d^2z}{dt^2} = m\text{R},$$

and using the sign of finite integrals, S, to denote the sum of all the equations thus formed for each body in the system,

$$\mathrm{S}m\frac{d^2x}{dt^2}=\mathrm{S}m\mathrm{P},\ \mathrm{S}m\frac{d^2y}{dt^2}=\mathrm{S}m\mathrm{Q},\ \mathrm{S}m\frac{d^2z}{dt^2}=\mathrm{S}m\mathrm{R}\quad .\ .\ (a),$$

where we suppose that m, m', &c. are each so small, that the co-ordinates of all their points may be regarded as the same: we shall also have

$$\left.\begin{aligned} d\,.\,\mathrm{S}m\left(\frac{xdy-ydx}{dt}\right)&=dt\,.\,\mathrm{S}m(\mathrm{Q}x-\mathrm{P}y)=d\mathrm{N}'',\\ d\,.\,\mathrm{S}m\left(\frac{zdx-xdz}{dt}\right)&=dt\,.\,\mathrm{S}m(\mathrm{P}z-\mathrm{R}x)=d\mathrm{N}',\\ d\,.\,\mathrm{S}m\left(\frac{ydz-zdy}{dt}\right)&=dt\,.\,\mathrm{S}m(\mathrm{R}y-\mathrm{Q}z)=d\mathrm{N}',\end{aligned}\right\}\quad .\ b)$$

which are the formulæ of rotation that we propose to transform into others which shall be more convenient in practice.

We shall now suppose that the co-ordinates of m, when referred to any other system, having the same origin, are x', y', z'; then if we suppose $x', y'\ z'$ as well as a, b, c, a', &c. are functions of t; we shall have from (1),

$$\left.\begin{aligned}\frac{dx}{dt}&=\frac{x'da+y'db+z'dc}{dt}+\frac{adx'+bdy'+cdz'}{dt},\\ \frac{dy}{dt}&=\frac{x'da'+y'db'+z'dc'}{dt}+\frac{a'dx'+b'dy'+c'dz'}{dt}\\ \frac{dz}{dt}&=\frac{x'da''+y'db''+z'dc''}{dt}+\frac{a''dx'+b''dy'+c''dz'}{dt}\end{aligned}\right\}\quad .\ (c).$$

Put

$cdb+c'db'+c''db''=pdt,\ adc+a'dc'+a''dc''=qdt,\ bda+b'da'+b''da''=rdt;$

then, by (2), we shall also have

$$\left.\begin{aligned} bdc+b'dc'+b''dc''&=-pdt,\\ cda+c'da'+c''da''&=-qdt,\\ adb+a'db'+a''db''&=-rdt,\end{aligned}\right\}\quad .\ .\ .\ .\ .\ (d);$$

and by substituting the values of a, b, c, a', &c., from (10) in (d), we have

$$\left.\begin{aligned}\sin\varphi\sin\theta d\chi-\cos\varphi d\theta&=pdt,\\ \cos\varphi\sin\theta d\chi+\sin\varphi d\theta&=qdt,\\ d\varphi-\cos\theta d\chi&=rdt,\end{aligned}\right\}\quad .\ .\ .\ .\ .\ .\ (e).$$

Put, $qz'-ry'+\dfrac{dx'}{dt}=\mathrm{L},\ rx'-pz'+\dfrac{dy'}{dt}=\mathrm{M}, py'-qx'+\dfrac{dz'}{dt}=\mathrm{N}\ .\ (f);$

multiply the equations in (c) by a, a', a'',; then by b, b', b''; then by c, c', c''; adding the respective results, and reducing by (d), (2), (4); we get

$$\frac{adx+a'dy+a''dz}{dt}=\mathrm{L},\ \frac{bdx+b'dy+b''dz}{dt}=\mathrm{M},\ \frac{cdx+c'dy+c''dz}{dt}=\mathrm{N}\ (g).$$

Multiply these severally by a, b, c; then by a', b', c'; then by a'', b'', c'', adding the products, and we get

$$\frac{dx}{dt}=a\mathrm{L}+b\mathrm{M}+c\mathrm{N},\ \frac{dy}{dt}=a'\mathrm{L}+b'\mathrm{M}+c'\mathrm{N},\ \frac{dz}{dt}=a''\mathrm{L}+b''\mathrm{M}+c''\mathrm{N}\ .\ (h).$$

The sum of the squares of these equations is

$$\frac{dx^2+dy^2+dz^2}{dt^2}=L^2+M^2+N^2$$

$$=\frac{dx'^2+dy'^2+dz'^2}{dt^2}+(x'^2+y'^2)r^2+(x'^2+z'^2)q^2+(y'^2+z'^2)p^2$$

$$-2x'y'pq-2x'z'pr-2y'z'qr$$

$$+2p\left(\frac{y'dz'-z'dy'}{dt}\right)+2q\left(\frac{z'dx'-x'dz'}{dt}\right)+2r\left(\frac{x'dy'-y'dx'}{dt}\right)\quad (i).$$

Put

$$\left.\begin{array}{l} S(y'^2+z'^2)m=A,\ Sy'z'm=D,\ Sm\left(\frac{x'dy'-y'dx'}{dt}\right)=A'; \\ S(x'^2+z'^2)m=B,\ Sx'z'm=E,\ Sm\left(\frac{z'dx'-x'dz'}{dt}\right)=B'; \\ S(x'^2+y'^2)m=C,\ Sx'y'm=F,\ Sm\left(\frac{y'dz'-z'dy'}{dt}\right)=C'; \\ Ap-Er-Fq=p',\ Bq-Dr-Fp=q',\ Cr-Dq-Ep=r' \end{array}\right\}(k),$$

then we easily deduce from (f), after reducing by (5) and by (k),

$Sm(Ny'-Mz')=p'+C'$, $Sm(Lz'-Nx')=q'+B'$, $Sm(Mx'-Ly')=r'+A'$,

and if we substitute these, together with the values of x, y, z from (1), and those of $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ from (h), in the three equations (b), we get

$$\left.\begin{array}{l} d.\left[a''(p'+C')+b''(q'+B')+c''(r'+A')\right]=dN''', \\ d.\left[a'(p'+C')+b'(q'+B')+c'(r'+A')\right]=dN'', \\ d.\left[a(p'+C')+b(q'+B')+c(r'+A')\right]=dN', \end{array}\right\}\quad . \quad (l);$$

hence

$$\left.\begin{array}{l} \frac{d(p'+C')}{dt}+q(r'+A')-r(q'+B')=\frac{a''dN'''+a'dN''+adN'}{dt}, \\ \frac{d(q'+B')}{dt}+r(p'+C')-p(r'+A')=\frac{b''dN'''+b'dN''+bdN'}{dt}, \\ \frac{d(r'+A')}{dt}+p(q'+B')-q(p'+C')=\frac{c''dN'''+c'dN''+cdN'}{dt}, \end{array}\right\}(m).$$

Put the second members of equations (m) $=N_{,}$, $N_{,,}$, $N_{,,,}$ respectively, then by (b),

$$N_{,}=Sm[(a'z-a''y)P+(a''x-az)Q+(ay-a'x)R],$$

and similarly the others. But, by (1) and (5),

$$a'z-a''y=cy'-bz',\ a''x-az=c'y'-b'z',\ ay-a'x=c''y'-b''z';$$

therefore put

$$cP+c'Q+c''R=R',\ bP+b'Q+b''R=Q',\ aP+a'Q+a''R=P' \quad . \quad . \ (n),$$

then $N_{,}=Sm(R'y'-Q'z')$, $N_{,,}=Sm(P'z'-R'x')$, $N_{,,,}=Sm(Q'x'-P'y')$. (o),

Multiply (i) by (m), and take the finite integrals relative to all the bodies of the system, putting $Sm\left(\frac{dx^2+dy^2+dz^2}{dt^2}\right)=2T=$ the living force of the system; then

$$\text{T} = \tfrac{1}{2}(\text{A}p^2 + \text{B}q^2 + \text{C}r^2) - \text{D}qr - \text{E}pr - \text{F}pq + \text{C}'p + \text{B}'q + \text{A}'r + \text{S}m\left(\frac{dx'^2 + dy'^2 + dz'^2}{2dt^2}\right) \quad . \quad . \; (p);$$

By taking the partial differential co-efficients of (p), relative to p, q, r, and noticing (k), we get

$$\frac{d\text{T}}{dp} = p' + \text{C}', \frac{d\text{T}}{dq} = q' + \text{B}', \frac{d\text{T}}{dr} = r' + \text{A}' \quad . \quad . \quad . \quad . \; (q);$$

hence (m) will be changed to

$$\left.\begin{aligned}
&\frac{d\left(\frac{d\text{T}}{dp}\right)}{dt} + q\left(\frac{d\text{T}}{dr}\right) - r\left(\frac{d\text{T}}{dq}\right) = \text{S}m(\text{R}'y' - \text{Q}'z'),\\
&\frac{d\left(\frac{d\text{T}}{dq}\right)}{dt} + r\left(\frac{d\text{T}}{dp}\right) - p\left(\frac{d\text{T}}{dr}\right) = \text{S}m(\text{P}'z' - \text{R}'x'),\\
&\frac{d\left(\frac{d\text{T}}{dr}\right)}{dt} + p\left(\frac{d\text{T}}{dq}\right) - q\left(\frac{d\text{T}}{dp}\right) = \text{S}m(\text{Q}'x' - \text{P}'y'),
\end{aligned}\right\} \quad . \quad . \; (r),$$

which agree with the equations given by La Grange, Mec. Analytique, vol. 2, p. 365, edition of 1815. Also, if we substitute (q) in (l), they become

$$\left.\begin{aligned}
&d\,.\left[a''\left(\frac{d\text{T}}{dp}\right) + b''\left(\frac{d\text{T}}{dq}\right) + c''\left(\frac{d\text{T}}{dr}\right)\right] = d\text{N}''',\\
&d\,.\left[a'\left(\frac{d\text{T}}{dp}\right) + b'\left(\frac{d\text{T}}{dq}\right) + c'\left(\frac{d\text{T}}{dr}\right)\right] = d\text{N}'',\\
&d\,.\left[a\left(\frac{d\text{T}}{dp}\right) + b\left(\frac{d\text{T}}{dq}\right) + c\left(\frac{d\text{T}}{dr}\right)\right] = d\text{N}',
\end{aligned}\right\} \quad . \quad . \quad . \quad . \; (s).$$

Now it may be proved, as in p. 55, &c. Vol. I., Mec. Cel., or Com. p. 104, &c. that (b) are independent of the mutual action of the bodies of the system on each other, and of any forces that are directed to or from the origin of co-ordinates; therefore if the system is not acted on by any foreign forces, except those passing through the origin of co-ordinates, and by the reciprocal action of the bodies that compose it; we shall have, by (b), and by integration,

$$d\,.\,\text{S}m\left(\frac{xdy - ydx}{dt}\right) = 0,\ \text{S}m\left(\frac{xdy - ydx}{dt}\right) = \text{A}'';$$

$$d\,.\,\text{S}m\left(\frac{xdz - zdx}{dt}\right) = 0,\ \text{S}m\left(\frac{xdz - zdx}{dt}\right) = \text{B}'';$$

$$d\,.\,\text{S}m\left(\frac{ydz - zdy}{dt}\right) = 0,\ \text{S}m\left(\frac{ydz - zdy}{dt}\right) = \text{C}'';$$

A'', B'', C'' being the arbitrary constants; but it is evident, by the method of obtaining (l), that we have

$$\frac{xdy - ydx}{dt} = a''(p' + \text{C}') + b''(q' + \text{B}') + c''(r' + \text{A}'),$$

and similarly for the like quantities; hence, if a, b, c, a' &c. are supposed invariable, and consequently $p'=0$, $q'=0$, $r'=0$, we have

$$a''\text{C}'+b''\text{B}'+c''\text{A}'=\text{A}'',\ a'\text{C}'+b'\text{B}'+c'\text{A}'=\text{B}'',\ a\text{C}'+b'\text{B}'+c'\text{A}'=\text{A}'' \quad . \quad (t).$$

But if we observe the values of A', B', C' in (K), and remark that, on the suppositions here made, the co-ordinates x', y', z' are fixed in position, and have the same origin as the other system of co-ordinates; we shall see that A', B', C' are constant, as we have before shown that A'', B'', C'' are so; and if we add the squares of (t)

$$\text{A}'^2+\text{B}'^2+\text{C}'^2=\text{A}''^2+\text{B}''^2+\text{C}''^2 \quad . \quad . \quad . \quad . \quad . \quad (u);$$

we also get from (t)

$$\text{A}'=c''\text{A}''+c'\text{B}''+c\text{C}'',\ \text{B}'=b''\text{A}''+b'\text{B}''+b\text{C}'',\ \text{C}'=a''\text{A}''+a'\text{B}''+a\text{C}'' \quad . \quad (v).$$

Now, the position of the plane $x'y'$ is arbitrary, let it be assumed so that

$$c''=\frac{\text{A}''}{\text{V}},\ c'=\frac{\text{B}''}{\text{V}},\ c=\frac{\text{C}''}{\text{V}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (w),$$

where $\text{V}=\sqrt{\text{A}''^2+\text{B}''^2+\text{C}''^2}$;

then, by (u), (v) and (w), we find

$$\text{A}'^2=\text{A}''^2+\text{B}''^2+\text{C}''^2,\ \text{B}'=0,\ \text{C}'=0;$$

and substituting (w) in (10), we get

$$\cos\theta=\frac{\text{A}''}{\text{V}},\ \sin\theta\cos\chi=\frac{\text{B}''}{\text{V}},\ \sin\theta\sin\chi=\frac{\text{C}''}{\text{V}} \quad . \quad . \quad . \quad . \quad (x).$$

The equations (w) agree with the formulæ given at p. 269, Vol. I., Mec. Anal., for the determination of the invariable plane, and (x) are given for the same purpose at p. 60, Vol. I., Mec. Cel., Com. p. 120.

Again, suppose the system to be rigid, and that the axes of x', y', z' are firmly connected with it, so that they do not vary with the time, and change their values only in passing from one body to another, then $\frac{dx'}{dt}$, $\frac{dy'}{dt}$, $\frac{dz'}{dt}$ each $=0$, therefore $\text{A}'=\text{B}'=\text{C}'=0$; hence (m) become

$$\frac{dp'}{dt}+qr'-rq'=\text{N}_{\prime},\ \frac{dq'}{dt}+rp'-pr'=\text{N}_{\prime\prime},\ \frac{dr'}{dt}+pq'-qp'=\text{N}_{\prime\prime\prime} \,.\ (y);$$

if the axes of x', y', z' are principal axes, $\text{D}=\text{E}=\text{F}=0$, and by restoring the values of p', q', r', equations (y) become

$$\text{A}\,.\frac{dp}{dt}+(\text{C}-\text{B})qr=\text{N}_{\prime},\ \text{B}\,.\frac{dq}{dt}+(\text{A}-\text{C})pr=\text{N}_{\prime\prime},\ \text{C}.\frac{dr}{dt}+(\text{B}-\text{A})pq=\text{N}_{\prime\prime\prime} \quad . \ (z),$$

or, writing the values of $\text{N}_{\prime}$, $\text{N}_{\prime\prime}$, $\text{N}_{\prime\prime\prime}$, as in (o), they become

$$\left.\begin{aligned}\text{A}\frac{dp}{dt}+(\text{C}-\text{B})qr&=\text{S}m(\text{R}'y'-\text{Q}'z'),\\ \text{B}\frac{dq}{dt}+(\text{A}-\text{C})pr&=\text{S}m(\text{P}'z'-\text{R}'x'),\\ \text{C}\frac{dr}{dt}+(\text{B}-\text{A})pq&=\text{S}m(\text{Q}'x'-\text{P}'y'),\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (z').$$

Since the position of the axes of x, in the plane xy, is arbitrary, we will suppose that it makes an indefinitely small angle with the line of intersection of the planes xy and $x'y'$; hence neglecting infinitely small

quantities of the second, &c., orders, $\cos\chi = 1$, $\sin\chi = \chi$. Therefore, neglecting quantities of the order χ, in a'', a', &c., as given in (10), restoring the values of $N_{,}$, $N_{,,}$, $N_{,,,}$, and multiply the equations in (z) by dt, they become

$$\left.\begin{array}{l} \text{A}dp + (\text{C}-\text{B})qrdt = -(\sin\theta d\text{N}''' - \cos\theta d\text{N}'')\sin\varphi + \cos\varphi d\text{N}', \\ \text{B}dq + (\text{A}-\text{C})prdt = -(\sin\theta d\text{N}''' - \cos\theta d\text{N}'')\cos\varphi - \sin\varphi d\text{N}', \\ \text{C}dr + (\text{B}-\text{A})pqdt = \quad \cos\theta d\text{N}''' + \sin\theta d\text{N}'' . \end{array}\right\} (z''),$$

which agree with (D) given at p. 74, Vol. I., Mec. Cel., or Com. p. 157. It is evident that all the formulæ which are applicable in the case of a rigid system, become applicable to the motion of a continuous solid, by changing m into dm, and then integrating relative to its mass in the expressions for A, B, &c., which in the rigid system depend on the integrals of finite differences.

Finally, it is evident that the system may be considered as having a momentary axis of rotation, and that whether it is a rigid system or a continuous solid; to find the momentary axis, we observe that, relative to it, we have $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$, $\frac{dz}{dt} = 0$; also since the system is rigid, or a solid, $\frac{dx'}{dt} = \frac{dy'}{dt} = \frac{dz'}{dt} = 0$; therefore, by ($g$), $\text{L} = \text{M} = \text{N} = 0$, or, by ($f$),

$$qz' - ry' = 0,\ rx' - pz' = 0,\ py' - qx' = 0;$$

which shows that the momentary axis passes through the origin of coordinates. If $a_{,}$, $b_{,}$, $c_{,}$ denote the cosines of the angles which the momentary axis makes with the axes of x', y', z', we shall have

$$\frac{p}{\sqrt{p^2+q^2+r^2}} = \frac{x'}{\sqrt{x'^2+y'^2+z'^2}} = a_{,},$$

$$\frac{q}{\sqrt{p^2+q^2+r^2}} = \frac{y'}{\sqrt{x'^2+y'^2+z'^2}} = b_{,},$$

$$\frac{r}{\sqrt{p^2+q^2+r^2}} = \frac{z'}{\sqrt{x'^2+y'^2+z'^2}} = c_{,};$$

which determine the position of this axis.

Since $\frac{dx'}{dt} = \frac{dy'}{dt} = \frac{dz'}{dt} = 0$, if we suppose $x' = y' = 0$, we get by (i),

$$z'\sqrt{p^2+q^2} = \frac{\sqrt{dx^2+dy^2+dz^2}}{dt},$$

for the velocity of a point on the axis of z', at the distance z' from the origin; but we have $\sqrt{\frac{p^2+q^2}{p^2+q^2+r^2}} = \sqrt{1-c_{,}^2}$ = sine of the angle made by the momentary axis with that of z'; therefore the perpendicular from the extremity of z' to the instantaneous axis is $= z'\sqrt{\frac{p^2+q^2}{p^2+q^2+r^2}}$; put w = the angular velocity round the momentary axis, and we shall have

$$w = z'(p^2+q^2) \div z'\sqrt{\frac{p^2+q^2}{p^2+q^2+r^2}} = \sqrt{p^2+q^2+r^2};$$

therefore $$p = a_{,}w,\ q = b_{,}w,\ r = c_{,}w,$$

where p, q, r are evidently the momentary rotations round the axes of x', y', z' severally; and therefore rotary velocities are compounded and resolved by the same rules as rectilineal ones.

Remarks. If in (a) we put $\mathrm{x}+x$, $\mathrm{y}+y$, $\mathrm{z}+z$ for x, y, z, and suppose $\mathrm{x}, \mathrm{y}, \mathrm{z}$ to denote the co-ordinates of the centre of gravity of the system, then, by the nature of that point, we have

$$\mathrm{S}mx = 0, \ \mathrm{S}my = 0, \ \mathrm{S}mz = 0;$$

therefore $$\mathrm{S}m\frac{d^2x}{dt^2} = 0, \ \mathrm{S}m\frac{d^2y}{dt^2} = 0, \ \mathrm{S}m\frac{d^2z}{dt^2} = 0 \quad . \ . \ . \ (a'),$$

and (a) become $$\frac{d^2\mathrm{x}}{dt^2} = \frac{\mathrm{S}m\mathrm{P}}{\mathrm{S}m}, \ \frac{d^2\mathrm{y}}{dt^2} = \frac{\mathrm{S}m\mathrm{Q}}{\mathrm{S}m}, \ \frac{d^2\mathrm{z}}{dt^2} = \frac{\mathrm{S}m\mathrm{R}}{\mathrm{S}m} \quad . \ . \ . \ (b')$$

and the centre of gravity moves as if all the bodies were united at that point, and the same forces were applied to them, and in the same manner.

Again if we substitute in (b), $\mathrm{x}+x$, $\mathrm{y}+y$, $\mathrm{z}+z$, for x, y, z; the terms depending on $\mathrm{x}, \mathrm{y}, \mathrm{z}$, will vanish from the equations, which will remain the same as before; and therefore placing the origin of co-ordinates, x, y, z in the centre of gravity of the system, the equations of rotation we have found will obtain, whether the centre of gravity is at rest or in motion.

Moreover, for a rigid system, in which x', y', z' are independent of t, if the values of $\frac{dx}{dt}$ given in (c) and (h) be compared, the values of $\mathrm{L}, \mathrm{M}, \mathrm{N}$ being substituted from (f), we shall have

$$x'\frac{da}{dt} + y'\frac{db}{dt} + z'\frac{dc}{dt} = (br - cq)x' + (cp - ar)y' + (aq - bp)z',$$

this, with the similar equations had by comparing the values of $\frac{dy}{dt}, \frac{dz}{dt}$, are identical equations, and the co-efficients of x', y', z' are equal; then

$$\left.\begin{array}{lll} da = (b\,r - c\,q)dt, & db = (c\,p - a\,r)dt, & dc = (a\,q - b\,p)dt, \\ da' = (b'r - c'q)dt, & db' = (c'p - a'r)dt, & dc' = (a'q - b'p)dt, \\ da'' = (b''r - c''q)dt, & db'' = (c''p - a''r)dt, & dc'' = (a''q - b''p)dt, \end{array}\right\} (c').$$

and hence we have

$$pda + qdb + rdc = 0, \ pda' + qdb' + rdc' = 0, \ pda'' + qdb'' + rdc'' = 0 \ (d'),$$

NOTE. The Title to the last Article should be "Motion of a System of Bodies round a fixed point."

We have also been obliged to defer Mr. Macully's Article on the "Summation of Trigonometrical Series." It will be inserted in the next Number.

METEOROLOGICAL OBSERVATIONS,

Made at St. Paul's College, near Flushing, L. I., for 37 successive hours, commencing at 6 A. M. of the 21st September, 1838, and ending at 6 P. M. of the following day.

Height of Barometer above low water in L. I. Sound 25 feet.
Lat. 40° 47′ 30″ N., Long. 73° 46′ W. nearly.

Hour.	Barometer Corrected.	Attached Therm'ter.	External Therm'ter.	Wet Bulb Therm'ter.	Winds —from—	Clouds —to—	Strength of wind.	REMARKS. Rain began at 7½ A. M.; continued to 2½ P. M.; ½ inch of rain fell.
6	30.104	63	62	61½	NE	SW	Gentle.	Stratus Clouds and mist.
7	.098	63	62½	62	"	"	"	"
8	.093	64	63	62½	"	"	"	Small rain and mist.
9	.091	64	64	63½	"	"	"	"
10	.088	65	64½	63¾	"	"	"	"
11	.077	66	66½	66	"	"	"	"
12	.071	67	67½	67	"	"	Light.	Heavy rain "
1	.071	67	68	67½	"	"	"	Light "
2	.054	68½	69½	69	"	"	"	" "
3	.037	69	71	70	SW	N	"	Clouds breaking.
4	.037	69	70	69½	SE	NW	Gentle.	Stratus Clouds.
5	.033	68½	68½	68	"	"	"	"
6	.032	68	67	66	"	"	Light.	"
7	.029	68	67	66	"	"	"	"
8	.029	68	67	66	"	"	"	"
9	.032	68	67	66	"	"	Fresh.	"
10	.030	68	67	66	"	"	Brisk.	"
11	.024	68	67	66	"	"	"	"
12	.017	68	67	66	"	"	High.	"
1	.007	68	67½	66½	"	"	Brisk.	"
2	29.998	68	67½	66½	"	"	"	"
3	.973	69	67½	67	"	"	"	"
4	.971	69	67½	67	"	"	Fresh.	"
5	.965	68	67½	67	"	"	"	"
6	.963	68	67	67	"	"	"	"
7	.958	70	69¾	69½	"	"	"	"
8	.954	70	70½	70	"	"	"	"
9	.946	72	76½	74	"	N	"	" breaking in E.
10	.943	73	76	74	"	"	"	"
11	.941	73	75	73	"	"	Gentle.	Cumuli.
12	.930	75	77	74½	"	"	"	"
1	.922	75	78	75	S	"	"	"
2	.918	76	78½	75	"	"	"	"
3	.918	75	76	72	"	"	"	Mostly clear.
4	.913	74	72½	70	"		"	Clear.
5	.913	73	71	68½	"		"	"
6	.912	72	69	67	"		"	"
	30.003	69	69	67¾		Means.		

METEOROLOGICAL OBSERVATIONS,

Made at St. Paul's College, near Flushing, L. I., for 37 successive hours, commencing at 6 A. M. of the 21st December, 1838, and ending at 6 P. M. of the following day.

Height of Barometer above low water in L. I. Sound, 25 feet.
Lat. 40° 47′ 30″ N., Long. 73° 46′ W. nearly.

Hour.	Barometer Corrected.	Attached Therm'ter.	External Therm'ter.	Wet Bulb Therm'ter.	Winds —from—	Clouds —to—	Strength of Wind.	REMARKS.
6	29.644	48	28	27	S. W.	N. E.	Gentle.	Stratus Clouds.
7	.642	48	28½	27½	"	"	"	"
8	.640	48	28	27½	"	"	"	"
9	.644	48	30	29½	"	"	"	"
10	.656	52	31½	31	"	"	"	"
11	.642	57	33	32	"	"	"	"
12	.620	67	33¼	32½	"	"	"	Cirrus Clouds.
1	.608	70	33½	32½	"	E.	"	"
2	.608	66	33	31½	"	"	"	"
3	.614	61	32½	31	"	"	"	"
4	.620	61	32	30½	"	"	"	Mostly clear.
5	.610	68	30	29	"	"	"	Thin Stratus Clouds.
6	.616	68	30	29	"	"	"	"
7	.612	69	29	28½	"	"	Light.	Thin misty clouds to the W.
8	.626	67	28	27½	"	"	"	[the stars visible through them.
9	.628	64	29	28½	"	"	"	"
10	.632	64	29	28½	"	"	"	"
11	.618	67	30	29	W.	"	"	Clouds darker.
12	.606	68	30½	29½	N.		"	"
1	.607	66	30½	29½	"		"	"
2	.595	65	31	30			Calm.	"
3	.567	62	32	31			"	"
4	.537	59	34	32½	E.		Light.	Bank of thin stratus clouds.
5	.531	58	34½	32½	"		"	"
6	.508	57	37	34	"	S. W.	Gentle.	"
7	.506	52	35	33½		"	Calm.	"
8	.496	53	35	34		N. W.	"	Cumulo—stratus clouds.
9	.491	50	36½	36		N. E.	"	"
10	.484	52	37	36½		"	"	"
11	.484	49	41½	41		"	"	"
12	.478	49	40	39½	WSW.	"	Light.	"
1	.457	51	38½	38		"	Calm.	"
2	.454	57	38¼	38	S. W.	"	Light.	Clouds lighter.
3	.453	62	38	37	"	"	"	Cirrus clouds.
4	.480	68	36¼	35	"	"	"	"
5	.493	69	32¼	31	"		"	Mostly clear.
6	.532	52	30½	29	"		"	"
	29.568	59¼	32¼	31¼	Means.			

METEOROLOGICAL OBSERVATIONS,

Made at St. Paul's College, near Flushing, L. I., for 37 successive hours, commencing at 6 A. M. of the 21st March, 1839, and ending at 6 P. M. of the following day.

Height of Barometer above low water in L. I. Sound, 25 *feet.*
Lat. 40° 47′ 30″ *N., Long.* 73° 46′ *W. nearly.*

Hour.	Barometer Corrected.	Attached Therm'ter.	External Therm'ter.	Wet Bulb Therm'ter.	Winds —from—	Clouds —to—	Strength of wind.	REMARKS. Rain began at 5 P. M. of the 20th, and ended at 1½ A. M. of the 22d.
6	30.056	55	34	33½	N. E.	S. W	Brisk.	Stratus Clouds—small rain.
7	.038	55	34¾	34	"	"	"	"
8	.023	55	36	35½	"	"	"	"
9	.016	56	35½	35	"	"	"	"
10	.011	58	36	35½	"	"	"	"
11	29.974	60	36	35½	"	"	"	"
12	.940	58	37	36½	"	"	"	"
1	.903	58	37½	37	"	"	"	"
2	.864	59	37½	37	"	"	"	"
3	.839	60	37	36½	"	"	"	"
4	.833	60	36½	36	"	"	High.	"
5	.833	60	36½	36	"	"	"	"
6	.822	58	36	35½	"	"	"	"
7	.820	56	36	35½	"	"	"	"
8	.812	56	36	35½	"	"	"	"
9	.804	56	36	35½	"	"	"	"
10	.787	54	36	35½	"	"	Brisk.	"
11	.775	52	36	35½	"	"	"	"
12	.767	51	36	35½	"	"	"	"
1	.752	50	36½	36	"	"	Gentle.	"
2	.744	49	36¾	36	"	"	"	Stratus Clouds.
3	.733	50	36¾	36	"	"	"	"
4	.735	49	36¾	36	"	"	"	"
5	.733	50	36¼	36	N. W.	S.	"	"
6	.772	50	37½	36½	W.	E.	"	"
7	.779	50	39½	39	S. W.		"	Clear.
8	.773	52	41	40	W.		Brisk.	"
9	.785	52	43	42	"	E.	Light.	Few Cirrus Clouds.
10	.776	58	47½	46	WNW	"	"	"
11	.767	64	53½	46½	"	"	Brisk.	Cumulus Clouds.
12	.758	65	56	49	W	"	"	"
1	.747	64	53½	47½	"	"	"	"
2	.738	62	56	49½	"	"	"	"
3	.738	62	56	50	"		Gentle.	Cirrus Clouds.
4	.743	61	53½	48	"		"	"
5	.746	60	48½	44½	"		Light.	Clear.
6	.752	59	44½	41½	"		"	Few Cirrus Clouds.
	29.824	56½	43¼	41½		Means.		

THE

MATHEMATICAL MISCELLANY.

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JUNIOR DEPARTMENT.

ARTICLE IV.

HINTS TO YOUNG STUDENTS.—No. VII.

35. An equation, not identical, among several quantities expresses a *relation* among these quantities; it makes any one of these quantities dependent on the others for its value, and it is thence called a *function* of these quantities. Thus, it is shown in Geometry that the three sides of a right triangle are connected by the relation

$$(12) \quad . \quad . \quad . \quad . \quad a^2 + b^2 = h^2,$$

h being the hypothenuse of the triangle; and here a is a function of b and h, inasmuch as its value is dependent upon theirs; or b is a function of a and h, or h of a and b. If one of these quantities, as h, be considered constant, then the equation expresses the relation between the two legs of all right triangles that have this hypothenuse, and either of them is a function of the other.

So if there be, among the m quantities x, y, z, &c., n independent relations, expressed analytically by so many equations, then any n of these quantities are dependent upon the other $m-n$ quantities for their values, or they are functions of these quantities. If $m=n$, the quantities are absolutely determinable from the equations, and therefore cannot vary.

36. When one or more quantities vary, any function of them varies also; the quantity and mode of this variation being dependent on the nature of the function. This kind of dependent variation is the first idea the student has to master in applying his analysis to any practicable purpose; and it will greatly facilitate his progress if he can acquire some knowledge of it from the analytical relations alone. Thus let a function a of b, be

$$a = 2b - 3;$$

then by imagining b to vary through all magnitudes, from $-\infty$ to $+\infty$, a will also vary through all magnitudes, from $-\infty$ to $+\infty$; this will be seen by substituting successively greater values for b, such as

$$-100, -10, -1, 0, 1, 10\ 100, \&c.$$

and the corresponding values of a will be

$$-203,\ -23,\ -5,\ -3,\ -1,\ 17,\ 197,\ \&c.$$

It is seen that a varies twice as fast as b does, and by taking values of b that have yet smaller differences, the mind can easily acquire the idea of the variation of both b and a through magnitudes having an insensible difference.

A variable quantity, in changing its sign, must pass either through zero or infinity; thus, while a varies by insensible differences between 1 and 10, b varies also by insensible differences between -1 and 17, and must during this variation pass through zero; this will evidently be when

$$2b-3=0, \text{ or } b=1\tfrac{1}{2}.$$

The converse, however, does not follow. That is, a variable function in passing through zero or infinity, does not necessarily change its sign; thus if the function a were

$$a=(2b-3)^2,$$

it could never change its sign, but while b varies from $-\infty$ to $+\infty$, a will decrease from $+\infty$ to 0, which value it would have when $b=1\frac{1}{2}$, and it afterwards increases again to $+\infty$.

37. If, in equation (12), h be considered constant, and a a function of b, the limiting values of b are $-h$ and $+h$; for if b were $<-h$ or $>+h$, a^2 would be <0, and therefore a imaginary. Making then, b vary from $-h$ to $+h$, there will be, for every value of b, two values of a, equal to each other, but with contrary signs, expressed by the forms

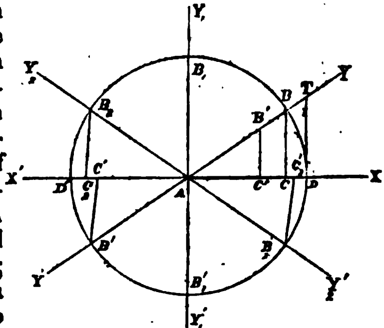
$$a=+\sqrt{h^2-b^2},\ a=-\sqrt{h^2-b^2},$$

these values increase numerically, from 0 when $b=-h$, to h when $b=0$, and then decrease again to 0 when $b=+h$.

On the contrary, if b were considered constant, and a a function of h, the limiting values of h would be $-b$ and $+b$; for if h were between these limits a^2 would be <0, and a imaginary. Making then h increase from $-\infty$ to $-b$, the two values of a, expressed as before, decrease numerically from ∞ to 0; and while h increases from b to ∞, the values of a increase from 0 to ∞.

The whole theory of the variation of variables and their functions, constitutes the science called the Differential Calculus; all that the student can do in the present state of his progress, when he is supposed to have acquired a pretty correct knowledge of the direct processes of Algebra, is to gain the idea of the variation of a magnitude between two fixed or finite limits, by successive, but insensibly small increments, and the consequent variation of magnitudes that depend upon it for their values by laws expressed in algebraic equations. A very few simple examples, like the above, which may be selected at pleasure, will familiarize the idea, especially if he calls to his aid some elementary ideas of motion, as I shall presently show.

38. "The *position* of a point," says Prof. Peirce, in his excellent treatise on Geometry, "is determined by its *Distance* and *Direction* from any known point." It may also be determined, on a plane by its distances from two fixed lines on that plane, or in space by its distances from three fixed planes. If A be the known point, the distance



of the point B from A is the length of the part AB, of the indefinite straight line Y′AY drawn through these two points, intercepted between them, and expressed in known units, as yards, or feet, &c. In order to distinguish on which *side* of the fixed point A the point B is situated, the distances on one side of A are marked with the sign +, and those on the other by the sign —; thus the distance of a point from a fixed point may vary from $-\infty$ to $+\infty$, and it is zero when the point coincides with A. The idea of this variation may be acquired by imagining the *progression* of the point B along the line Y′AY, its motion beginning at an infinite distance from A on the negative side AY′, and moving through A to an infinite distance from A on the positive side AY. Two consecutive positions of the point in its motion are infinitely near each other; and this minute distance is the *infinitely small increment* of the varying distance. The distance of a point B from a fixed line AX, is the length of a perpendicular BC, drawn from the point to the line; these distances being marked + when the point is on one side of the line, and — when it is on the other.

The direction of the point B from A on a plane, is determined in reference to a known or *fixed* direction, such as that of the line AX, which is called the *Angular axis*, and the difference of direction of the two straight lines AY and AX is called an *angle*. The variation of angular magnitude may be conceived by imagining the indefinite line AY to revolve about the fixed point A, beginning from coincidence with AX the angular axis, in which position the angle A is zero, since the lines have then no difference of direction. As the revolution of the line continues, the angle A increases by insensible differences. When AY becomes perpendicular to AX, $A = 90° = \frac{1}{2}\pi$; when AY takes the position AX′, which is the prolongation of AX, $A = 180° = \pi$; when AY becomes again perpendicular to AX, or takes the position AY′, $A = \frac{3}{2}\pi$; and when it coincides again with AX, $A = 360° = 2\pi$. By continuing the revolution, the line passes over the same positions as before; but the angles are 360° greater in any position than during the first revolution, and thus the angle may be conceived to increase from 0° to ∞, by an indefinite number of revolutions. Negative angles are counted in the opposite direction from the angular axis, and correspond to a rotation of AY about A, from left to right, instead of from right to left as before, varying in the same manner from 0° to $-\infty$. Thus if the angle $\text{YAX} = \varphi$, φ being $< 360°$, the position of the line AY, is determined either from the positive value $n \,.\, 2\pi + \varphi$, or the negative value $-(n+1)\,2\pi + \varphi$, n representing the number of complete revolutions that the line has made.

39. If from different points, B, B′, &c. of the line AY, perpendiculars be let fall upon the angular axis AX, they will form with AY and AX right angled triangles ABC, AB′C′, &c., which have a common angle A; they

are therefore similar, and the ratios of all their homologous sides is the same: these ratios are therefore functions of the angle A, and enable us to connect angular with linear magnitude. If the hypothenuse of any one of these right triangles, as ABC, be represented by h, and the two sides, BC by a, and AC by b; these three sides have six different ratios, which are the six *trigonometrical functions* of the angle A, and are named thus:—

(13). $\frac{a}{h} = \sin. A$, or the *sine* of the angle A,

(14). $\frac{b}{h} = \cos. A$, or the *cosine* of the angle A,

(15). $\frac{a}{b} = \tan. A$, or the *tangent* of the angle A,

(16). $\frac{h}{a} = \text{cosec.} A$, or the *cosecant* of the angle A,

(17). $\frac{h}{b} = \sec. A$, or the *secant* of the angle A,

(18). $\frac{b}{a} = \cot. A$, or the *cotangent* of the angle A.

These six functions are not independent of each other, since three quantities have but two independent ratios, and these three quantities are also related to each other as in equation (12); it follows that any five of these ratios is dependent upon the sixth. To find the equations which express their relations, multiply (13), (14), (15), severally by (16), (17), (18), then

$$(19). \quad \begin{cases} 1 = \sin A \text{ cosec } A, \\ 1 = \cos A \sec A, \\ 1 = \tan A \cot A; \end{cases}$$

Divide (13) by (14), member by member, and

$$(20). \quad \frac{a}{b} = \tan A = \frac{\sin A}{\cos A},$$

consequently,

$$(21). \quad \cot A = \frac{1}{\tan A} = \frac{\cos A}{\sin A}.$$

Divide the terms of equation (12), by h^2, b^2, a^2, severally;

$$\frac{a^2}{h^2} + \frac{b^2}{h^2} = 1,$$

$$\frac{a^2}{b^2} + 1 = \frac{h^2}{b^2},$$

$$1 + \frac{b^2}{a^2} = \frac{h^2}{a^2},$$

or writing the proper functions of the angle for these ratios,

$$(22). \quad \begin{cases} \sin^2 A + \cos^2 A = 1, \\ \tan^2 A + 1 \quad = \sec^2 A, \\ \cot^2 A + 1 \quad = \text{cosec}^2 A. \end{cases}$$

The student should be able to express any one of these six functions of the angle A in terms of any other one. For instance, in terms of the *sine:*

$$\cos A = \sqrt{1 - \sin^2 A}$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\sqrt{1 - \sin^2 A}},$$

$$\cot A = \frac{\cos A}{\sin A} = \frac{\sqrt{1 - \sin^2 A}}{\sin A},$$

$$\operatorname{cosec} A = \frac{1}{\sin A},$$

$$\sec A = \frac{1}{\cos A} = \frac{1}{\sqrt{1 - \sin^2 A}}.$$

40. From the relation between the sine and cosine of an angle in the first of (22), and from the discussion of a similar one in § 37, it is evident that the sine and cosine vary only between the limits -1 and $+1$. But these quantities are functions of the angle, and therefore must vary with the angle; the dependency will be more immediately seen by finding a system of straight lines, proportionable to them. If, with centre A and radius = AB = AD = the linear unit, a circle be described, and from the point B, where the circumference intersects the line AY in any of its positions, the perpendicular BC be let fall on the angular axis, and at D the line DT be drawn tangent to the circle and intersecting AY in T, we have from the definitions in (13, (14) . . . .,

$$(23), \quad \left\{\begin{array}{l} \sin A = \dfrac{BC}{AB} = \dfrac{BC}{1} = BC, \\ \cos A = \dfrac{AC}{AB} = \dfrac{AC}{1} = AC, \\ \tan A = \dfrac{DT}{AD} = \dfrac{DT}{1} = DT, \\ \sec A = \dfrac{AT}{AD} = \dfrac{AT}{1} = AT. \end{array}\right.$$

These lines are therefore numerically represented by the same quantities as the functions to which they stand opposite, and are proportional, to them; but it must be recollected that these lines are expressed in denominate numbers, and are in fact linear magnitudes, while the trigonometrical functions are ratios of lines, or abstract numbers; thus, if radius AB = 1 yard and BC = $\frac{2}{3}$ yard, we have also $\sin A = \dfrac{\frac{2}{3}\text{ yard}}{1\text{ yard}} = \frac{2}{3}$.

Now while the line AY revolves about the point A, the point of intersection B, moves from coincidence with D round the circumference, passing through BB, when $A = \frac{1}{2}\pi$, through D′ when $A = \pi$, through BB′, when $A = \frac{3}{2}\pi$, through D when $A = 2\pi$, and similarly for the successive revolutions. The line BC = sin A is the *distance* of the point B from the angular axis, and is therefore + when B is above AX, and — when it is below it. This settled, it is at once seen that

When $A = 0$, $\sin A = 0$;
While A varies from 0 to $\frac{1}{2}\pi$, $\sin A$ increases from 0 to 1;
When $A = \frac{1}{2}\pi$, $\sin A = 1$;
While A varies from $\frac{1}{2}\pi$ to π, $\sin A$ decreases from 1 to 0;
When $A = \pi$, $\sin A = 0$;
While A varies from π to $\frac{3}{2}\pi$, $\sin A$ decreases from 0 to -1;
When $A = \frac{3}{2}\pi$, $\sin A = -1$;
While A varies from $\frac{3}{2}\pi$ to 2π, $\sin A$ increases from -1 to 0;
When $A = 2\pi$, $\sin A = 0$;

and so on for successive revolutions. It follows that, if φ be any angle, and n any integer.

$$(24)\quad \begin{cases} \sin\varphi > 0, \text{ when } \varphi > 2n\pi \text{ and } \varphi < (2n+1)\pi; \\ \sin\varphi < 0, \text{ when } \varphi > (2n-1)\pi \text{ and } \varphi < 2n\pi; \\ \sin\varphi = 0, \text{ when } \varphi = n\pi; \\ \sin\varphi = 1, \text{ when } \varphi = (2n+\frac{1}{2})\pi; \\ \sin\varphi = -1, \text{ when } \varphi = (2n-\frac{1}{2})\pi; \end{cases}$$

Similarly, the line $AC = \cos A$ is the distance from the foot of the sine to the centre, and is therefore $+$ when C is to the right of A, and $-$ when it is to the left; hence

When $A = 0$, $\cos A = 1$;
While A varies from 0 to $\frac{1}{2}\pi$, $\cos A$ decreases from 1 to 0;
When $A = \frac{1}{2}\pi$, $\cos A = 0$;
While A varies from $\frac{1}{2}\pi$ to π, $\cos A$ decreases from 0 to -1
When $A = \pi$, $\cos A = -1$;
While A varies from π to $\frac{3}{2}\pi$, $\cos A$ increases from -1 to 0;
When $A = \frac{3}{2}\pi$, $\cos A = 0$;
While A varies from $\frac{3}{2}\pi$ to 2π, $\cos A$ increases from 0 to 1;
When $A = 2\pi$, $\cos A = 1$;

and so on for successive revolutions; hence

$$(25)\quad \begin{cases} \cos\varphi > 0, \text{ when } \varphi > (2n-\frac{1}{2})\pi \text{ and } \varphi < (2n+\frac{1}{2})\pi; \\ \cos\varphi < 0, \text{ when } \varphi > (2n+\frac{1}{2})\pi \text{ and } \varphi < (2n+\frac{3}{2})\pi; \\ \cos\varphi = 0, \text{ when } \varphi = (n+\frac{1}{2})\pi; \\ \cos\varphi = 1, \text{ when } \varphi = 2n\pi; \\ \cos\varphi = -1, \text{ when } \varphi = (2n+1)\pi. \end{cases}$$

From the variation of the line $DT = \tan A$, or from (20), (24), and (25);

$$(26)\quad \begin{cases} \tan\varphi > 0, \text{ when } \varphi > n\pi \text{ and } \varphi < (n+\frac{1}{2})\pi; \\ \tan\varphi < 0, \text{ when } \varphi > (n-\frac{1}{2})\pi \text{ and } \varphi < n\pi; \\ \tan\varphi = 0. \text{ when } \varphi = n\pi; \\ \tan\varphi = +\infty, \text{ when } \varphi = (2n+\frac{1}{2})\pi; \\ \tan\varphi = -\infty, \text{ when } \varphi = (2n-\frac{1}{2})\pi; \end{cases}$$

By (19) it is apparent that the cosecant has the same sign as the sine, the secant as the cosine, and the cotangent as the tangent.

41. If k be a given number, positive or negative, which is either equal to -1 or $+1$, or is comprehended between these limits; there is some angle φ, either equal to $-\frac{1}{2}\pi$ or $+\frac{1}{2}\pi$, or is comprehended between these limits, and which may be found from the inspection of a table of sines, such that

$$\sin\varphi = k;$$

but there are an infinite number of other angles which have the same sine, and if any one of these angles be represented by θ, so that

$$\sin\theta = \sin\varphi;$$
then, $\sin\theta - \sin\varphi = 0$,
or, $2\cos\frac{1}{2}(\theta+\varphi)\sin\frac{1}{2}(\theta-\varphi) = 0$;
which is satisfied by making, either
$\cos\frac{1}{2}(\theta+\varphi) = 0$, and by (25), $\frac{1}{2}(\theta+\varphi) = (n+\frac{1}{2})\pi$;
or, $\sin\frac{1}{2}(\theta-\varphi) = 0$, and by (24), $\frac{1}{2}(\theta-\varphi) = n\pi$; hence
(27), $\theta = 2n\pi + \varphi$, or $= (2n+1)\pi - \varphi$,
where n is any integer. Similarly there is an angle φ, either equal to π or to 0 or is comprehended between these limits, as may be found from the tables, such that
$$\cos\varphi = k;$$
and if θ be any other angle having the same cosine, or such that
$$\cos\theta = \varphi;$$
then $\cos\theta - \cos\varphi = 0$,
or $2\sin\frac{1}{2}(\theta+\varphi)\sin\frac{1}{2}(\theta-\varphi) = 0$;
and therefore, either
$\sin\frac{1}{2}(\theta+\varphi) = 0$, and by (24), $\frac{1}{2}(\theta+\varphi) = n\pi$,
or $\sin\frac{1}{2}(\theta-\varphi) = 0$, and by (24), $\frac{1}{2}(\theta-\varphi) = n\pi$; and
(28) $\theta = 2n\pi \pm \varphi$,
n being any integer. Again, if k be any number whatever, positive or negative, an angle, φ, may be found from the tables, either equal to $-\frac{1}{2}\pi$ or $+\frac{1}{2}\pi$, or is comprehended between these limits, such that
$$\tan\varphi = k,$$
and if θ be any other angle having the same tangent, so that
$$\tan\theta = \tan\varphi;$$
then $\tan\theta - \tan\varphi = 0$,
or $\dfrac{\sin(\theta-\varphi)}{\cos\theta\cos\varphi} = 0$,
or $\sin(\theta-\varphi) = 0$;
and, by (24), $\theta - \varphi = n\pi$, hence
(29) $\theta = n\pi + \varphi$,
where n is any integer. Equations (27), (28), (29), give all the angles which have the same sine, cosine or tangent.

ARTICLE V.

SOLUTIONS TO THE QUESTIONS PROPOSED IN NUMBER VII.

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(43). QUESTION I. *By* ——.

Transform the numbers 25 and 389 into a system of notation whose base, or scale of relation is 3; multiply them in that state, exhibiting the process, and transform the result to the decimal scale.

SOLUTION. *By Mr. W. B. Benedict, Upperville, Va.*

It is obvious that, whatever be the scale of relation of the particular numbers employed, in adding, multiplying and dividing, we must divide

by the radix, setting down the excess. By the process explained in my solution to question (32).

$$389 = 112102, \text{ in the ternary scale,}$$
$$25 = \quad 221 \quad "$$

```
   112102
  1001211
 1001211
 ---------
111100012  their product;
```

and $111100012 = 1.3^8 + 1.3^7 + 1.3^6 + 1.3^5 + 1.3 + 2 = 9725.$

(44.) QUESTION IV. *By* ——.

Prove that
$$(n+1)(n+2)\ldots\ldots(n+n) = 2^n \times 1.3.5\ldots\ldots(2n-1).$$

SOLUTION. *By Mr. Alfred Birdsall, Clinton Liberal Institute.*

We have
$$(n+1)(n+2)\ldots\ldots 2n = \frac{1.2.3\ldots n}{1.2.3\ldots n} \times (n+1)(n+2)\ldots\ldots 2n$$
$$= \frac{1.2.3\ldots 2n}{1.2.3\ldots n}$$
$$= \frac{2.4.6\ldots 2n \times 1.3.5\ldots\ldots(2n-1)}{1.2.3\ldots\ldots n}$$
$$= \frac{2^n \times 1.2.3\ldots n \times 1.3.5\ldots(2n-1)}{1.2.3\ldots\ldots n}$$
$$= 2^n \times 1.3.5\ldots\ldots(2n-1).$$

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(45). QUESTION III. *By L., Murray Co., Geo.*

To find x, y, z, there are given the three equations

(1), $\quad xz = y^2,$

(2), $\quad 2xz - x - y - z = 1,$

(3), $\quad x^2 + y^2 + z^2 = 3x + 3y + 3z.$

SOLUTION. *By Mr. Daniel Kirkwood, York, Pa.*

We have, from (1), $\quad y = \sqrt{xz}, \quad$ and by substitution

(4), $\quad 2xz - x - \sqrt{xz} - z = 1,$

(5), $\quad x^2 + xz + z^2 = 3(x + \sqrt{xz} + z);$

dividing (5) by $x + \sqrt{xz} + z$,

(6), $\quad x - \sqrt{xz} + z = 3,$

by adding (4) and (6), $\quad xz - \sqrt{xz} = 2;$

hence, $\quad \sqrt{xz} = 2 \text{ or } -1;$

then $\quad x - 2\sqrt{xz} + z = 1 \text{ or } 4;$

$x + 2\sqrt{xz} + z = 9 \text{ or } 0;$

$\sqrt{x} - \sqrt{z} = \pm 1 \text{ or } \pm 2;$

$\sqrt{x} + \sqrt{z} = \pm 3 \text{ or } 0;$

$\sqrt{x} = \pm 2 \text{ or } \pm 1,$

$\sqrt{z} = \pm 1 \text{ or } \pm 1,$

$x = 4 \text{ or } 1,$

$z = 1 \text{ or } 1,$

$y = \sqrt{xz} = \pm 2 \text{ or } \pm 1.$

—— There are also imaginary roots found from the equation

$$x + \sqrt{xz} + z = 0.$$

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(46). QUESTION IV. *By Mr. E. H. Delafield.*

Find what relation must have place among the co-efficients of the equation

$$x^4 + \text{A}x^3 + \text{B}x^2 + \text{C}x + \text{D} = 0,$$

so that the process for taking away its second term, may also take away its fourth term at the same time.

SOLUTION. *By Mr. J. K. Anderson, St. Paul's College.*

Let $\qquad x = y + a,$ and the equation becomes

$$\begin{array}{l|l|l|l}
y^4 + 4a & y^3 + 6a^2 & y^2 + 4a^3 & y + a^4 = 0, \\
\quad + \text{A} & \quad + 3\text{A}a & \quad + 3\text{A}a^2 & \quad + \text{A}a^3 \\
 & \quad + \text{B} & \quad + 2\text{B}a & \quad + \text{B}a^2 \\
 & & \quad + \text{C} & \quad + \text{C}a \\
 & & & \quad + \text{D}
\end{array}$$

and, in order that the second and fourth terms may be taken away by the same operation, we must have

$$4a + \text{A} = 0,$$

$$4a^3 + 3\text{A}a^2 + 2\text{B}a + \text{C} = 0,$$

and by eliminating a, we find

$$\text{A}^3 - 4\text{AB} + 8\text{C} = 0;$$

this is the same relation as in question (36), page 17; and therefore when this relation has place among the co-efficients of an equation of the fourth degree, it can be put into any one of the three forms

$$(x^2 + ax)^2 + b(x^2 + ax) + c = 0,$$

$$(x^2 + a'x + b')^2 + c' = 0,$$

$$(x - a'')^4 + b''(x - a'')^2 + c'' = 0.$$

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(47). QUESTION V. *By* ——.

Let $\quad \mathrm{x}_0 = 1,\ 2\mathrm{x}_1 = x + \dfrac{1}{x},\ 2\mathrm{x}_2 = x^2 + \dfrac{1}{x^2},\ 2\mathrm{x}^3 = x^3 + \dfrac{1}{x^3},$ &c.;

prove that

$$\mathrm{x}_{n+1} + \mathrm{x}_{n-1} = 2\mathrm{x}_n\mathrm{x}_1.$$

SOLUTION. *By Mr. W. B. Benedict.*

By the notation, $\qquad 2\mathrm{x}_{n+1} = x^{n+1} + x^{-n-1}$

$\qquad\qquad\qquad\qquad 2\mathrm{x}_{n-1} = x^{n-1} + x^{-n+1}$

then $\qquad 2\mathrm{x}_{n+1} + 2\mathrm{x}_{n-1} = x^{n+1} + x^{-n-1} + x^{n-1} + x^{-n+1}$

$$\begin{aligned}
&= x^n.x + x^{-n}.x^{-1} + x^n.x^{-1} + x^{-n}.x \\
&= (x^n + x^{-n})\,x + (x^{-n} + x^n)\,x^{-1} \\
&= (x^n + x^{-n})\,(x + x^{-1}) \\
&= 2\mathrm{x}_n.2\mathrm{x}_1
\end{aligned}$$

and $\qquad \mathrm{x}_{n+1} + \mathrm{x}_{n-1} = 2\mathrm{x}_n\mathrm{x}_1.$

(48). QUESTION VI. *By* ——.

Adapt the relations of the sides and angles of a plane triangle to the case where the sides are in arithmetical progression, and find the area of the triangle.

SOLUTION. *By Mr. J. K. Anderson.*

When the sides are in arithmetical progression,

$$a-b=b-c, \text{ or}$$
$$(1) \qquad 2b=a+c.$$

The general relation, $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$, being written in (1) gives

$$(2) \qquad 2\sin B=\sin A+\sin C;$$

or the sines of the angles are in arithmetical progression: eliminating any one of the three angles between this and the relation.

$$A+B+C=\pi,$$
$$(3) \qquad \begin{cases} 2\cos\frac{1}{2}(A+C)=\cos\frac{1}{2}(A-C), \\ 2\sin\frac{1}{2}B=\sin(A+\frac{1}{2}B)=\sin(C+\frac{1}{2}B), \end{cases}$$

and from these, if one angle be given, the others can be found. If (1) be written in the general relation

$$b^2=a^2+c^2-2ac\cos B.$$

it becomes $\quad (4) \qquad 3b^2=4ac\cos^2\frac{1}{2}B;$

and for the area,

$$(5) \qquad S=\tfrac{1}{2}ac\sin B=\frac{3b^2\sin B}{8\cos^2\frac{1}{2}B}$$
$$=\tfrac{3}{4}b^2\tan\tfrac{1}{2}B.$$

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(49). QUESTION VII. *By* β.

Prove that, θ being any angle,

$$\operatorname{cosec}\theta-2\operatorname{cosec}3\theta=\cot 3\theta\sec\theta.$$

SOLUTION. *By Mr. J. V. Campbell, St. Paul's College.*

Using the known formulas

$$\sin 3\theta=\sin\theta(2\cos 2\theta+1),$$
$$\cos 3\theta=\cos\theta(2\cos 2\theta-1):$$

we have $\operatorname{cosec}\theta-2\operatorname{cosec}3\theta=\frac{1}{\sin\theta}-\frac{2}{\sin 3\theta}$

$$=\frac{2\cos 2\theta+1}{\sin 3\theta}-\frac{2}{\sin 3\theta}$$
$$=\frac{2\cos 2\theta-1}{\sin 3\theta}$$
$$=\frac{\cos\theta(2\cos 2\theta-1)}{\cos\theta\sin 3\theta}$$
$$=\frac{\cos 3\theta}{\cos\theta\sin 3\theta}$$
$$=\cot 3\theta\sec\theta$$

(50). QUESTION VIII. *By* ———.

In Navigation, find the bearing and distance from a given place on the earth's surface to another one, differing from the former 10° in latitude and 10° in longitude.

SOLUTION. *By Mr. E. H. Delafield, St Paul's College.*

Denote the distance of the two places from the north pole by d and $d + 10°$; then their middle latitude will be $90° - (d + 5°)$, and their diff. long. will be $= 10° = 600$ miles; and we have

$$\text{departure} = \text{diff. long.} \times \text{cos. mid. lat.}$$
$$= 600 + \sin(d + 5°),$$
$$\tan \text{bearing} = \frac{\text{dep.}}{\text{diff. lat.}} = \frac{600 \sin (d + 5°)}{600} = \sin (d + 5°);$$
$$\text{distance} = \text{diff. lat} \times \text{sec. bearing}$$
$$= 600 \sqrt{1 + \sin^2 (d + 5°)}.$$

When the places are near either pole, it is necessary to use the formulas in Art. I. Vol. I. Math. Miscel., which give

$$\text{cot. course} = \frac{\text{RM}}{\text{diff. long.}} . \log \left(\frac{\tan\frac{1}{2}\ (90° - l)}{\tan\frac{1}{2}\ (90° - \text{L})}\right)$$
$$= \frac{\text{RM}}{600} . \log \left(\frac{\tan \frac{1}{2} d}{\tan (\frac{1}{2} d + 5°)}\right),$$
$$\text{distance} = 600 \times \text{sec. course}.$$

---

(51). QUESTION IX. *By* ———.

The earth being supposed a perfect sphere, draw a great circle arc between any two points on the surface which differ from each other 10° in latitude and 10° in longitude; find its length, and the angle it makes with the meridian of either place.

SOLUTION. *By β.*

The polar distances d and $d + 10°$ will be the two sides b and a of a spherical triangle, the included angle $\text{C} = 10°$ being at the north pole; hence for the third side c, or the length of the arc drawn between them,

$$\cos c = \cos a \ \cos b + \sin a \sin b \cos \text{C}$$
$$= \cos (a - b) \cos^2 \tfrac{1}{2}\text{C} + \cos (a + b) \sin^2 \tfrac{1}{2}\text{C}$$
$$= \cos 10° \cos^2 5° + \cos (2d + 10°) \sin^2 5°;$$
$$\therefore \sin^2 \tfrac{1}{2} c = \sin^2 5° \ (\cos^2 5° + \sin^2 \overline{d + 5°});$$

and, from Napier's Analogies,

$$\tan\tfrac{1}{2}\ (\text{A} - \text{B}) = \cot\tfrac{1}{2}\text{C} . \frac{\sin\frac{1}{2}\ (a - b)}{\sin\frac{1}{2}\ (a + b)} = \frac{\cos 5°}{\sin\ (d + 5°)},$$
$$\tan\tfrac{1}{2}\ (\text{A} + \text{B}) = \cot\tfrac{1}{2}\text{C} . \frac{\cos\frac{1}{2}\ (a - b)}{\cos\frac{1}{2}\ (a + b)} = \frac{\cos^2\ 5°}{\sin 5° \sin (d + 5°)},$$

from which the two angles may be determined. After these are found c may be had by logarithms; for

$$\sin (d + 5°) = \cos 5° \ \cot \tfrac{1}{2}\ (\text{A} - \text{B}),$$
$$\therefore \sin^2 \tfrac{1}{2} c = \sin^2 5° \ (\cos^2 5° + \cos^2 5° \cot^2 \tfrac{1}{2}\ (\text{A} - \text{B}))$$
$$= \sin^2 5° \cos^2 5° \ \text{cosec}^2 \tfrac{1}{2}\ (\text{A} - \text{B})$$

$$\text{and} \qquad \sin\tfrac{1}{2}c = \frac{\sin 10^\circ}{2\sin\tfrac{1}{2}(A - B)}$$

---

(52). QUESTION X. *By* ——.

Find the points of intersection of the two ellipses

$$7y^2 + 4x^2 = 28,$$
$$6y^2 + 5x^2 = 30,$$

related to the same axes of co-ordinates, and determine the angles they make with each other at these points.

SOLUTION. *By L., Murray Co., Geo.*

By solving the two equations for y and x, we shall have the co-ordinates of the four points of intersection,

$$y = \pm 2\sqrt{\tfrac{5}{11}}, \qquad x = \pm\sqrt{\tfrac{42}{11}};$$

let α and α' be the angles, the tangents of these curves at one of these points, make with the axis of x, and ν the angle they, or the curves, make with each other; the equations of the tangents are

$$14\sqrt{\tfrac{5}{11}}\,.\,y + 4\sqrt{\tfrac{42}{11}}\,.\,x = 28,$$
$$12\sqrt{\tfrac{5}{11}}\,.\,y + 5\sqrt{\tfrac{42}{11}}\,.\,x = 30;$$

$$\text{hence} \qquad \tan\alpha = -\tfrac{2}{7}\sqrt{\tfrac{42}{5}}, \ \tan\alpha' = -\tfrac{5}{42}\sqrt{\tfrac{42}{5}},$$

$$\text{and} \qquad \tan\nu = \tan(\alpha - \alpha') = \frac{\tan\alpha - \tan\alpha'}{1 + \tan\alpha'\tan\alpha} = \tfrac{11}{84}\sqrt{\tfrac{21}{10}}$$

$$\nu = 10^\circ\ 44'\ 43''.$$

---

(53). QUESTION XI. *By Mr. H. Clay.*

Find when $\varphi = 0$, the value of the expression

$$\frac{1}{\varphi^2} - \frac{1}{\tan^2\varphi}.$$

SOLUTION. *By L.*

$$\text{Since} \qquad \tan\varphi = \varphi + \tfrac{1}{3}\varphi^3 + \tfrac{2}{15}\varphi^5 + \&c.,$$
$$\tan^2\varphi = \varphi^2 + \tfrac{2}{3}\varphi^4 + \tfrac{17}{45}\varphi^6 + \&c.$$

$$\text{and} \quad \frac{1}{\varphi^2} - \frac{1}{\tan^2\varphi} = \frac{1}{\varphi^2} - \frac{1}{\varphi^2 + \tfrac{2}{3}\varphi^4 + \tfrac{17}{45}\varphi^6 + \&c.}$$

$$= \frac{\tfrac{2}{3}\varphi^2 + \tfrac{17}{45}\varphi^4 + \&c.}{\varphi^2 + \tfrac{2}{3}\varphi^4 + \tfrac{17}{45}\varphi^6\ \&c.}$$

$$= \frac{\tfrac{2}{3} + \tfrac{17}{45}\varphi^2 + \&c.}{1 + \tfrac{2}{3}\varphi^2 + \tfrac{17}{45}\varphi^4 + \&c.}$$

and if we now make $\varphi = 0$, we have

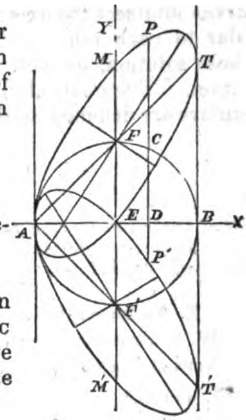
$$\frac{1}{\varphi^2} - \frac{1}{\tan^2\varphi} = \tfrac{2}{3}.$$

---

(54). QUESTION XII. *By* ——.

AB is the diameter of a given circle, and C any point in the circumference; from C let fall CD perpendicular to AB, and upon it take CP = AD; find the curve in which the point P is always found.

SOLUTION. *By Mr. Warren Colburn, St. Paul's College.*



Let R be the radius of the circle, the diameter AB being the axis of a, and the centre E the origin of co-ordinates. Let x, y' be the co-ordinates of the point C, and x, y those of the point P; then we have

$$y' = \sqrt{R^2 - x^2},$$

$$y = y' \pm (R + x) = \sqrt{R^2 - x^2} \pm (R + x),$$

and this equation when properly ordered becomes

$$(1) \quad y^2 + 2x^2 \mp 2yx \mp 2Ry + 2Rx = 0,$$

the upper signs being used when CP is taken on CD produced, and the lower when taken from C towards D. Using first the upper signs, the curve will necessarily pass through the origin since there is no absolute term, and if we make

$$x = R, \text{ we find } (y - 2R)^2 = 0, \; y = 2R;$$

$$\text{if } x = -R, \text{ we find } \quad y^2 = 0, \; y = 0;$$

the curve is therefore bounded by the perpendiculars to AB at A and B, touching them at A and at T, so that BT = AB. Comparing the equation (1) with the general equation of the second degree

$$(2) \quad Ay^2 + Bx^2 + 2Cxy + 2Dy + 2Ex = K,$$

we find $\quad A = 1, B = 2, C = -1, D = -E = -R, K = 0$; hence

$$AB - C^2 = 1 > 0,$$

and the curve is an ellipse, the co-ordinates of its centre being

$$k = R, \quad l = 0,$$

or the centre is at the point F where the axis of y cuts the circle; hence the line EFM = 2R is a diameter of the ellipse, and if θ be the angle which a diameter conjugate to this makes with the axis of x, we shall find $\tan\theta = 1$, and the equation of this diameter is

$$y - R = x,$$

which therefore passes through the points A and T, and the diameters AFT, EFM are conjugate to each other.

The equation of the major axis of this ellipse is

$$y - R = \tfrac{1}{2}(\sqrt{5} + 1)x,$$

which therefore cuts the axis of x at a distance

$$x = -\tfrac{1}{2}R(\sqrt{5} - 1)$$

from the origin and at an angle φ, such that

$$\tan\varphi = \tfrac{1}{2}(\sqrt{5} + 1), \text{ and } \varphi = 58^\circ 17'.$$

The lengths of the semiaxes, a and b, are the positive roots of the equation

$$\left(\frac{R^2}{r^2} - 1\right)\left(\frac{R^2}{r^2} - 2\right) - 1 = 0,$$

and therefore $\quad a = \tfrac{1}{2}R(\sqrt{5} + 1), \; b = \tfrac{1}{2}R(\sqrt{5} - 1),$

In a similar manner, the other locus is found to be an equal ellipse, posited in the same way on the opposite side of the axis.

—— The tangents at E to the two curves are respectively parallel to

the diameters AT, AT′ conjugate to the diameters through E, hence the curves intersect the axis of x at E in an angle of 45°, and are perpendicular to each other at this point. This is shown by Mr. Campbell, whose solution, as well as that of Mr. Deslond, are well worthy of insertion. The general discussion of equation (2), from which these particulars are deduced, will be inserted in a subsequent article.

List of Contributors to the Junior Department, and of Questions answered by each. The figures refer to the number of the questions, as marked in Number VII, Article III, page 24, Vol. II.

J. K. ANDERSON, Freshman Class, St. Paul's College, ans. 3, 4, 5, 6, 7.
W. B. BENEDICT, Upperville, Va., ans. 1, 3, 4, 5, 12.
ALFRED BIRDSALL, Clinton Liberal Institute, ans. 1, 2, 3, 4.
β, ans. 7, 9.
J. V. CAMPBELL, Sophomore Class, St. Paul's College, ans. 5, 7, 12.
WARREN COLBURN, Sophomore Class, St. Paul's College, ans. 12.
E. H. DELAFIELD, Freshman Class, St. Paul's College, ans. 1, 3, 4, 5, 7, 8.
A. DESLOND, Sophomore Class, St. Paul's College, ans. 12.
DANIEL KIRKWOOD, York, Pa., ans. 3.
L., Murray Co., Geo., ans. all the questions.

*** The communication of "Numerator," containing correct solutions to questions 3, 4 and 7, did not come to hand until the previous part of the copy was in the printer's hands.

ARTICLE VI.

QUESTIONS TO BE ANSWERED IN NUMBER IX.

Their solutions must arrive before February 1st, 1840.

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(55). QUESTION I. *By* ——.

Prove that the product of two numbers which are, each of them, less than unity, is less than either of them.

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(56). QUESTION II. *By Numerator.*

Given that

$$(1+\sqrt{-1})^7 = 8(1-\sqrt{-1}), \text{ and } (1-\sqrt{-1})^7 = 8(1+\sqrt{-1});$$

show that one value of

$$(1+\sqrt{-1})^{\frac{1}{7}}+(1-\sqrt{-1})^{\frac{1}{7}} \text{ is } (16)^{\frac{1}{7}}.$$

(57). QUESTION III. *By* ——.

It is required to find the roots of the equation

$$x^3 - \frac{a^2 - ab + b^2}{a^{\frac{1}{2}}b^{\frac{3}{2}}} . x^2 - \frac{a^2 - ab + b^2}{b^2} . x + \left(\frac{a}{b}\right)^{\frac{3}{2}} = 0,$$

which are in geometrical progression.

---

(58). QUESTION IV. *By Mr. D. Kirkwood.*

To find x, y, z, u, then are given the four equations

$$\begin{array}{ll} (1), & x^2y^2 + uxyz + u^2z^2 = 460, \\ (2), & \sqrt{1 + 2xy - y^2} = x, \\ (3), & \dfrac{u^2z^2}{xy} + xy^2 + uz = 38\tfrac{1}{3}, \\ (4), & u^2 + u^2z^2 = 5. \end{array}$$

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(59). QUESTION V. *By* ——.

Find x from the equation

$$\sqrt{abx - \frac{bc}{x}} + \sqrt{ac \quad \frac{bc}{x}} = ax.$$

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(60). QUESTION VI. *By* ——.

In any plane triangle ABC, if the angle made by the line drawn from the angle C to the middle point of the side c be denoted by θ, prove that

$$2 \cot \theta = \cot \text{A} - \cot \text{B}.$$

---

(61). QUESTION VII. *By Eltinge, New-Brunswick.*

It is required to find a point in a given equilateral triangle from which if straight lines, the lengths of which are denoted by k, l, m, be drawn to the three vertices of the triangle, we may have

$$\tfrac{1}{3}(k + l) + m = \tfrac{1}{4}(k + m) + l = \tfrac{1}{5}(l + m) + k.$$

---

(62). QUESTION VIII. *By* ——.

Eliminate θ between the two equations

$$x \sin \theta + a \operatorname{cosec} \theta = c,$$
$$x \cos \theta + b \quad \sec \theta = c.$$

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(63). QUESTION IX. *By L., Murray Co., Geo.*

Prove that

$$\cot \theta - \tan \theta = 2 \cot 2\theta,$$

and thence show that

$$\tan \theta + 2 \tan 2\theta + \ldots\ldots + 2^{n-1} \tan 2^{n-1} \theta = \cot \theta - 2^n \cot 2^n \theta.$$

(64). QUESTION X. *By* ——.

Find the values of x corresponding to the *maxima* or *minima* values of the expression

$$2x^3 - 3(a+b)x^2 + 6abx + c$$

and distinguish the *maxima* from the *minima* values.

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(65). QUESTION XI. *By* ——.

Given the equation

$$(\text{R}-r)^2y^2 + \{(\text{R}-r)^2 - d^2\}x^2 - \{(\text{R}-r)^2 - d^2\}dx = \tfrac{1}{4}\{(\text{R}-r)^2 - d^2\}^2 r$$

it is required to find its form when R and d become infinite, but have a given finite difference.

---

(66). QUESTION XII. *By* ——.

When the general equation of the second degree

$$\text{A}y^2 + \text{B}x^2 + 2\text{C}xy + 2\text{D}y + 2\text{E}x = \text{K},$$

represents a parabola, it is required to find its parameter, and the position of its axis, focus and vertex, in terms of the co-efficients.

SENIOR DEPARTMENT.

ARTICLE IV.

SOLUTIONS TO THE QUESTIONS PROPOSED IN NUMBER VI.

(98.) QUESTION I. *By an Engineer.*

a, b, c, d are four points on a hill which is to be reduced to a level of 10 feet below a; the surface nearly coincides with the planes drawn through a, b, c, and through a, c, d. It is required to find the quantity of earth to be removed from this part of the hill; the relative position of the points being given, as below:

Stations.	Bearing.	Distance.	Elevation.
a to b	S. 23° 17′E.	51 feet 3 in.	— 5° 25′
b to c	S. 54° 38′W.	79 " 10 "	+ 8° 37′
c to d	N. 10° 15′W.	63 " 5 "	+ 10° 9′
d to a	———	———	———

SOLUTION *By L., Murray Co., Geo.*

a', b', c', d', are the projections of a, b, c, d on a horizontal plane drawn 10 feet below a; by the process of question (85) we find

$$\begin{aligned} a'b' &= 51{,}0211, & aa' &= 10, \\ b'c' &= 78{,}9322, & bb' &= 5{,}1621, \\ c'd' &= 62{,}4242, & cc' &= 17{,}1229, \\ a'c' &= 102{,}565\ , & dd' &= 28{,}2965; \end{aligned}$$

hence, area of $a'b'c' = 1968,994$ square feet,
area of $a'd'c' = 1871,563$ do.

Pass a verticle plane through a, c, a', c' it will divide the solid into two truncated triangular prisms, and the

solidity of $abca'b'c' = \frac{1}{3}a'b'c'(aa' + bb' + cc') = 21189,465$ cu. ft
solidity of $adca'd'c' = \frac{1}{3}a'd'c'(aa' + dd' + cc') = 34574,85$ do.

and their sum is the whole solid $= 2065,345$ cu. yds. $= 55764,315$.

(99). Question II. *By Wm. Lenhart, Esq.*

Show how to find those integers whose cubes terminate with the three digits 048.

First Solution. *By Prof. Peirce.*

This question is to solve the equation

$$x^3 - 1000n = 48, \text{ or } x^3 \equiv 48 \text{ (mod. 1000)},$$

so that x must satisfy the congruences,

$$x^3 \equiv 48 \text{ (mod. 8)}, \quad \text{and } x^3 \equiv 48 \text{ (mod. 125)}.$$

Now the only root of the first of these inequalities is

$$x \equiv 0 \text{ (mod. 2)}, \text{ or } x = 2x',$$

in which x' is any integer. We have then to satisfy the inequality

$$8x'^3 \equiv 48 \text{ (mod. 125)}, \text{ or } x'^3 \equiv 6 \text{ (mod. 125)};$$

which involves $\quad x' \equiv 1$ (mod. 5), or $x' = 5x'' + 1$,
and again we have $\quad (1 + 5x'')^3 \equiv 6$ (mod. 125),
or $\quad 3x'' + 15x''^2 + 25x'''^3 \equiv 1$ (mod. 25),
which involves $\quad 3x'' \equiv 1$ (mod. 5), or $x'' = 5x''' + 2$;
so that the last congruence to be satisfied is

$$15x''' + 66 \equiv 1 \text{ (mod. 25)}, \text{ or } 3x'' + 13 \equiv 0 \text{ (mod. 5)},$$
$$\text{or } x''' + 1 \equiv 0 \text{ (mod. 5)}, \text{ whence } x''' = 4 + 5m;$$
$$x'' = 22 + 25m, x' = 111 + 125m, x = 222 + 250m;$$

and the numbers required are the arithmetical series 222, 472, &c.

Second Solution. *By Mr. Alfred Birdsall, Clinton Liberal Institute.*

Since the cube terminates with 048, it may be represented by $1000\,a + 48$, and the root evidently terminating with 2, may be represented by $10n + 2$, then

$$(10n + 2)^3 = 1000n^3 + 600\,n^2 + 120n + 8 = 1000\,a + 48,$$
$$\text{and} \quad 25n^3 + 15n^2 + 3n - 1 = 25a;$$

then we must have $3n - 1$ divisible by 5, which will be the case if $n = 2 + 5u$, then the preceding equation becomes

$$5\,(2 + 5u)^3 + 3\,(2 + 5u)^2 + 3u + 1 = 5a;$$

so that a will be an integer for all values of u that make

$$3u + 13, \text{ or } 3u + 3, \text{ or } u + 1 \text{ divisible by } 5,$$

and this will be so when $u = 5w + 4$; whence the number is

$$10n + 2 = 50u + 22 = 250w + 222.$$

If $w = 0$, the least root is 222.

——— Mr. Lenhart, the proposer, mentions that these numbers have several singular properties, one of which is, that

$$4\,(250m + 222)^3 - (250m + 223)^3 = 5^3\,.\,\text{A}.$$

Mr. Perkins also notices the singular property that all numbers terminating with the figures 12890625, have all their integral powers terminating with the same figures; or that

$$(\ldots\ldots 12890625)^n = \ldots\ldots\ldots 12890625.$$

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(100). QUESTION III. *Generalized from Peirce's Algebra.*

n men play together on the condition that he who loses shall give to all the rest as much as they already have. They play n games, and each loses in his turn, after which it is found that they have given sums of money. How much had each when they began to play?

SOLUTION. *By Mr. E. H. Delafield, St Paul's College.*

Let x_m denote the sum the m^{th} man had when they began to play,
a_m the sum he had at the end of the n^{th} game;
and let $s = a_1 + a_2 + \ldots + a_m \ldots + a_n$.
Then this man, who wins the m^{th} game, would have, at the end of the

$$\begin{array}{ll} \text{1st game} & \ldots\ldots 2x_m, \\ \text{2d} \quad " & \ldots\ldots 2^2 x_m, \\ & \&\text{c.} \\ (m-1)^{st} \quad " & \ldots\ldots 2^{m-1} x_m, \\ m^{th} \quad " & \ldots\ldots 2^m x_m - s, \\ (m+1)^{st} \quad " & \ldots\ldots 2^{m+1} x_m - 2s, \\ & \&\text{c.} \\ (m+k)^{th} \quad " & \ldots\ldots 2^{m+k} x_m - 2^k s, \end{array}$$

Let $m + k = n$, then, at the end of the n^{th} game, he will have

$$2^n x_m - 2^{n-m} s = a_m,$$

$$x_m = 2^{-m} s + 2^{-n} a_m.$$

——— The solution of Mr. Kirkwood was also very neat.

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(101). QUESTION IV. *By P.*

A hemisphere and cone are fastened with their equal bases together. It is required to find the height of the cone, so that the whole solid may be in equilibrium on any point of the curve surface of the hemisphere.

SOLUTION. *By Mr. M. P. Barton, Jun., Esperance, N. Y.*

It is evident that the centre of gravity of the cone and hemisphere, considered as one body of uniform density, must be in the centre of the sphere of which the hemisphere is a part. Put r = radius of the hemisphere, and x = altitude of the cone; then

$$\tfrac{2}{3}\pi r^3 = \text{content of the hemisphere},$$

$$\tfrac{2}{3}\pi r^3 \times \tfrac{3}{8} r = \tfrac{1}{4}\pi r^4 = \text{its moment},$$

$$\tfrac{1}{3}\pi r^2 x = \text{content of the cone},$$

$$\tfrac{1}{3}\pi r^2 x \times \tfrac{1}{4}\, x = \tfrac{1}{12}\, \pi r^2 x^2 = \text{its moment},$$

hence

$$\tfrac{1}{4}\pi r^4 = \tfrac{1}{12}\pi r^2 x^2$$

$$x = r\sqrt{3},$$

and the moments being equal when the axis is horizontal, they will be

equal in all positions. The whole solid may evidently be generated by the revolution of a semicircle and equilateral triangle described on different sides of the same line, about a perpendicular to this line through its middle point.

(102.) QUESTION V. *By* ψ.

Given the equation

$$y^4 - 9y^3x + 2x^2 = 0;$$

to express y in a series of monomials, arranged 1°. according to the ascending, and 2°. according to the descending powers, of x.

SOLUTION. *By Prof. M. Catlin, Hamilton College, Clinton.*

Let $y = \mathrm{A}x^a + \mathrm{B}x^b + \mathrm{C}x^c + \&c.$, then the equation becomes

$$\begin{aligned} 0 = {} & \mathrm{A}^4x^{4a} + 4\mathrm{A}^3\mathrm{B}x^{3a+b} + 6\mathrm{A}^2\mathrm{B}^2x^{2a+2b} + 4\mathrm{A}\mathrm{B}^3x^{a+3b} + \&c. \\ & - 9\mathrm{A}^3x^{3a+1} - 27\mathrm{A}^2\mathrm{B}x^{2a+b+1} - 27\mathrm{A}\mathrm{B}^2x^{a+2b+1} - 27\mathrm{A}^2\mathrm{C}x^{2a+c+1} + \&c. \\ & + 2x^2 \quad \text{- - - - - - -} \quad (1). \end{aligned}$$

If $a = \frac{1}{2}$, then $b = 1$, $c = \frac{3}{2}$, $d = 2$, &c.; and

$$0 = \begin{array}{l|l|l|l} \mathrm{A}^4 & x^2 + 4\mathrm{A}^3\mathrm{B} & x^{\frac{5}{2}} + 6\mathrm{A}^2\mathrm{B}^2 & x^3 + 12\mathrm{A}^2\mathrm{BC} \\ +2 & \quad - 9\mathrm{A}^3 & \quad + 4\mathrm{A}^3\mathrm{C} & \quad + 4\mathrm{A}\mathrm{B}^3 \\ & & \quad - 27\mathrm{A}^2\mathrm{B} & \quad + 4\mathrm{A}^3\mathrm{D} \\ & & & \quad - 27\mathrm{A}^2\mathrm{C} \\ & & & \quad - 27\mathrm{A}\mathrm{B}^2 \end{array} x^{\frac{7}{2}} + \&c. \quad . \quad (2);$$

hence $\mathrm{A} = (-2)^{\frac{1}{4}}$, $\mathrm{B} = \frac{9}{4}$, $\mathrm{C} = \frac{3}{2}\cdot\frac{9^2}{4^2}\cdot\frac{1}{(-2)^{\frac{1}{4}}}$, $\mathrm{D} = \frac{6\,.\,2}{2\,.\,3}\cdot\frac{9^3}{4^3}\cdot\frac{1}{(-2)^{\frac{1}{2}}}$,

$\mathrm{E} = \frac{9\,.\,5\,.\,1}{2\,.\,3\,.\,4}\cdot\frac{9^4}{4^4}\cdot\frac{1}{(-2)^{\frac{3}{4}}}$, $\mathrm{F} = \frac{12\,.\,8\,.\,4\,.\,0}{2\,.\,3\,.\,4\,.\,5}\cdot\frac{9^5}{4^5}\cdot\frac{1}{-2}$, &c. and

$$y = (-2)^{\frac{1}{4}}x^{\frac{1}{2}} + \tfrac{9}{4}x + \tfrac{3}{2}\cdot\frac{9^2}{4^2}\cdot\frac{1}{(-2)^{\frac{1}{4}}}x^{\frac{3}{2}} + \frac{6\,.\,2}{2\,.\,3}\cdot\frac{9^3}{4^3}\cdot\frac{1}{(-2)^{\frac{1}{2}}}x^2 + \&c. (3)$$

Equation (1) may also be satisfied by putting $4a = 3a + 1$, $a = 1$; $\therefore b = -1$, $c = -3$, $d = -5$, &c. Therefore we shall have

$$0 = \begin{array}{l|l|l|l} \mathrm{A}^4 & x^4 + 4\mathrm{A}^3\mathrm{B} & x^2 + 6\mathrm{A}^2\mathrm{B}^2 & x^0 + 4\mathrm{A}\mathrm{B}^3 \\ -9\mathrm{A}^3 & \quad - 18\mathrm{A}^2\mathrm{B} & \quad + 4\mathrm{A}^3\mathrm{C} & \quad + 12\mathrm{A}^2\mathrm{BC} \\ & \quad - 9\mathrm{B}\mathrm{A}^2 & \quad - 27\mathrm{A}\mathrm{B}^2 & \quad + 4\mathrm{A}^3\mathrm{D} \\ & \quad + 2 & \quad - 27\mathrm{A}^2\mathrm{C} & \quad - 9\mathrm{B}^3 \\ & & & \quad - 54\,\mathrm{ABC} \\ & & & \quad - 27\,\mathrm{A}^2\mathrm{D} \end{array} x^{-2} + \&c, \quad (4);$$

Equating the co-efficients with zero, we shall obtain

$\mathrm{A} = 9$, $\mathrm{B} = -\frac{2}{9^3}$, $\mathrm{C} = -\frac{3\,.\,4}{9^7}$, $\mathrm{D} = -\frac{9\,.\,10}{2\,.\,3}\cdot\frac{2^3}{9^{11}}$, &c.

$$y = 9x - \frac{2}{9^3}x^{-1} - \frac{6}{2}\cdot\frac{2^2}{9^7}x^{-3} - \frac{9\,.\,10}{2\,.\,3}\cdot\frac{2^3}{9^{11}}x^{-5} - \frac{12\,.\,13\,.\,14}{2\,.\,3\,.\,4}\frac{2^4}{9^{15}}x^{-7} - \&c.$$

——— Mr. Perkins notices that in the first development, (3), all the terms standing in the order $4n + 2$, n being any integer, vanish from having 0 for a factor, and the terms between the vanishing terms are all alternately positive and negative.

(103). QUESTION VI. *By* ——.

It is required to place a given parabola so as to touch a given line at a given point in it, and to intersect a second given line at a given angle.

FIRST SOLUTION. *By James F. Macully, Esq. New-York.*

Let the first given line be the angular axis, the given point being the pole; the equation of the given parabola (M. Misc., Vol. I. page 12,) is

$$(1).\qquad r=\frac{p\sin\omega}{\sin\varepsilon\sin^2(\omega-\varepsilon)},$$

p being the parameter, ε the angle the axis of the parabola makes with the angular axis. Also (Misc., Vol, II. page 29), the equation of the second given line is

$$(2).\qquad r=\mathrm{P}\sec(\omega-\alpha);$$

and the equation of a tangent to the parabola at the point $r\omega$, is

$$r_{,}\left\{\cos(\omega_{,}-\omega)-\sin(\omega_{,}-\omega)\frac{dr}{r\,d\omega}\right\}=r.$$

If β be the angle this line makes with the angular axis, the angle $\alpha-\frac{1}{2}\pi-\beta$ is given, therefore β is given, and it is evidently such that

$$\cos(\beta-\omega)-\sin(\beta-\omega)\frac{dr}{rd\omega}=0$$

or, since, by (1), $\quad\dfrac{dr}{rd\omega}=-\dfrac{\sin\varepsilon+\sin\omega\cos(\omega-\varepsilon)}{\sin\omega\sin(\omega-\varepsilon)},$

this equation becomes

$$(3).\qquad \sin\omega\sin(\beta-\varepsilon)+\sin\varepsilon\sin(\beta-\omega)=0$$

Eliminating r and ω between the equations (1), (2), (3), we have

$$(4).\quad 4\mathrm{P}\sin^2\varepsilon\sin^2(\beta-\varepsilon)=p\sin\beta\left\{\sin\varepsilon\cos(\alpha+\beta)+\sin(\varepsilon-\beta)\cos\alpha\right\},$$

from which ε may be found, which determines the position of the parabola.

SECOND SOLUTION. *By Mr. B. Birdsall, Clinton Liberal Institute.*

By using the usual formulas of transformation, the equation of the parabola will be

$$\left\{(y-\beta)\cos\theta-(x-\alpha)\sin\theta\right\}^2=2p\left\{(y-\beta)\sin\theta+(x-\alpha)\cos\theta\right\}\quad(1),$$

α and β being the co-ordinates of its vertex, and θ the angle, its axis makes with the axis of x, which we will suppose to be the second given line of the question, the origin being at the intersection of the two given lines, and therefore the equation of the first is

$$y'=ax'.\quad .\quad .\quad .\quad .\quad (2).$$

This line is to be tangent to the parabola, at the given point $y'x'$, and therefore,

$$\frac{dy}{dx}=\frac{dy'}{dx'}=\frac{\left\{(y'-\beta)\cos\theta-(x'-\alpha)\sin\theta\right\}\sin\theta+p\cos\theta}{\left\{(y'-\beta)\cos\theta-(x'-\alpha)\sin\theta\right\}\cos\theta-p\sin\theta}=a\qquad(3).$$

This given point being in the parabola

$$\left\{(y'-\beta)\cos\theta-(x'-\alpha)\sin\theta\right\}^2=2p\left\{(y'-\beta)\sin\theta+(x'-\alpha)\cos\theta\right\}\ (4).$$

But the line is to cut the axis of x, at an angle whose tangent is b, and therefore, for this point, since $y=0$,

$$\left\{\beta\cos\theta+(x-\alpha)\sin\theta\right\}^2=2p\left\{-\beta\sin\theta+(x-\alpha)\cos\theta\right\}\ (5),$$

$$\frac{dy}{dx}=\frac{\left\{\beta\cos\theta+(x-\alpha)\sin\theta\right\}\sin\theta-p\cos\theta}{\left\{\beta\cos\theta+(x-\alpha)\sin\theta\right\}\cos\theta+p\sin\theta}=b\ .\qquad(6).$$

Equations (3), (4), (5), (6) are sufficient to determine the unknown quantities x, α, β, θ, and thence the position of the parabola.

——— The most interesting case of the question is that where (in the first solution) $\alpha = \frac{1}{2}\pi$, or the given straight lines are parallel, and also $\beta = \frac{1}{2}\pi$; then equation (4) becomes

$$\sin^3 \varepsilon - \sin \varepsilon + \frac{p}{4\mathrm{P}} = 0,$$

It follows that if $$8p > 3\mathrm{P}\sqrt{3}$$
the parabola can be placed but in one position; but if
$$8p = \text{or} < 3\mathrm{P}\sqrt{3},$$
it can be placed in three positions, such that it touches one given line at a given point, and is perpendicular to a parallel line.

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(104). QUESTION VII. *By Prof. G. R. Perkins, Utica Academy.*

Given the sum of the squares, and the sum of the fourth powers of four lines drawn from a point to the four vertices of a regular tetraedron, to find the side of the tetraedron.

SOLUTION. *By the Proposer.*

If we refer the corners of the tetraedron to three rectangular axes through its centre, the plane of xy being parallel to one of the faces, and the axis of x parallel to one of the edges of this face, we shall have the co-ordinates of the vertices

$$\begin{array}{ccc} -x, & -\frac{1}{3}x\sqrt{3}, & -\frac{1}{6}x\sqrt{6} \\ x, & -\frac{1}{3}x\sqrt{3}, & -\frac{1}{6}x\sqrt{6} \\ 0, & \frac{2}{3}x\sqrt{3}, & -\frac{1}{6}x\sqrt{6} \\ 0, & 0\ , & \frac{1}{2}x\sqrt{6} \end{array}$$

where x is half the required side of the tetraedron. Now if we represent the co-ordinates of the point by x', y', z', and the four distances of this point from the vertices by a, b, c, d, we shall find

$$(x'+x)^2 + (y'+\tfrac{1}{3}x\sqrt{3})^2 + (z'+\tfrac{1}{6}x\sqrt{6})^2 = a^2 \quad . \quad . \quad (1),$$
$$(x'-x)^2 + (y'+\tfrac{1}{3}x\sqrt{3})^2 + (z'+\tfrac{1}{6}x\sqrt{6})^2 = b^2 \quad . \quad . \quad (2),$$
$$x'^2 + (y'-\tfrac{2}{3}x\sqrt{3})^2 + (z'+\tfrac{1}{6}x\sqrt{6})^2 = c^2 \quad . \quad . \quad (3),$$
$$x'^2 + y'^2 + (z'-\tfrac{1}{2}x\sqrt{6})^2 = d^2 \quad . \quad . \quad (4).$$

By eliminating x', y', z', we find

$$12x^2 = a^2+b^2+c^2+d^2 \pm 2\sqrt{(a^2+b^2+c^2+d^2)^2 - 3(a^4+b^4+c^4+d^4)} \ (5),$$

Hence, if we put $\mathrm{S} = 2x$, the required side,
$$\mathrm{A} = a^2+b^2+c^2+d^2,$$
$$\mathrm{B} = a^4+b^4+c^4+d^4;$$

then $$\mathrm{S} = \sqrt{\tfrac{1}{3}}\,.\,\{\mathrm{A} \pm 2\sqrt{\mathrm{A}^2 - 3\mathrm{B}}\}^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad (6).$$

If r be the radius of the sphere passing through the vertices of the tetraedron, and ρ the distance of the point from the centre

$$r^2 = \tfrac{3}{8}\mathrm{S}^2 = \tfrac{1}{8}\{\ \mathrm{A} \pm 2\sqrt{\mathrm{A}^2 - 3\mathrm{B}}\};$$

and by adding together the equations (1), (2), (3), (4),

$$x'^2 + y'^2 + z'^2 = \rho^2 = \tfrac{1}{4}(\mathrm{A} - 6x^2) = \tfrac{1}{8}\{\mathrm{A} \mp 2\sqrt{\mathrm{A}^2 - 3\mathrm{B}}\}.$$

Now, we have always $A > 2\sqrt{A^2 - 3B}$, because

$$A^2 - 4(A^2 - 3B), \quad \text{or} \quad 3(4B - A^2)$$

can be put into the form

$$(3d^2 - a^2 - b^2 - c^2)^2 + 2(2c^2 - a^2 - b^2)^2 + 6(a^2 - b^2)^2,$$

and is therefore > 0; it follows that there are always two tetraedrons which solve the problem, the point from which the lines are drawn for one tetraedron being any where in the surface of the sphere circumscribed about the other. The case of

$$A^2 - 3B = 0,$$

is an exception; the two tetraedrons and spheres being then confounded with each other. If $a = b = c = d$, one of the tetraedrons is reduced to a point coinciding with the centre of the other.

(105.) QUESTION VIII. *By Wm. Lenhart, Esq. York, Penn.*

It is required to find n numbers such that their sum increased by the sum of their cubes shall be equal to the sum of n other numbers increased by the sum of their cubes.

FIRST SOLUTION. *By the Proposer.*

Suppose that two numbers were required, then we should have

$$x^3 + y^3 + x + y = v^3 + w^3 + v + w \qquad (1).$$

Or, supposing $x^3 + y^3$ and $v^3 + w^3$ to be equal to tabular numbers t and t' respectively,

$$t - t' = (v + w) - (x + y) \qquad (2).$$

by which we perceive that when $t > t'$, the sum of the roots of the cubes which compose t', is greater than the sum of the roots of the cubes composing t; therefore all we have to do is to look into the table of numbers composed of two cubes, for two numbers whose difference is equal to the difference of the sums of the roots of the component cubes, as in (2), and these roots will be the numbers required. To illustrate: By Table we have $t = 344 = (7)^3 + (1)^3$, and $t' = 341 = (5)^3 + (6)^3$; and since $344 - 341 = (5 + 6) - (7 + 1)$ therefore $x = 5$, $y = 6$, and $v = 7$, $w = 1$; the numbers required. If n numbers be required then

$$x^3 + y^3 + z^3 \text{ \&c.} + x + y + z \text{ \&c.} = v^3 + w^3 + u^3 \text{ \&c.} + v + w + u \text{ \&c.} \quad (3),$$

in which it is evident that all the numbers, with the exception of two on each side of the equation, may be assumed. This being done the equation takes the form of

$$x^3 + y^3 + x + y + m = v^3 + w^3 + v + w + \eta. \quad . \quad (4),$$

in which since $x^3 + y^3$ or t is supposed to be greater than $v^3 + w^3$ or t', we suppose $\eta > m$, and thence putting $\eta - m = \eta'$, we shall have

$$t - t' = (v + w) - (x + y) + \eta' \quad . \quad . \quad (5),$$

or, if $t = t'$

$$x + y = v + w + \eta'. \quad . \quad . \quad . \quad (6),$$

which are readily answered by a simple inspection of the Table of numbers and their two component cubes. For instance: suppose three numbers were required; then assuming $z = 1$ and $u = 2$, we shall have $z^3 + z = m = 2$, $u^3 + u = \eta = 10$, $\eta - m = \eta' = 8$, and

$$t - t' = (v + w) - (x + y) + 8 \quad . \quad . \quad (7).$$

Now, by the Table,

$$t = 1853 = (5)^3 + (12)^3 \quad \text{and} \quad t' = 1843 = (8)^3 + (11)^3$$

therefore $\quad t - t' = 10 = (8 + 11) - (5 + 12) + 8$

and consequently $x = 12$, $y = 5$, $z = 1$; and $v = 11$, $w = 8$ and $u = 2$. In the same way we readily find the four numbers 14, 13, 11, 8; and the other four 17, 12, 5 and 3; also the five numbers 21, 14, 10, 4, 1; and the other five 20, 17, 5, 3 and 2.

We may, however, from having several sets of numbers to answer the first case of the question, find n numbers to answer independently of assuming any of the numbers. Suppose, for instance, we have

$$x^3 + y^3 + x + y = v^3 + w^3 + v + w,$$

and also $\quad x'^3 + y'^3 + x' + y' = v'^3 + w'^3 + v' + w',$

then by adding these equations together the reader will perceive that we shall succeed in obtaining four different numbers on each side: Or, if $x = v'$, or $y = w'$ or $x = v$, or $y' = w$, by cancelling either of the equal values, we shall get three different numbers on each side. And it is obvious that in the same manner n numbers may be found.

SECOND SOLUTION. *By Prof. Peirce.*

Let the m^{th} number of the first set be $\quad a_m x + b_m$,
and the m^{th} number of the second set $\quad a_m x + b_{n-m+1}$,
and we have for the sums

$$x^3 s\,.\,a^3_m + 3x^2 s\,.\,a^2_m b_m + xs\,.\,(3a_m b^2_m + a_m) + s\,.\,(b^3_m + b_m)$$
$$= x^3 s.\,a^3_m + 3x^2 s.\,a^2_m b_{n-m+1} + xs.\,(3a_m b^2_{n-m+1} + a_m) + s.\,(b^3_m + b_m)$$

so that $\quad x = \dfrac{s.\,a_m\,(b^2_{n-m+1} - b^2_m)}{s.\,a^2_m\,(b_m - b_{n-m+1})}$,

whence the required numbers are found.

(106.) QUESTION IX. *By J. F. Macully, Esq. New-York.*

It is required to find the sum of the series

$$\frac{1 + 4\cos^4\theta}{\cos^2 2\theta \cos^2\theta} + \frac{1}{4^2}.\frac{1 + 4\cos^4\frac{1}{4}\theta}{\cos^2\frac{1}{2}\theta\cos^2\frac{1}{4}\theta} + \frac{1}{4^4}.\frac{1 + 4\cos^4\frac{1}{16}}{\cos^2\frac{1}{8}\theta\cos^2\frac{1}{16}\theta} + \&c.$$

FIRST SOLUTION. *By Prof. C. Avery, Hamilton College, Clinton.*

Here $\quad 1 + 4\cos^4\theta = 1 + (1 + \cos 2\theta)^2 = 2(1 + \cos 2\theta) + \cos^2 2\theta$

$$= 4\cos^2\theta + \cos^2 2\theta,$$
$$1 + 4\cos^4\tfrac{1}{4}\theta = 4\cos^2\tfrac{1}{4}\theta + \cos^2\tfrac{1}{2}\theta,$$
$$1 + 4\cos^4\tfrac{1}{16}\theta = 4\cos^2\tfrac{1}{16}\theta + \cos^2\tfrac{1}{8}\theta,$$
$$\&c.;$$

and, by substitution, the given series is changed to

$$s_n = \frac{4}{\cos^2 2\theta} + \frac{1}{\cos^2\theta} + \frac{1}{4\cos^2\frac{1}{2}\theta} + \dots \frac{1}{4^{2n-2}\cos^2 4^{-n+1}\theta}$$

$$= \frac{d.}{d\theta}[2\tan 2\theta + \tan\theta + \tfrac{1}{2}\tan\tfrac{1}{2}\theta + \dots 4^{-n+1}\tan 4^{-n+1}\theta]$$

$$= \frac{d.}{d\theta}[4^{-n+1}\cot 4^{-n+1}\theta - 4\cot 4\theta]$$

$$= \frac{16}{\sin^2 4\theta} - \frac{1}{4^{2n-2}\sin^2 4^{-n+1}\theta}.$$

And when $n = \infty$,

$$s = \frac{16}{\sin^2 4\theta} - \frac{1}{\theta^2}.$$

SECOND SOLUTION. *By L., Murray Co., Geo.*

We have
$$u_{n+1} = 4^{-2n}.\frac{1+4\cos^4 4^{-n}\theta}{\cos^2 4^{-n}\theta \cos^2 4^{-n}.2\theta}$$
$$= 4^{-2n}.\frac{4\cos^2 4^{-n}\theta + \cos^2 4^{-n}.2\theta}{\cos^2 4^{-n}\theta\cos^2 4^{-n}.2\theta}$$
$$= \frac{4^{-2n+1}}{\cos^2 4^{-n}.2\theta} + \frac{4^{-2n}}{\cos^2 4^{-n}\theta}$$
$$= \frac{4^{-2n+2}\sin^2 4^{-n}.2\theta}{\sin^2 4^{-n+1}\theta} + \frac{4^{-2n+1}\sin^2 4^{-n}\theta}{\sin^2 4^{-n}.2\theta}$$
$$= \frac{4^{-2n+2}\,(1-\cos^2 4^{-n}.2\theta)}{\sin^2 4^{-n+1}\theta} + \frac{4^{-2n+1}(1-\cos^2 4^{-n}\theta)}{\sin^2 4^{-n}.2\theta}$$
$$= \frac{4^{-2n+2}}{\sin^2 4^{-n+1}\theta} - \frac{4^{-2n+1}}{\sin^2 4^{-n}.2\theta} + \frac{4^{-2n+1}}{\sin^2 4^{-n}.2\theta} - \frac{4^{-2n}}{\sin^2 4^{-n}\theta}$$
$$= -\Delta\,.\,\frac{4^{-2n+2}}{\sin^2 4^{-n+1}\theta}$$

Hence
$$s_n = \text{const.} + \Sigma\, u_{n+1} = \text{const.} - \frac{4^{-2n+2}}{\sin^2 4^{-n+1}\theta}$$
$$s_0 = 0 = \text{const.} - \frac{16}{\sin^2 4\theta};$$
$$s_n = \frac{16}{\sin^2 4\theta} - \frac{4^{-2n+2}}{\sin^2 4^{-n+1}\theta}.$$
$$s_\infty = \frac{16}{\sin^2 4\theta} - \frac{1}{\theta^2}.$$

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(107). QUESTION X. *By P.*

It is required to solve question (75) when, instead of the area and vertical angle, there are given the area and the side opposite the fixed extremity of the base.

SOLUTION. *By the Proposer.*

Retaining the notation in the solution of question (75); let BC $= a$, and the perpendicular from A upon BC $= h$; then

the equation of BC is $\quad y\,(x_2 - x_{,}) = y_2\,(x - x_{,}),$

that of the perp. from A upon CC, $\quad yy_2 = (x_{,} - x_2)\,x \quad . \; . \; . \quad (1),$

that of the perp. from C upon AB, $\quad x = x_2 \quad . \; . \; . \quad (2).$

Moreover $\quad y_2^{\,2} + (x' - x_2)^2 = a^2 \quad . \; . \; . \quad (3),$

and $\quad y^q\, x_{,} - ah = 2\,\Delta\,\text{ABC} \quad . \; . \; . \; . \quad (4),$

By eliminating x_1, x_2, y_2 among these four equations, we get

$$x^4(x^2+y^2) = h^2(x^2+y^2)^2 - 2ahxy(x^2+y^2) + a^2x^2y^2 \quad . \quad (5),$$

for the equation of the line, which is of the sixth order.

If we change these into polar co-ordinates, A being the pole, and the positive axis of x the angular axis, it becomes

$$v\cos^2\varphi = \pm(h - a\sin\varphi\cos\varphi) \quad . \quad . \quad . \quad (6).$$

The double sign indicates that the curve is the same in the opposite right angles, and we shall therefore use the upper one, and trace the curve from $\varphi = 0$ to $\varphi = \pi$; assume also k and β such that

$$k = \frac{a}{h}, \text{ and } \sin 2\beta = \frac{2}{k};$$

then

$$v = h\,.\,\frac{1 - \frac{1}{2}k\sin 2\varphi}{\cos^2\varphi},$$

$$x = h\,.\,\frac{1 - \frac{1}{2}k\sin 2\varphi}{\cos\varphi},$$

$$y = h\,.\,\frac{\sin\varphi - \frac{1}{2}k\sin\varphi\sin 2\varphi}{\cos^2\varphi},$$

$$\frac{dv}{d\varphi} = h\,.\,\frac{2\sin\varphi - k\cos\varphi}{\cos^3\varphi},$$

$$\frac{dx}{d\varphi} = h\,.\,\frac{\sin\varphi - k\cos^3\varphi}{\cos^2\varphi},$$

$$\frac{dy}{d\varphi} = h\,.\,\frac{1 + \sin^2\varphi - \frac{1}{2}k\sin 2\varphi(1 + \cos^2\varphi)}{\cos^3\varphi},$$

$$\frac{dy}{dx} = \frac{1 + \sin^2\varphi - \frac{1}{2}k\sin 2\varphi(1 + \cos^2\varphi)}{\cos\varphi(\sin\varphi - k\cos^3\varphi)}.$$

$$\frac{d^2y}{dx^2} = \frac{2 - 3\cos^2\varphi - 6k\sin\varphi\cos^3\varphi + k^2\cos^4\varphi(3 - \cos^2\varphi)}{h(\sin\varphi - k\cos^3\varphi)^3}.$$

The form of the curve will be seen by tracing it through the following points:—

1°. When $\varphi = 0°$, $v = x = h$, and if ψ be the angle made by the tangent at any point in the curve with the axis of x, for this point

$$\tan\psi = \frac{dy}{dx} = -\frac{1}{k}, \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{1 - 2k^2}{hk},$$

As φ increases, v decreases, until

2°. When $k\sin 2\varphi = 2$, or $\varphi = \beta$, $v = x = y = 0$, $\tan\psi = \dfrac{dy}{dx} = \tan\beta$;

and the curve passes through the origin in the direction of the radius vector. As φ increases, v continues to decrease, until

3°. When $\dfrac{dv}{dx} = 0$, or $\tan\varphi = \frac{1}{2}k$; then

$$v = h(1 - \tfrac{1}{4}k^2), \tan\psi = \frac{dy}{dx} = -\frac{2}{k} = -\cot\varphi, \frac{d^2y}{dx^2} = \frac{(2k^4 + 20k^2 + 16)(k^2+4)^{\frac{3}{2}}}{hk^3\,(k^2-4)^2},$$

or the curve is perpendicular to the radius vector. Now v increases until

4°. When $k\sin 2\varphi = 2$ again, or $\varphi = \frac{1}{2}\pi - \beta$, then

$$v = x = y = 0,\ \tan\psi = \frac{dy}{dx} = \cot\beta,$$

and the curve again passes through the origin in the direction of the radius vector, and then v still continues to increase until

5°. When $\varphi = \frac{1}{2}\pi$, then $v = x = y = \infty$, $\tan\psi = \frac{dy}{dx} = \infty$, and there is no assymptote to this infinite branch. As φ increases from $\frac{1}{2}\pi$ to π, v decreases, until

6°. When $\varphi = \pi$, then

$$v = -x = h, \quad \tan\psi = \frac{dy}{dx} = -\frac{1}{k}, \quad \frac{d^2y}{dx^2} = \frac{2k^2-1}{hk},$$

7°. The curve is perpendicular to the axis of x, and x is a minimum when $\frac{dx}{d\varphi} = 0$, or, if $\tan\varphi = t$, when

$$t^3 + t - k = 0 \quad . \quad . \quad . \quad . \qquad (7),$$

an equation which has never more than one real root.

8°. The curve is parallel to the axis of x, and y is a maximum or minimum, when $\frac{dy}{d\varphi} = 0$, or when

$$2t^4 - kt^3 + 3t^2 - 2kt + 1 = 0, \quad . \quad . \quad . \qquad (8);$$

and examining this equation, by Sturm's Theorem, we find that if

$$X_1 = -32k^6 - 795k^4 + 2388k^2 + 32,$$

the equation has no real roots, when $X > 0$, or $k^2 < 2{,}720732$,
two equal roots, when $X_1 = 0$, or $k^2 = 2{,}720732$,
two unequal roots, when $X_1 < 0$, or $k > 2{,}720732$;

and the roots are both positive, the corresponding values of φ being between 0 and $\frac{1}{2}\pi$.

9°. There are points of inflexion in the curve, when $\frac{d^2y}{dx^2} = 0$, or

$$2t^6 + 3t^4 - 6kt^3 + 3k^2t^2 - 6kt + 2k^2 - 1 = 0 \quad . \qquad (9),$$

an equation which has no real roots, when $k > 2$,
two equal roots, when $k = 2$,
two unequal roots, when $k < 2$;

which two roots are both positive when $k > \frac{1}{2}$, one of them is zero, when $k^2 = \frac{1}{2}$, they have different signs when $k < \frac{1}{2}$, and when $k = 0$, these roots $= \pm\sqrt{\frac{1}{2}}$.

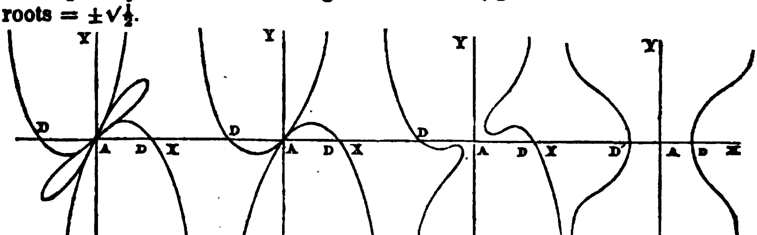

Fig. 1. Fig. 2. Fig. 3. Fig. 4.

Every relation between the constants a and h can be expressed by imagining $h = AD = AD'$ to remain constant, while their ratio, k, varies

from ∞ to 0. When $k=\infty$ the curve is confounded with the axes of co-ordinates; as k decreases, the curve takes the form of fig. 1, the loop continually decreasing in magnitude until $k=2$, when it disappears, and the curve is as in fig. 2, when $k<2$, the curve does not pass through the origin, and the curve will be as in fig. 3, approaching to the form of fig. 4, as k approaches to zero, which shape it finally assumes when $k=0$, and the equation of this curve is

$$x^4=k^2\,(x^2+y^2) \quad . \quad . \quad . \quad . \quad (10).$$

There are no points of inflection in fig. 1, the roots of equation (9), not being real when $k>2$; the two real roots of equation (8) determine the position of the maximum ordinates in the loop and in the branch AD. In fig. 2, where $k=2$; the points determined in 2°, 3°, 4°, 5°, 6°, 7°, one of of the points in 8°, and the two points in 9°, all coincide in the point A; the curve therefore does not, in its course, pass through this point, but there are two cusps whose vertices meet there. In fig. 3, the two points of inflection are on contrary sides of the point of least distance to A, and they recede from it as k decreases, one of them coinciding with the point D when $k^2=\frac{1}{2}$, after which it is below the axis.

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(108). QUESTION XI. *From Legendre's Theorie des Nombres, Vol. 2., p. 144.*
(Communicated by Mr. Geo. R. Perkins.)

A	B	C	D
E	F	G	H
I	K	L	M
N	O	P	Q

"In a square, divided into 16 spaces, as in the adjoining figure inscribe 16 numbers, A, B, C, ... Q, which will satisfy the following conditions:

1°. That the sum of the squares of the numbers may be equal in each of the four horizontal lines, also equal in each of the four vertical lines, and in the two diagonals.

2°. That the sum of the products, taken two and two, such as AE + BF + CG + DH may be equal to nothing with regard to the first two horizontal lines, as well as with regard to any two horizontal lines whatever, and that this may be the same also with regard to any two vertical lines."

FIRST SOLUTION. *By Prof. C. Avery.*

Let a, a', a'', a'''; b, b', b'', b'''; c, c', c'', c'''; d, d', d'', d'' be the required numbers, then

$$\left.\begin{aligned} A&=a'^2+a'^2+a''^2+a'''^2=b^2+b'^2+b''^2+b'''^2\\ &=c^2+c'^2+c''^2+c'''^2=d^2+d'^2+d''^2+d'''^2\end{aligned}\right\} \quad . \quad (1),$$

$$\left.\begin{aligned} A&=a^2+b^2+c^2+d^2=a'^2+b'^2+c'^2+d'^2\\ &=a''^2+b''^2+c''^2+d''^2=a'''^2+b'''^2+c'''^2+d'''^2\end{aligned}\right\} \quad . \quad (2),$$

$$A=a^2+b'^2+c''^2+d'''^2=a'''^2+b''^2+c'^2+d^2 \quad . \quad . \quad (3),$$

$$\left.\begin{aligned} 0&=ab+a'b'+a''b''+a'''b'''=ac+a'c'+a''c''+a'''c'''\\ &=ad+a'd'+a''d''+a'''d'''=bc+b'c'+b''c''+b'''c'''\\ &=bd+b'd'+b''d''+b'''d'''=cd+c'd'+c''d''+c'''d'''\end{aligned}\right\} \quad . \quad . \quad . \quad (4),$$

$$\left.\begin{aligned} 0&=aa'+bb'+cc'+dd'=aa''+bb''+cc''+dd''\\ &=aa'''+bb'''+cc'''+dd'''=a'a''+b'b''+c'c''+d'd''\\ &=a'a'''+b'b'''+c'c'''+d'd'''=a''a'''+b''b'''+c''c'''+d''d'''\end{aligned}\right\} \quad . \quad (5),$$

If (1) and (4) be satisfied, (2) and (4) will be :—See La Place, page 115. In order to satisfy (1) and (4) we use the well known principles demonstrated by Barlow, in his Theory of Numbers, (page 179); to wit

$$(p^2+q^2+r^2+s^2)(p'^2+q'^2+r'^2+s'^2)=(pp'+qq'+rr'+ss')^2$$
$$+(pq'-qp'+rs'-sr')^2+(pr'-rp'+sq'-qs')^2+(ps'-sp'+qr'-rq')^2,$$

and where the signs of the simple quantities may be changed at pleasure. Assume

$$\left.\begin{array}{ll} a = pr'+qs'+rp'+sq', & a' = -pp'+qq'+rr'-ss' \\ a'' = ps'-qr'+rq'-sp', & a''' = -pq'-qp'+rs'+sr' \end{array}\right\} (6),$$
$$\left.\begin{array}{ll} b = -pq'+qp'-rs'+sr', & b' = ps'+qr'-rp'-sp' \\ b'' = pp'+qq'+rr'+ss', & b''' = -pr'+qs'+rp'-sq' \end{array}\right\} (7),$$
$$\left.\begin{array}{ll} c = pp'+qq'-rr'-ss', & c' = pr'-qs'+rp'-sq' \\ c'' = pq'-qp'-rs'+sr', & c''' = ps'+qr'+rq'+sp' \end{array}\right\} (8),$$
$$\left.\begin{array}{ll} d = -ps'+qr'+rq'-sp', & d' = -pq'-qp'-rs'-sr' \\ d'' = pr'+qs'-rp'-sq', & d''' = pp'-qq'+rr'-ss' \end{array}\right\} (9).$$

Then will all the equations be satisfied, save (3). From (1) and (3)
$0=b'^2+c''^2+d'''^2-a'^2-a''^2-a'''^2=b''^2+c'^2+d^2-a^2-a'^2-a''^2$ (10).

If we write the previous values of a, a', &c. in these two equations, and take their sum and difference, putting $s=0$, we shall find

$$p'r'=q's', \quad \text{and} \quad p(p'q'-r's')=r(p's'-q'r') \quad . \quad (11),$$

Whence $p'=\dfrac{q's'}{r'}$, and $p=\dfrac{rq'}{s'}\cdot\dfrac{s'^2-r'^2}{q'^2-r'^2}$ (12),

and $q'r's'$, &c. may be taken at pleasure. For instance, if $r=5$, $s=0$, $q'=3$, $r'=2$, $s'=4$; then $p=9$, $p'=6$, and the numbers placed in their appropriate cells will be

$48+4q$	$-44+3q$	$51-2q$	$-7-6q$
$-47+6q$	$21+2q$	$64+3q$	$12+4q$
$44+3q$	$48-4q$	$7-6q$	$51+2q$
$-21+2q$	$-47+6q$	$-12+4q$	$64-3q$

If $q=0$, 1, 2, &c., we have different Tables which will satisfy. When $q=5$ we obtain Euler's numbers

68	—29	41	—37
—17	31	79	32
59	28	—23	61
—11	—77	8	49

Second Solution. *By Prof. G. R. Perkins, Utica Academy.*

a	a'	a''	a'''
b	b'	b''	b'''
c	c'	c''	c'''
d	d'	d''	d'''

For greater symmetry, we will represent the numbers as in the adjoining figure.

Now by the conditions of the question, we must satisfy the following equations, where A denotes any constant quantity:

$$\left.\begin{array}{ll} a^2+a'^2+a''^2+a'''^2=A, & b^2+b'^2+b''^2+b'''^2=A, \\ c^2+c'^2+c''^2+c'''^2=A, & d^2+d'^2+d''^2+d'''^2=A, \end{array}\right\} \quad (1)$$

$$\left.\begin{array}{ll} a^2+b^2+c^2+d^2=A, & a'^2+b'^2+c'^2+d'^2=A, \\ a''^2+b''^2+c''^2+d''^2=A, & a'''^2+b'''^2+c'''^2+d'''^2=A, \end{array}\right\} \quad (2),$$

$$\left.\begin{array}{ll} ab+a'b'+a''b''+a'''b'''=0, & ac+a'c'+a''c''+a'''c'''=0, \\ ad+a'd'+a''d''+a'''d'''=0, & bc+b'c'+b''c''+b'''c'''=0, \\ bd+b'd'+b''d''+b'''d'''=0, & cd+c'd'+c''d''+c'''d'''=0, \end{array}\right\} \quad (3),$$

$$\left.\begin{array}{ll} aa'+bb'+cc'+dd'=0, & aa''+bb''+cc''+dd''=0, \\ aa'''+bb'''+cc'''+dd'''=0, & a'a''+b'b''+c'c''+d'd''=0, \\ a'a'''+b'b'''+c'c'''+d'd'''=0. & a''a'''+b''b'''+c''c'''+d''d'''=0, \end{array}\right\} \quad (4),$$

$$a^2+b'^2+c''^2+d'''^2=A, \qquad a'''^2+b''^2+c'^2+d^2=A, \qquad (5),$$

We will first show that if equations (1), (3) be satisfied, then will (2) and (4) be also satisfied: for assume

$$\left.\begin{array}{ll} w=as+bt+cu+dv, & x=a's+b't+c'u+d'v \\ y=a''s+b''t+c''u+d''v, & z=a'''s+b'''t+c'''u+d'''v \end{array}\right\} \quad (6).$$

If we take the sum of the squares of these four equations, we shall have in virtue of (4) and (3)

$$w^2+x^2+y^2+z^2=A\,(s^2+t^2+u^2+v^2) \quad . \; . \; . \quad (7).$$

From (6) we get, by having to (1) and (3),

$$\left.\begin{array}{ll} A\,s=aw+a'x+a''y+a'''z, & A\,t=bw+b'x+b''y+b'''z, \\ A\,u=cw+c'x+c''y+c'''z, & A\,v=dw+d'x+d''y+d'''z, \end{array}\right\} \quad (8).$$

If we substitute these values in (7), and compare the like co-efficients, we shall obtain (2) and (4). Hence we need only seek to satisfy (1), (3) and (5).

On page 213, Vol. 1., of *Theorie des Nombres*, we find this identical equation given by Euler,

$$(p^2+q^2+r^2+s^2)(p'^2+q'^2+r'^2+s'^2)=(pp'+qq'+rr'+ss')^2$$
$$+(pr'+qs'-rp'-sq')^2+(ps'-qr'+rq'-sp')^2+(pq'-qp'-rs'+sr')^2 \; (9)$$

This equation will still hold good, when the signs of any of the letters are changed. The four terms on the right hand member of (9) may have their sign so changed as to correspond with the values of a, a', a'', a''', and of b, b', b'', b''', and of c, c', c'', c''', and of d, d', d'', d''' in succession. This is done in the following manner: Form a square of 16 cells, and write the four terms in the same order which they have in (9) for the first horizontal line; then for the second horizontal line, write these same terms in a reverse order, observing to change the signs of r, s in the first two terms, and the signs of p, q in the last two terms. The remaining cells are filled by taking the terms in the first horizontal line for the third line, those in the second line for the fourth, placing each term, either one vertical column to the right, or one to the left, but not crossing the middle vertical line, observing to change the signs of q, s, when the term is moved to the right, and to change the signs of p, r when it is moved to the left.

$pp'+qq'+rr'+ss'$	$pr'+qs'-rp'-sq'$	$ps'-qr'+rq'-sp'$	$pq'-qp'-rs'+sr'$
$-pq'+qp'-rs'+sr'$	$-ps'+qr'+rq'-sp'$	$pr'+qs'+rp'+sq'$	$pp'+qq'-rr'-ss'$
$-pr'+qs'+rp'-sq'$	$pp'-qq'+rr'-ss'$	$-pq'-qp'+rs'+sr'$	$ps'+qr'+rq'+sp'$
$ps'+qr'-rq'-sp'$	$-pq'-qp'-rs'-sr'$	$-pp'+qq'+rr'-ss'$	$pr'-qs'+rp'-sq'$

$pr'+qs'+rp'+sq'$	$-pp'+qq'+rr'-ss'$	$ps'-qr'+rq'-sp'$	$-pq'-qp'+rs'+sr'$
$-pq'+qp'-rs'+sr'$	$ps'+qr'-rq'-sp$	$pp'+qq'+rr'+ss'$	$-pr'+qs'+rp'-sq'$
$pp'+qq'-rr'-ss'$	$pr'-qs'+rp'-sq'$	$pq'-qp'-rs'+sr'$	$ps'+qr'+rq'+sp'$
$-ps'+qr'+rq'-sp'$	$-pq'-qp'-rs'-sr$	$pr'+qs'-rp'-sq'$	$pp'-qq'+rr'-ss'$

The terms may be permuted in a great variety of ways, by observing the above law of filling the cells; either of the above squares will satisfy the conditions of the question, except (5).

It remains to satisfy equations (5); for this purpose we will subtract the first of equations (1) from equations (5), and we get

$$b'^2+c''^2+d'''^2-a'^2-a''^2-a'''^2=0,\ b''^2+c'^2+d^2-a^2-a'^2-a''^2=0 \quad \ldots\ldots (10).$$

Substituting for these values their corresponding values in the last square, and taking the sum and difference of the resulting equations, we get

$$\left.\begin{aligned} pq(p'q'-r's')+ps(p's'-q'r')+qr(q'r'-p's')+rs(r's'-p'q)=0 \\ pr(p'r'-q's')+qs(q's'-p'r')=0 \end{aligned}\right\} \quad \ldots\ldots (11).$$

If we make $s=0$, these become

$$\left.\begin{aligned} p'r'=q's' \\ p(p'q'-r's')=r(p's'-q'r') \end{aligned}\right\} \quad \ldots\ldots (12).$$

68	—29	41	—37
—17	31	70	32
59	28	—23	61
—11	—77	8	49

Equations (12) may be satisfied in an infinite number of ways. If we take

$$p=9,\ q=5,\ r=5,\ s=0,$$
$$p'=6,\ q'=3,\ r'=2,\ s'=4;$$

we shall get the adjoining square, which are smaller than the numbers given by Euler.

$42+2q$	$-11+4q$	$24-q$	$2-8q$
$-18+8q$	$-16+q$	$21+4q$	$38+2q$
$11+4q$	$42-2q$	$-2-8q$	$24+q$
$16+q$	$-18-8q$	$-38+2q$	$21-4q$

If $p=2$, $q=q$, $r=5$, $s=0$, $p'=8$, $q'=4$, $r'=1$, $s'=2$; the sixteen numbers will be as in the margin, and in which q is entirely arbitrary. If $q=3$, we get the first of the succeeding sets; and if $p=9$, $q=2$, $r=5$, $s=0$, $p'=6$, $q'=3$, $r'=2$, $s'=4$, we have the second of the sets, both less than the numbers of Euler :

48	1	21	−22
6	−13	33	44
23	36	−26	27
19	−42	−32	9

56	−38	47	−19
−35	25	70	20
50	40	−5	55
−17	−59	−4	58

It should be observed, that these numbers, besides fulfilling the conditions of the question, satisfy the following :

1°. The sum of the squares of the four central terms, as well as the sum of the squares of the four corner terms, is equal to the sum of the squares of any four terms in a row.

2°. The sum of the squares of the four terms, at the extremities of the two middle horizontal rows, as well as the sum of the squares of the four terms, at the extremities of the two middle vertical rows, is the same as the sum of the squares of any four terms in a row.

(109). Question XII. *By* ——.

It is required to find the locus of the centres, and the *envelope*, of all the spheres that can be made to touch the surface of a given sphere, and also two planes, given in position.

Solution. *By the Proposer.*

The centres of the spheres will be in a plane bisecting the angle 2θ, of the given planes; let this plane be the plane of xy, and let the plane of xz pass through the centre of the sphere, the axis of y being the intersection of the planes. Let the centre of the given sphere be $a, 0, c$ and its radius R; the centre of one of the tangent spheres $x, y, 0$ and its radius r. Then, since the sphere is tangent to the planes,

(1), $$r = x \sin\theta,$$

and since they are tangent to each other

(2) $$(x-a)^2+y^2+c^2=(\text{R}+r)^2\,;$$

and eliminating r between these equations, we get the equation of the locus of the centres of the touching spheres

(3), $$y^2+x^2\cos^2\theta - 2kx - \text{A}^2 = 0\,;$$

which is therefore an ellipse, having its major axis on the axis of x, the distance of its centre from the origin being $k\sec^2\theta$, and the length of its semiaxes

$$\text{M}\sec\theta \text{ and } \text{M}\sec^2\theta, \text{ where}$$

(4), $k=a+\text{R}\sin\theta, \text{A}^2=\text{R}^2-a^2-c^2, \text{M}^2=k^2+\text{A}^2\cos^2\theta=(\text{R}+a\sin\theta)^2-c^2\cos^2\theta$

If the given sphere is either enveloped by, or envelopes, the tangent

spheres, R must be taken negative in these equations. If the given sphere intersects either or both the planes, there will be tangent spheres having their centres in the plane of yz, and the locus of their centres will be had by writing z for x and $\frac{1}{2}\pi-\theta$ for θ in equation (3).

The equation of the touching sphere whose centre is yx, is

(5). $$(\mathrm{X}-x)^2+(\mathrm{Y}-y)^2+\mathrm{Z}^2=r^2=x^2\sin^2\theta,$$

and that of the next one whose centre is $(y+dy),(x+dx)$, is

$$(\mathrm{X}-x-dx)^2+(\mathrm{Y}-y-dy)^2+\mathrm{Z}^2=(x+dx)^2\sin^2\theta$$

and therefore, for the intersection of these two spheres,

$$(\mathrm{Y}-y)\,dy+(\mathrm{X}-x\cos^2\theta)\,dx=0;$$

but by (3), $$y\,dy-(k-x\cos^2\theta)\,dx=0,$$

and by eliminating dy and dx.

(6), $$k\,(\mathrm{Y}-y)-\mathrm{Y}x\cos^2\theta+\mathrm{X}y=0.$$

(5) and (6) are the equations of the intersection of two consecutive tangent spheres, and eliminating y and x between (3), (5), (6), we get the equation of the envelope of all the tangent spheres, and which can be put into either of the two forms

(7), $$\{\mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2+\mathrm{A}^2)\cos^2\theta+2k\,(k-\mathrm{X})\}^2-4\mathrm{M}^2\{(k-\mathrm{X})^2+\mathrm{Y}^2\cos^2\theta\}=0,$$

(8), $$\{(\mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2-\mathrm{A}^2)\cos^2\theta-2k\mathrm{X}\}^2-4\mathrm{M}^2\{\mathrm{X}^2\sin^2\theta-\mathrm{Z}^2\cos^2\theta\}=0$$

When $\mathrm{X}=0$, equation (8) becomes

$$(\mathrm{Y}^2+\mathrm{Z}^2-\mathrm{A}^2)^2\cos^4\theta+4\mathrm{M}^2\mathrm{Z}^2\cos^2\theta=0,$$

which can only be satisfied by the values

$$\mathrm{Z}=0,\ \mathrm{Y}=\pm\mathrm{A}=\pm\sqrt{\mathrm{R}^2-a^2-c^2},$$

and these two points are the intersection of the surface with the plane YZ; they only exist when

$$\mathrm{R}^2=\text{or}>a^2+c^2,$$

or when the axis of Y is either a tangent or chord of the given sphere.

When $\mathrm{Z}=0$, equation (8) represents the intersection of the surface with the plane of XY; its first member is divisible into two factors, and the equation is satisfied by either

(9), $$\mathrm{Y}^2\cos^4\theta+(\mathrm{X}\cos^2\theta-k-\mathrm{M}\sin\theta)^2=(\mathrm{M}+k\sin\theta)^2,$$

(10), $$\mathrm{Y}^2\cos^4\theta+(\mathrm{X}\cos^2\theta-k+\mathrm{M}\sin\theta)^2=(\mathrm{M}-k\sin\theta)^2;$$

which are two circles such that, R_1, R_2 being their radii, and D the distance of their centres,

(11), $$\mathrm{R}_1=\frac{\mathrm{M}+k\sin\theta}{\cos^2\theta},\ \mathrm{R}_2=\pm\frac{\mathrm{M}-k\sin\theta}{\cos^2\theta},\ \mathrm{D}=\frac{2\mathrm{M}\sin\theta}{\cos^2\theta}.$$

The upper sign is used when

$$\mathrm{M}>k\sin\theta,$$
$$k^2+\mathrm{A}^2\cos^2\theta>k^2\sin^2\theta,$$
$$k^2+\mathrm{A}^2>0,$$
$$\mathrm{R}^2\,(1+\sin^2\theta)+2a\mathrm{R}\sin\theta-c^2>0;$$

and in this case

$$\mathrm{R}_1+\mathrm{R}_2=\frac{2\mathrm{M}}{\cos^2\theta}>d,$$

$$\mathrm{R}_1-\mathrm{R}_2=\frac{2k\sin\theta}{\cos^2\theta}=\frac{k\mathrm{D}}{\mathrm{M}};$$

hence the circles intersect each other when

$$k<\mathrm{M},\ \text{or}\ \mathrm{R}^2>a^2+c^2,$$

they touch each other when

$$k = \text{M}, \text{ or } \text{R}^2 = a^2 + c^2,$$

and they envelope each other when

$$k > \text{M}, \text{ or } \text{R}^2 < a^2 + c^2.$$

The under sign is used when

$$\text{M} < k \sin\theta,$$

$$\text{R}^2 (1 + \sin^2\theta) + 2a\text{R}\sin\theta - c^2 < 0;$$

and then $$\text{R}_1 - \text{R}_2 = \frac{2\text{M}}{\cos^2\theta} > \text{D},$$

or the circles envelope each other. One circle becomes a point when

$$\text{M} = k \sin\theta,$$

or $$\text{R}^2 (1 + \sin^2\theta) + 2a\text{R}\sin\theta - c^2 = 0.$$

If, then, it were required under any circumstances, to make spheres touch a given sphere and two planes; it would be sufficient to make circles touch the two circles (9) and (10), under the same circumstances, and the spheres of which these touching circles are the great sections, would evidently be the ones required. For instance, if spheres were described on the tangent circles in question (50), they would all touch a sphere and two planes whose positions may be found. It follows also that, having given two planes and a sphere, the sphere may be so placed that n spheres can be described each touching two of the others as well as the given sphere and the two planes; it is obviously sufficient that R_1, R_2, D may have the relation of R, r, d in equation (25), p. 248, Vol. I.; and this will be the case when the centre of the sphere is placed on the circumference of either the ellipse or the hyperbola, on the plane of xz, whose equations are

$$c^2\left(1 - \cos^2\theta\cos^2\frac{i\pi}{n}\right) + a^2\sin^2\theta\cos^2\frac{i\pi}{n} - 2a\text{R}\sin\theta\sin^2\frac{i\pi}{n} = \text{R}^2\left(\sin^2\theta + \sin^2\frac{i\pi}{n}\right),$$

$$c^2\left(1 - \sin^2\theta\cos^2\frac{i\pi}{n}\right) - a^2\sin^2\theta\cos^2\frac{i\pi}{n} - 2a\text{R}\sin\theta = \text{R}^2\left(1 + \sin^2\theta\sin^2\frac{i\pi}{n}\right).$$

Again, when Y = 0, the equation (7) represents the intersection of the surface with the plane of XY; and it is satisfied by either

$$(12),\quad \text{Z}^2\cos^4\theta + (\text{X}\cos^2\theta - \text{M} - k)^2 = (\text{M} + k)^2\sin^2\theta,$$

$$(13),\quad \text{Z}^2\cos^4\theta + (\text{X}\cos^2\theta + \text{M} - k)^2 = (\text{M} - k)^2\sin^2\theta;$$

which are also two circles, and if r_1, r_2 be their radii, d the distance of their centres we shall have

$$(14),\quad r_1 = \frac{(\text{M}+k)\sin\theta}{\cos^2\theta},\quad r_2 = \frac{(\text{M}-k)\sin\theta}{\cos^2\theta},\quad d = \frac{2\text{M}}{\cos^2\theta};$$

and we find that

when $\text{M} > k \sin\theta$, these circles are without each other;
when $\text{M} = k \sin\theta$, they touch externally;
when $\text{M} < k \sin\theta$, they intersect each other.

The surface is evidently an elliptic ring, its axes being the ellipse (3)

on the plane of XY, and all the sections perpendicular to this ellipse are circles, whose equations are (5) and (6).

(110.) QUESTION XIII. *By* ψ.

See Dr. Bowditch's Commentary on the Mécanique Céleste, Vol. I, page 304, equations (i). It is required to be determined whether these equations cannot be reduced to the forms given in page 313, equations (H), in a more simple manner than has been done in that admirable work.

FIRST SOLUTION. *By Dr. T. Strong, New-Brunswick, N. J.*

The equations (i) are

$$\frac{d^2x}{dt^2}=\frac{dQ}{dx},\qquad \frac{d^2y}{dt^2}=\frac{dQ}{dy},\qquad \frac{d^2z}{dt^2}=\frac{dQ}{dz}\quad . \qquad (1),$$

Q being a function of x, y, z; and at p. 306 of that work,

$$x=r\cos\theta\cos v,\qquad y=r\cos\theta\sin v,\qquad z=r\sin\theta\quad . \qquad (2);$$

hence Q is a function of r, θ, v; therefore

$$\frac{dQ}{dx}dx+\frac{dQ}{dy}dy+\frac{dQ}{dz}dz=\frac{dQ}{dr}dr+\frac{dQ}{d\theta}d\theta+\frac{dQ}{dv}dv\quad . \quad (3);$$

or, by substituting the values of dx, dy, dz from (2) in (3), we get

$$-\left[\frac{dQ}{dx}r\sin\theta\cos v+\frac{dQ}{dy}r\sin\theta\sin v-\frac{dQ}{dz}r\cos\theta\right]d\theta$$

$$+\left[\frac{dQ}{dy}r\cos\theta\cos v-\frac{dQ}{dx}r\cos\theta\sin v\right]+\left[\frac{x}{r}\cdot\frac{dQ}{dx}+\frac{y}{r}\cdot\frac{dQ}{dy}+\frac{z}{r}\cdot\frac{dQ}{dz}\right]dr$$

$$=\frac{dQ}{d\theta}d\theta+\frac{dQ}{dv}dv+\frac{dQ}{dr}dr;$$

therefore by the method of indeterminate co-efficients, equating the co-efficients of $d\theta$, dv, dr, we have

$$\left.\begin{aligned}
&\frac{dQ}{dz}r\cos\theta-\frac{dQ}{dx}r\sin\theta\cos v-\frac{dQ}{dy}r\sin\theta\sin v=\frac{dQ}{d\theta},\\
&\frac{dQ}{dy}r\cos\theta\cos v-\frac{dQ}{dx}r\cos\theta\sin v=x\frac{dQ}{dy}-y\frac{dQ}{dx}=\frac{dQ}{dv},\\
\text{or, by (1),}\quad &\frac{xd^2y-yd^2x}{dt^2}=\frac{d\,(xdy-ydx)}{dt^2}=\frac{dQ}{dv},\\
&x\frac{dQ}{dx}+y\frac{dQ}{dy}+z\frac{dQ}{dz}=r\frac{dQ}{dr}
\end{aligned}\right\}\quad (4).$$

By (1), the last of (4) becomes

$$\frac{xd^2x+yd^2y+zd^2z}{dt^2}=r\frac{dQ}{dr}\quad . \quad . \quad . \quad . \quad . \qquad (5),$$

and, by (2), $\quad x^2+y^2+z^2=r^2$, therefore $xdx+ydy+zdz=rdr$;

and $\quad xd^2x+yd^2y+zd^2z=rd^2r+dr^2-(dx^2+dy^2+dz^2)$,

or, since, by (2), $\quad dx^2+dy^2+dz^2=dr^2+r^2\cos^2\theta\,dv^2+r^2d\theta^2$,

$$xd^2x+yd^2y+zd^2z=rd^2r-r^2\,(\cos^2\theta\,dv^2+d\theta^2);$$

hence (5) is changed to $\quad \dfrac{d^2r-r\cos^2\theta\,dv^2-rd\theta^2}{dt^2}=\dfrac{dQ}{dr}\quad . \quad . \quad (6),$

which is the first of the equations (H). Again by (2),

$\frac{y}{x} = \tan v$, $\therefore \frac{xdy - ydx}{x^2} = \frac{dv}{\cos^2 v}$, or $xdy - ydx = r^2 \cos^2\theta\, dv$,
hence the last but one of equations (4) becomes

$$\frac{d\,(r^2 \cos^2 \theta dv)}{dt^2} = \frac{dQ}{dv} \quad . \quad . \quad . \quad . \quad (7),$$

which is the second of equations (H). The first of (4) is reduced by (1) to

$$\frac{r\cos\theta\, d^2 z - r\sin\theta \cos v d^2 x - r\sin\theta\, \sin v d^2 y}{dt^2} = \frac{dQ}{d\theta}$$

the first member of which is reduced by (2), to

$$\frac{r\cos^2\theta\, d^2 z - \sin\theta\,(xd^2 x + yd^2 y)}{dt^2 \cos\theta} = \frac{rd^2 z - \sin\theta\,(xd^2 x + yd^2 y + zd^2 z)}{dt^2 \cos\theta}$$

$$= \frac{rd^2 z - \sin\theta\,[rd^2 r - r^2 \cos\theta^2 dv^2 - r^2 d\theta^2]}{dt^2 \cos\theta}$$

$$= \frac{r^2 d^2\theta + 2rdr\, d\theta + r^2 \sin\theta \cos\theta\, dv^2}{dt^2}$$

hence $$\frac{d\,(r^2 d\theta) + r^2 \sin\theta\, \cos\theta\, dv^2}{dt^2} = \frac{dQ}{d\theta} \quad . \quad . \quad . \quad (8),$$

which is the third and last of equations (H). In conclusion we would remark that these equations could be obtained with the greatest facility by the very ingenious method of transformation given by Pontecoulant at p. 207, Vol. I, of his Systeme du Monde, which comes to the same thing as the method given by La Grange at p. 304, etc., of his Mec. Anal. Vol. I.

Second Solution. *By Prof. M. Catlin, Hamilton College.*

Put $p = \cos\theta$, $p' = \sin\theta$, $q = \cos v$, $q' = \sin v$. Then equations (*i*) are reduced, by the method of Dr. Bowditch, to the following:

$$\left.\begin{array}{ll} [503a] & \left(\frac{dQ}{dr}\right) = pq\frac{d^2 x}{dt^2} + pq'\frac{d^2 y}{dt^2} + p'\frac{d^2 z}{dt^2} \\ [506a] & \left(\frac{dQ}{dv}\right) = -rpq'\frac{d^2 x}{dt^2} + rpq\,\frac{d^2 y}{dt^2} \\ [507a] & \left(\frac{dQ}{d\theta}\right) = -rp'\,\frac{d^2 x}{dt^2} - rp'q'\frac{d^2 y}{dt^2} + rp\frac{d^2 z}{dt^2} \end{array}\right\} \quad (1),$$

and, (501), $\quad x = rpq, \quad y = rpq', \quad z = rp' \quad . \quad . \quad (2),$

$$\therefore \left.\begin{array}{l} d^2 x = pqd^2 r + prd^2 q + qrd^2 p + 2pdqdr + 2qdpdr + 2rdpdq \\ d^2 y = pq'd^2 r + prd^2 q' + q'rd^2 p + 2pdq'dr + 2q'dpdr + 2rdpdq' \\ d^2 z = p'd^2 r + rd^2 p' + 2dp'dr \end{array}\right\} (3).$$

Substitute (3) in (1), observing that $p^2 + p'^2 = 1$, $q^2 + q'^2 = 1$,

$$\left.\begin{array}{l} \left(\frac{dQ}{dr}\right) = \frac{d^2 r}{dt^2} - rp^2.\,\frac{dv^2}{dt^2} - r\frac{d\theta^2}{dt^2} \\ \left(\frac{dQ}{dv}\right) = \frac{d(r^2 p^2 dv)}{dt^2} \\ \left(\frac{dQ}{d\theta}\right) = r^2\,\frac{d^2\theta}{dt^2} + r^2\,pp'.\,\frac{dv^2}{dt^2} + 2r\frac{dr}{dt}.\,\frac{d\theta}{dt} \end{array}\right\} \quad . \quad . \quad (4);$$

which are the equations (H) required.

(111.) QUESTION XIV. *By Professor B. Peirce, Harvard University.*

Calling the evolute of a curve its first evolute, the evolute of the first evolute the second evolute, that of the second evolute the third evolute, and that of the third evolute the fourth evolute; to find a curve whose fourth evolute is the curve placed in a position parallel to its original one: *i. e.* one in which the equation is the same when referred to rectangular axes parallel in the one case to those in the other.

SOLUTION. *By the Proposer.*

Let $s =$ arc of given curve,

$\varrho =$ its radius of curvature,

$\varphi =$ the angle which ϱ makes with the axis;

and let s', ϱ', φ' be the corresponding quantities for the fourth evolute. We find as in question (94),

$$\varphi' = \varphi + n\,.\,360^\circ., \qquad \varrho' = \frac{d^4\varrho}{d\varphi^4},$$

in which n is any integer whatever; so that if the curve is determined by the equation $\qquad \varrho = \mathrm{f}(\varphi),$
we have

$$\mathrm{f}(\varphi + n\,.\,360^\circ) = \frac{d^4\,.\,\mathrm{f}(\varphi)}{d\varphi^4}.$$

If, now, we suppose

$$\varrho = \mathrm{f}(\varphi) = \mathrm{A}e^{m\varphi} + \mathrm{A}'e^{m'\varphi} + \text{\&c.},$$

we have A, A′, &c., arbitrary, and m, m', &c., are the different roots of the equation $\qquad e^{mn.360^\circ} = m^4.$

General Case.—Let $\qquad n = 0,$
we have $\qquad m = \pm 1,$ and $\pm\sqrt{-1},$

$$\varrho = \mathrm{A}e^{\varphi} + \mathrm{A}'e^{-\varphi} + \mathrm{B}\sin(\varphi + \beta),$$

in which A, A′, B, β are arbitrary.

Case 1. Let $\mathrm{A}' = \mathrm{B} = 0$, the curve is a logarithmic spiral.

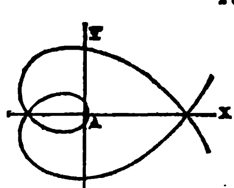
Case 2. Let $\mathrm{A} = \mathrm{A}' = 0$, the curve is a cycloid.

Case 3. Let $\mathrm{B} = 0$, the curve is a variety of Example 3, quest. (94).

Case 4. Let $\mathrm{A} = \mathrm{A}' = -\frac{1}{2}\mathrm{B},\ \beta = \frac{1}{2}\pi$, then

$$\varrho = \mathrm{A}\,(e^{\varphi} + e^{-\varphi} - 2\cos\varphi),$$

$$\begin{aligned} y &= \int \varrho d\varphi\,.\cos\varphi \\ &= \mathrm{A}\sqrt{\tfrac{1}{2}}\{e^{\varphi}\sin(\varphi + 45^\circ) + e^{-\varphi}\cos(\varphi - 45^\circ)\} \\ &\qquad - \mathrm{A}\,(\varphi + \tfrac{1}{2}\sin 2\varphi) \\ x &= -\int \varrho d\varphi\,.\sin\varphi \\ &= \mathrm{A}\sqrt{\tfrac{1}{2}}\{e^{\varphi}\cos(\varphi + 45^\circ) + e^{-\varphi}\cos(\varphi - 45^\circ)\} - \mathrm{A}\cos^2\varphi. \end{aligned}$$



When $\varphi = 0, \varrho = y = x = 0,\ \dfrac{dy}{dx} = \infty$, the axis of y is a tangent to the curve; but we shall find also for two numerically equal values of φ, the one positive, the other negative,

$$y_{\varphi} = -y_{-\varphi}\,,\quad x_{\varphi} = x_{-\varphi}\,,\quad \varrho_{\varphi} = \varrho_{-\varphi}\,;$$

and therefore there is no cusp at the origin A.
The curve makes an infinite number of revolutions round this point in both directions, receding very rapidly from it.

(112). QUESTION XV. *By Prof. B. Peirce.*

Integrate the equations

$$\frac{d^2y}{dx^2}+\frac{A}{x}.\frac{dy}{dx}-B^2\,x^n y=0,$$

$$\frac{d^2y}{dx^2}+A.\ \frac{dy}{dx}-B^2\,e^{nx}y=0;$$

in which A, B and n are constants, and e is the base of the Naperian system of logarithms.

FIRST SOLUTION. *By the Proposer.*

I. Since these equations are linear with respect to y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, we have only to find two particular values to satisfy them. If we denote these values by y' and y'', and take C and C' for two arbitrary constants, we have, for the complete value of y,

$$y=\text{C}\,y'+\text{C}'\,y''.$$

To find y' and y'' we will take the general equation of the second order

$$\frac{d^2y}{dx^2}+\text{P}\frac{dy}{dx}+\text{Q}\,y=0,$$

in which P and Q are functions of x. Suppose, then,

$$y'=\int e^{-\theta z}\varphi d\theta,$$

where z is a function of x, φ of θ. and the integral is definite. Then

$$\frac{dy'}{dx}=-\int\frac{dz}{dx}\,e^{-\theta z}\varphi\theta d\theta,$$

$$\frac{d^2y'}{dx^2}=-\int\frac{d^2z}{dx^2}\,e^{-\theta z}\,\varphi\theta d\theta+\int\frac{dz^2}{dx^2}\,e^{-\theta z}\,\varphi\theta^2\,d\theta$$

$$0=\frac{d^2y'}{dx^2}+\text{P}\frac{dy'}{dx}+\text{Q}y'$$

$$=\frac{dz^2}{dx^2}\int e^{-\theta z}\varphi\theta^2 d\theta-\left(\frac{d^2z}{dx^2}+\text{P}\,\frac{dz}{dx}\right)\int e^{-\theta z}\varphi\theta\,d\theta+\text{Q}\int e^{-\theta z}\,\varphi\,d\theta.$$

We will suppose the second member of this last equation

$=\text{X}\,.\,\Phi\,.\,e^{-\theta z}$,

in which X is a function x, and Φ of θ. Differentiation, relative to θ gives

$$\frac{dz^2}{dx^2}\,\varphi\theta^2-\left(\frac{d^2z}{dx^2}+\text{P}\,\frac{dz}{dx}\right)\varphi\,\theta+\text{Q}=\text{X}\left(\frac{d\Phi}{d\theta}-\Phi\,z\right).$$

If we again suppose $\Phi=a\varphi+b\varphi\theta+c\varphi\theta^2$,

$$\frac{d\Phi}{d\theta}=a'\varphi+b'\varphi\theta+c'\varphi\theta^2;$$

in which a, b, c, a', b', c' are constants, φ is determined by the equation

$$\frac{d\varphi}{\varphi}=\frac{a'-b+(b'-2c)\,\theta+c'\theta^2}{a+b\theta+c\theta^2}.\,d\theta\quad.\ .\ .\ .\ .\ (\text{I});$$

and if we put the co-efficients of θ and θ^2 equal to zero, we have

$$\frac{dz^2}{dx^2} = \mathrm{X}\,(c' - cz),$$

$$\frac{d^2z}{dx^2} + \mathrm{P}\frac{dz}{dx} = -\mathrm{X}\,(b' - bz),$$

$$\mathrm{Q} = \mathrm{X}\,(a' - az);$$

and, by the elimination of X,

$$\frac{dz^2}{dx^2}(a' - az) = \mathrm{Q}\,(c' - cz) \quad . \quad . \quad . \quad . \quad \text{(II)},$$

$$\frac{d^2z}{dx^2} + \mathrm{P}\frac{dz}{dx} = -\frac{dz^2}{dx^2}\,\frac{b' - bz}{c' - cz} \quad . \quad . \quad . \quad \text{(III)}.$$

Equations (II) and (III) serve to determine the value of z, when P and Q are given so as to satisfy the equation of condition involved in these equations, or to find values of P and Q for given values of z.

Case 1. Suppose, in (II) and (III), $z = x^n$,

and they give
$$\mathrm{Q} = n^2 x^{2n-2}.\frac{a' - ax^n}{c' - cx^n},$$

$$\mathrm{P} = -nx^{n-1}.\frac{b' - bx^n}{c' - cx^n} - (n-1)x^{-1};$$

which, when $a' = b = c' = 0$, become

$$\mathrm{Q} = \frac{a}{c}n^2.x^{2n-2}, \qquad \mathrm{P} = \frac{b'n - c\,(n-1)}{cx};$$

which coincides with the first of the given equations by changing $2n-2$ into n, and putting

$$\mathrm{A} = \frac{b'(n+2) - cn}{2c}, \qquad \mathrm{B}^2 = -\frac{a}{c}\,(\tfrac{1}{2}n+1)^2,$$

which requires that a and c be of opposite signs.

Case 2. Suppose $z = e^{nx}$,

we find
$$\mathrm{Q} = n^2 e^{2nx}.\frac{a' - ae^{nx}}{c' - ce^{nx}},$$

$$\mathrm{P} = -ne^{nx}.\frac{b' - be^{nx}}{c' - ce^{nx}} - n;$$

which, when $a' = b = c' = 0$, become

$$\mathrm{Q} = \frac{a}{c}\,n^2\,e^{2nx}, \qquad \mathrm{P} = \left(\frac{b'}{c} - 1\right)n;$$

and these coincide with the second given equation, by changing n into $\frac{1}{2}n$, and making

$$\mathrm{A} = \tfrac{1}{2}\left(\frac{b'}{c} - 1\right)n, \qquad \mathrm{B}^2 = -\frac{a}{4c}.\,n^2,$$

so that a and c must have opposite signs.

II. We will now consider the values of φ determined by equation (I), and the limits of the integration determined by the condition that

$$\Phi e^{-\theta z} = 0 \quad . \quad . \quad . \quad . \quad . \quad \text{(IV)}.$$

We will limit ourselves to the case involved in each of the preceding cases, that $a'=b=c'=0$, and $ac < 0$. Then (I) becomes

$$\frac{d\varphi}{\varphi}=\frac{(b' \quad 2c)\,\theta d\theta}{a+c\theta^2}, \text{ or, if } a=-cm^2,\ b'=2ch;\ \frac{d\varphi}{\varphi}=\frac{2\,(h-1)\,\theta d\theta}{\theta^2-m^2};$$

the integral of which gives

$$\varphi=(\theta^2-m^2)^{h-1},\ \text{or}=(m^2-\theta^2)^{h-1};$$

$$\Phi=c\varphi\,(\theta^2-m^2)=c\,(\theta^2-m^2)^h,\ \text{or}=-c\,(m^2-\theta^2)^h;$$

so that, when h is positive, (IV) is satisfied by $\theta=\pm m$; and, when z is positive, by $\theta=\infty$, when z is negative by $\theta=-\infty$.

We may then, for y', use the first value of φ, and the limits

$$\theta=\pm m \text{ to } \theta=\pm\infty,$$

the upper sign being used when z is positive, the lower when z is negative; and, for y'', we may use the second value of φ and the limits

$$\theta=-m \text{ to } \theta=+m.$$

SECOND SOLUTION. *By Dr. T. Strong, New-Brunswick.*

If in the first given equation we put $u=x^{n+2}=x^m$, and in the second $u'=e^{nx}$, and suppose $du=$ const., they will become

$$\frac{d^2y}{du^2}+\left(1+\frac{A-1}{m}\right)\frac{dy}{udu}-\frac{B^2}{m^2}\cdot\frac{y}{u}=0 \quad . \quad . \quad . \quad (1),$$

$$\frac{d^2y}{du^2}+\left(1+\frac{A}{n}\right)\frac{dy}{udu}-\frac{B^2}{n^2}\cdot\frac{y}{u}=0 \quad . \quad . \quad . \quad (2);$$

which have the same form. The differential equation

$$\frac{d^2y}{du^2}+(2pq-q+1)\ \frac{dy}{udu}+q^2a^2b^2u^{2q-2}\,y=0 \quad . \quad (3),$$

is satisfied by $\quad y=c\int dv\,(a^2-v^2)^{p-1}\cos buv \quad . \quad . \quad . \quad (4)$, the integral being taken between the limits $v=0$, $v=\pm a$, c being an arbitrary constant, and $p>0$; see Lacroix Calcul. Diff. et Int., Vol. III, p. 537. By comparing (1) and (3) we have $q=\frac{1}{2}$, $\frac{1}{4}a^2b^2=-B^2m^{-2}$;

$\therefore$ by assuming $a^2=1$, we get $b=\frac{2B}{m}\sqrt{-1}=b'\sqrt{-1}$; suppose also

$p+\frac{1}{2}=1+\frac{A-1}{m}$, or $p=\frac{2A+m-2}{2m}$, $p-1=\frac{2A-m-2}{2m}$; then we get

$$y=c\int dv(1-v^2)^{p-1}\cos b'u^{\frac{1}{2}}v\sqrt{-1}=c'\int dv(1-v^2)^{p-1}(e^{b'u^{\frac{1}{2}}v}+e^{-b'u^{\frac{1}{2}}v}).(5),$$

where $c=2c'$, and the integral is taken from $v=0$ to $v=1$. Again, if we put $v=v'\sqrt{-1}$, and $c=-c''\sqrt{-1}$, we have

$$y=c''\int dv'\,(1+v'^2)^{p-1}.\cos b'\,u^{\frac{1}{2}}v' \quad . \quad . \quad (6),$$

the limits of the integral being $v=0$, $v=\frac{\pi}{b'u^{\frac{1}{2}}}$; hence the complete value of y is the sum of the two values given by (5) and (6), c',c'' being the two arbitrary constants which the integral requires.

When $p<0$, the limits of the integral in (6) may also be $v'=0$ and $v'=\infty$, hence in this case the complete integral of (1) is

$$y=c'\int dv'\ (1+v'^2)^{p-1}\cos b'u^{\frac{1}{2}}v'+c''\int dv'\ (1+v'^2)^{p-1}\cos b'u^{\frac{1}{2}}v' \ . \ (7),$$

the first integral being taken between the limits $v' = 0$, $v' = \frac{\pi}{b'u}$ the second between the limits $v' = 0$, $v' = \infty$.

Again if we use the same values of q and a as before, and put $p = \frac{2A + n}{2n}$, $b' = \frac{2B}{n}$, then the complete value of y which satisfies (2) is the sum of the values in (5) and (6) when $p > 0$, or the value in (7) when $p < 0$.

Remark. See Mec. Cel., Vol. IV, supp. on capillary attraction, page 60, eq. (5) which is of the form

$$\frac{d^2 z}{du^2} + \frac{dz}{udu} - 2\left(a'z + \frac{1}{b''}\right) = 0,$$

which, by putting $\quad z + \frac{1}{a'b''} = y$ becomes

$$\frac{d^2 y}{du^2} + \frac{dy}{udu} - 2a'y = 0,$$

and if this be compared with (4) it gives $q = 1$, $p = \frac{1}{2}$, $a^2 = 1$, $b = \sqrt{-2a'}$, or $b' = \sqrt{2a'}$; and if these values be written in (5) and (6), the sum of these values will give the complete integral of La Place's equation which he failed to find; nor do I perceive that the deficiency has been supplied in Dr. Bowditch's Commentary. The partial integral which he finds, and which is equivalent to (5), is however sufficient for the particular purpose he had in view.

——— The following equations, which are frequently met with in Physico-Mathematical researches, will be found to be immediately dependant on each other, and their solutions, as well as those of many others, are comprehended in Prof. Peirce's Analysis :—

$$\frac{d^2 y}{dx^2} + \frac{A}{x} \cdot \frac{dy}{dx} - B^2 x^n y = 0,$$

$$\frac{d^2 y}{dx^2} + A \cdot \frac{dy}{dx} - B^2 e^{mx} y = 0,$$

$$\frac{d^2 y}{dx^2} - (A^2 + B^2 e^{2x})\, y = 0,$$

$$\frac{d^2 y}{dx^2} + \frac{A}{x} \cdot \frac{dy}{dx} - B^2 y = 0.$$

$$\frac{d^2 y}{dx^2} + \frac{A}{x}\,\frac{dy}{dx} - \frac{B^2}{x} \cdot y = 0,$$

$$\frac{d^2 y}{dx^2} + \left(\frac{A}{x^2} - B^2\right) y = 0,$$

$$\frac{d^2 y}{dx^2} - B^2 x^n y = 0;$$

and on the last depends the equation of Ricatti.

List of Contributors, and of Questions answered by each. The figures refer to the number of the questions, as marked in Number VI., Article XXV.

Prof. C. Avery, Hamilton College, N. Y., ans. all the questions.
P. Barton, Jun., Esperance, N. Y., ans. 1, 2, 3, 4.
B. Birdsall, Clinton Liberal Institute, N. Y., ans. 1 to 12, 15.
Prof. M. Catlin, Hamilton College, Clinton, N. Y., ans. all the questions.
E. H. Delafield, St. Paul's College, N. Y., ans. 3.
Engineer, ans. 1.
D. Kirkwood, York, Pa., ans. 3.
Wm. Lenhart, York, Pa., ans. 2, 3, 8.
L., Murray Co., Geo., ans. 1, 2, 3, 4, 9.
J. F. Macully, Teacher of Mathematics, New-York, ans. 1 to 10.
Omicron, Jun., Chapel Hill, N. C., ans. 3.
Prof. B. Peirce, Harvard University, Cambridge, ans. all the questions.
Prof. G. R. Perkins, Utica Academy, N. Y., ans. all the questions.
P., ans. 3, 10.
ψ, ans. 5, 13.
Prof. T. Strong, L. L. D., New Brunswick, N. J., ans. all the questions.

*** All communications for Number IX, which will be published on the first day of May, 1840, must be post paid, addressed to the Editor, *College Point*, N. Y., and must arrive before the first of February, 1840. New Questions must be accompanied with their solutions.

The Editor begs to thank Mr. Spiller for a copy of his beautiful translation of Sturm's Theorem, and its demonstration.

A correspondent is right in supposing that the results of Mr. Young are erroneous in the problem referred to.

ARTICLE V.

NEW QUESTIONS TO BE ANSWERED IN NUMBER X.

Their solutions must arrive before August 1st, 1840.

(128). QUESTION I. *By P.*

Transform the value of x

$$x = a \sin \varphi + b \cos \varphi.$$

into a real form, when a and φ become imaginary.

(129). QUESTION II. *By P.*

Find all the values of

$$(1+\sqrt{-1})^{\frac{1}{7}} + (1-\sqrt{-1})^{\frac{1}{7}}.$$

(130). QUESTION III. *By Wm. Lenhart, Esq., York, Pa.*

Find x, y, x', y', so that

$x^3 + y^3 + xy$ may be a square,

$x'^3 + y'^3 + x'y'$ may be a cube.

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(131). QUESTION IV. *By Mr. P. Barton, Jun., Esperance, N. Y.*

Let $x + y + z - 10 = 0$, be the equation of a plane, $(2, 4, 4)$ the centre of a circle in that plane, whose radius is 6; to find the least distance from the point $(1, 2, 3)$ to the periphery of that circle.

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(132). QUESTION V. *By J. F. Macully, Esq. New-York.*

On a given regular polygon, as base, to construct a right pyramid, which shall have each of the solid angles at the base equal to m times the solid angle at the vertex.

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(133). QUESTION VI. *By Prof. W. McCartney, Easton, Pa.*

The centre of a circle moves along a given straight line, while its radius varies as the square of the distance of the centre from a given point in that line. It is required to find the line to which the circle is always a tangent.

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(134). QUESTION VII. *By J. F. Macully, Esq.*

Find the sum of n terms of the series

$$\frac{\sin^7\theta}{\cos\theta\cos2\theta} + \frac{8\sin^7\frac{1}{2}\theta}{\cos\frac{1}{2}\theta\cos\theta} + \frac{8^2\sin^7\frac{1}{4}\theta}{\cos\frac{1}{4}\theta\cos\frac{1}{2}\theta} + \&c.$$

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(135). QUESTION VIII. *By M. Terquem.*

To find the number of normals to a given algebraic surface of the degree m, that can be drawn through a given point.

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(136). QUESTION IX. *By M. Terquem.*

To find the locus of the points from which the sum of the squares of all the normals that can be drawn to a given surface of the second degree is a given constant quantity.

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(137). QUESTION X. *By ——.*

If P_x, P_{x+1}, P_{x+2} represent the consecutive terms of a series of integers which are at once polygonal numbers of both the m^{th} and n^{th} orders, they will be connected by the equation

$$P_{x+2} - 2aP_{x+1} + P_x = b;$$

where a and b are constants depending on the numbers m and n (see question (44), Math. Misc., Vol. I). It is required to express P_x in terms of x.

(138). QUESTION XI. *From the Dublin Problems.*

If any two polygons, having the same number of sides, be circumscribed about a given ellipse, so that the points of contact may be the middle points of each side; then, of these two polygons,

1°. Their areas are equal.

2°. The sum of the squares of their sides are equal.

3°. The sum of the squares of the semidiameters drawn to the points of contact of the sides, are equal.

4°. The sum of the squares of the lines drawn from the centre to the angular points of the polygons are equal.

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(139). QUESTION XII. *By* Δ.

Find the quickest path between two given points on the surface of a sphere; the force residing in another given point of the surface, and varying as some function of the distance.

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(140). QUESTION XIII. *By M. Poisson.*

Three players A, B, C, play, two by two, a series of games; each new game is played by the one who has won the preceding game, with the one who did not play, and the two who play the first game is determined by lot. The party is terminated when one of the three players has won two successive games, and this player gains the party. It is required to determine, for each player, the probability he has of winning the party, having given his chance of winning a game from either of his opponents.

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(141). QUESTION XIV. *By Investigator.*

M. Jacobi has shown that there exists a class of homogeneous liquid ellipsoids with three unequal axes, susceptible of equilibrium when revolving round one of these axes, with a centrifugal force It is required to determine "what are the precise limits within which this extension of the problem is possible."

Mr. Ivory and M. Liouville arrive at contradictory results on this subject.

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(142). QUESTION XV. *By Prof. B. Peirce, Harvard University.*

Professor ——— found it impossible to spin a top upon a hard steel floor, the point being perfectly sharp like that of a needle; while he found no difficulty in spinning one whose point was blunted and nearly hemispherical. Can this difference be explained by analysis?

Prof. P. desires it to be stated, that he has not attempted the solution of this question.

ARTICLE VI.

SUMMATION OF TRIGONOMETRICAL SERIES.

By J. F. Macully, Esq., New-York.

LITTLE has yet been done in the Inverse method of Finite Differences. Herschel's examples contain the most complete collection I have seen; and in these there is no approach to any classification of forms, beyond those of the most simple Algebraical and Circular Functions. It would seem that, like all Converse methods of operation, examples of the direct process must be multiplied to a great extent, before the results can be grouped in such a way as to lead to a direct knowledge of the primitive function from which a given Difference is derived. My aim in drawing up the following article, has been to contribute my quota to such a collection of examples. I have only inserted a few of the more remarkable results under the different classes, finding that to insert a greater number, as I originally designed, would have encroached too much on the limits of the Miscellany. The Classes marked (IV.), (V.), (VI.), (VII.), (VIII.) are particularly deficient, while they comprehend many results worth noting. If what I have done meets with the approbation of the readers of the Miscellany, I may at a future time add other examples together with forms derived from the combination of two or more of these Classes. I have added a few examples of series and products, the sums of which are derived from these examples.

Examples of the finite Differences of Trigonometrical Functions.

(I).
$$\begin{aligned}\Delta.\ h^x \sin k^x\theta &= h^{x+1} \sin k^{x+1}\theta - h^x \sin k^x\theta\\ &= h^x(h \sin k^{x+1}\theta - \sin k^x\theta),\end{aligned}$$

where h, k, θ are constant, and $\Delta x = 1$.

Example 1. Let $k = 2$, then since

$$\sin 2^{x+1}\theta = \sin 2\,.\,2^x\theta = 2 \sin 2^x\theta \cos 2^x\theta,$$

(a)
$$\begin{aligned}\Delta.\, h^x \sin 2^x\theta &= h^x(h \sin 2^{x+1}\theta - \sin 2^x\theta)\\ &= h^x \sin 2^x\theta(2h \cos 2^x\theta - 1).\end{aligned}$$

Thus, 1.
$$\begin{aligned}\Delta \sin 2^x\theta &= \sin 2^{x+1}\theta - \sin 2^x\theta\\ &= 2 \sin 2^{x-1}\theta \cos 2^{x-1}.3\theta,\end{aligned}$$

2.
$$\begin{aligned}\Delta 2^{-x} \sin 2^x\theta &= 2^{-x} \sin 2^x\theta(\cos 2^x\theta - 1)\\ &= -2^{-x+1} \sin 2^x\theta \sin^2 2^{x-1}\theta.\end{aligned}$$

Example 2. Let $k = \frac{1}{2}$, then

(b)
$$\begin{aligned}\Delta.h^x \sin 2^{-x}\theta &= h^x(h \sin 2^{-x-1}\theta - \sin 2^{-x}\theta)\\ &= h^x \sin 2^{-x-1}\theta(h - 2 \cos 2^{-x-1}\theta).\end{aligned}$$

Thus, 3.
$$\begin{aligned}\Delta \sin 2^{-x}\theta &= \sin 2^{-x-1}\theta - \sin 2^{-x}\theta\\ &= -2 \sin 2^{-x-2}\theta \cos 2^{-x-2}.3\theta,\end{aligned}$$

4.
$$\begin{aligned}\Delta.2^x \sin 2^{-x}\theta &= 2^{x+1} \sin 2^{-x-1}\theta(1 - \cos 2^{-x-1}\theta)\\ &= 2^{x+2} \sin 2^{-x-1}\theta \sin^2 2^{-x-2}\theta.\end{aligned}$$

Example 3. Let $k = 3$, then, since

$$\begin{aligned}\sin 3^{x+1}\theta &= \sin 3\,.\,3^x\theta\\ &= \sin 3^x\theta(2 \cos 3^x.2\theta + 1),\end{aligned}$$

(c)
$$\begin{aligned}\Delta.\, h^x \sin 3^x\theta &= h^x(h \sin 3^{x+1}\theta - \sin 3^x\theta)\\ &= h^x \sin 3^x\theta(2h \cos 3^x\,.\,2\theta + h - 1).\end{aligned}$$

Thus, 5. $\Delta \sin 3^x\theta = 2\sin 3^x\theta \cos 3^x . 2\theta,$

6. $\Delta\, 2^{-x} \sin 3^x\theta = 2^{-x-1} \sin 3^x\theta(2\cos 3^x . 2\theta - 1)$

$$= 2^{-x-1} \sin 3^x\theta . \frac{\cos 3^{x+1}\theta}{\cos 3^x\theta}$$

$$= 2^{-x-1} \tan 3^x\theta \cos 3^{x+1}\theta,$$

7. $\Delta .(-1)^x \sin 3^x\theta = (-1)^{x+1} . 2\sin 3^x\theta(\cos 3^x . 2\theta + 1)$

$$= (-1)^{x+1} . 4 \sin 3^x\theta \cos 3^x\theta$$

$$= (-1)^{x+1} . 2 \sin 3^x . 2\theta \cos 3^x\theta$$

Example 4. Let $k = \frac{1}{3}$, then

(*d*) $\Delta . h^x \sin 3^{-x}\theta = h^x (h \sin 3^{-x-1}\theta - \sin 3^{-x}\theta)$

$$= h^x \sin 3^{-x-1}\theta(h - 1 - 2\cos 3^{-x-1} . 2\theta).$$

Thus 8. $\Delta \sin 3^{-x}\theta = -2 \sin 3^{-x-1} . \theta \cos 3^{-x-1} . 2\theta,$

9. $\Delta (-1)^x \sin 3^{-x}\theta = -4\,(-1)^x \sin 3^{-x-1}\theta \cos^2 3^{-x-1}\theta,$

10. $\Delta\, 2^x \sin 3^{-x}\theta = 2^x \sin 3^{-x-1}\theta\,(1 - 2\cos 3^{-x-1} . 2\theta)$

$$= -2^x \cos 3^{-x}\theta \tan 3^{-x-1}\theta,$$

11. $\Delta\, 3^x \sin 3^{-x}\theta = 4 . 3^x \sin^3 3^{-x-1}\theta.$

Example 5. Let $h = 1$, $k = \frac{1}{5}$,

12. $\Delta \sin 5^{-x}\theta = \sin 5^{-x-1}\theta - \sin 5^{-x}\theta$

$$= -2\sin\tfrac{1}{2}(5^{-x} - 5^{-x-1})\theta \cos\tfrac{1}{2}(5^{-x} + 5^{-x-1})\theta$$

$$= -2\sin 2 . 5^{-x-1}\theta \cos 3 . 5^{-x-1}\theta.$$

(II.) $\Delta\, h^x \cos k^x\theta = h^x (h \cos k^{x+1}\theta - \cos k^x\theta).$

Example 1. Let $k = 2$, then, since

$$\cos 2^{x+1}\theta = \cos 2 . 2^x\theta = 2\cos^2 2^x\theta - 1,$$

(*e*) $\Delta . h^x \cos 2^x\theta = h^x(2h \cos^2 2^x\theta - \cos 2^x\theta - h).$

Thus; 13. $\Delta \cos 2^x\theta = 2 \sin 3 . 2^{x-1}\theta \sin 2^{x-1}\theta,$

14. $\Delta (-1)^x \cos 2^x\theta = (-1)^{x+1} . 2\cos 3 . 2^{x-1}\theta \cos 2^{x-1}\theta.$

Example 2. Let $k = \frac{1}{2}$, then

(*f*) $\Delta . h^x \cos 2^{-x}\theta = h^x (h \cos 2^{-x-1}\theta - \cos 2^{-x}\theta)$

$$= h^x (h \cos 2^{-x-1}\theta - 2\cos^2 2^{-x-1}\theta + 1).$$

Thus, 15. $\Delta \cos 2^{-x}\theta = 2 \sin 2^{-x-2}\theta \sin 2^{-x-2} . 3\theta,$

16. $\Delta .(-1)^x \cos 2^{-x}\theta = (-1)^{x+1} . 2\cos 2^{-x-2}\theta \cos 2^{-x-2} . 3\theta,$

17. $\Delta .\, 2^x \cos 2^{-x}\theta = 2^x(1 + 4\cos 2^{-x-1}\theta \sin^2 2^{-x-2}\theta).$

Example 3. Let $k = 3$, then

(*g*) $\Delta . h^x \cos 3^x\theta = h^x (h \cos 3^{x+1}\theta - \cos 3^x\theta)$

$$= h^x \cos 3^x\theta\,(2h \cos 2 . 3^x\theta - h - 1).$$

Thus; 18. $\Delta . \cos 3^x\theta = -4 \sin^2 3^x\theta \cos 3^x\theta,$

19. $\Delta . (-1)^x \cos 3^x\theta = 2\,(-1)^{x+1} \cos 3^x\theta \cos 2 . 3^x\theta,$

20. $\Delta\, 2^x \cos 3^x\theta = 2^x \cos 3^x\theta\,(1 - 8\sin^2 3^x\theta),$

21. $\Delta .(-2)^{-x} \cos 3^x\theta = (-2)^{-x-1} \cot 3^x\theta \sin 3^{x+1}\theta,$

22. $\Delta .(-3)^{-x} \cos 3^x\theta = (-3)^{-x-1} . 4\cos 3^x\theta \sin^2 3^x\theta.$

Example 4. Let $k = \frac{1}{3}$, then

(*h*) $\Delta . h^x \cos 3^{-x}\theta = h^x \cos 3^{-x-1}\theta\,(h + 1 - 2\cos 2 . 3^{-x-1}\theta).$

Thus, 23. $\Delta \cos 3^{-x}\theta = 4\cos 3^{-x-1}\theta \sin^2 3^{-x-1}\theta$

$$= 2\sin 3^{-x-1}\theta \sin 3^{-x-1} . 2\theta,$$

24. $\Delta (-1)^x \cos 3^{-x}\theta = (-1)^{x+1} . 2\cos 3^{-x-1}\theta \cos 2 . 3^{-x-1}\theta,$

25. $\Delta (-2)^x \cos 3^{-x}\theta = -(-2)^x \sin 3^{-x}\theta \cot 3^{-x-1}\theta,$

26. $\Delta (-3)^x \cos 3^{-x}\theta = -(-3)^x . 4\cos^3 3^{-x-1}\theta.$

(III). $\Delta.\dfrac{1}{h^x\sin k^x\theta} = -\dfrac{\Delta.\,h^x\sin k^x\theta}{h^{2x+1}\sin k^x\theta\,\sin k^{x+1}\theta}.$

Thus, 27. $\Delta.\dfrac{1}{\sin 2^x\theta} = \dfrac{-2\sin 2^{x-1}\theta\,\cos 2^{x-1}.3\theta}{\sin 2^x\theta\,\sin 2^{x+1}\theta} = -\dfrac{\cos 2^{x-1}.\,3\theta}{\cos 2^{x-1}\theta\,\sin 2^{x+1}\theta}.$

28. $\Delta\,\dfrac{2^x}{\sin 2^x\theta} = -\dfrac{2^x\tan 2^{x-1}\theta}{\cos 2^x\theta} = \dfrac{2^{x+1}\sin^2 2^{x-1}\theta}{\sin 2^{x+1}\theta},$

29. $\Delta.\dfrac{1}{\sin 2^{-x}\theta} = \dfrac{2\sin 2^{-x-2}\theta\,\cos 2^{-x-2}.\,3\theta}{\sin 2^{-x}\theta\,\sin 2^{-x-1}\theta} = \dfrac{\cos 2^{-x-2}.\,3\theta}{\sin 2^{-x}\theta\,\cos 2^{-x-2}\theta}.$

30. $\Delta.\dfrac{1}{2^x\sin 2^{-x}\theta} = \dfrac{-\sin^2 2^{-x-2}\theta}{2^{x-1}\sin 2^{-x}\theta},$

31. $\Delta.\dfrac{1}{\sin 3^x\theta} = \dfrac{-2\cos 3^x.\,2\theta}{\sin 3^{x+1}\theta},$

32. $\Delta.\dfrac{(-1)^x}{\sin 3^x\theta} = \dfrac{4.\,(-1)^{x+1}\cos^2 3^x\theta}{\sin 3^{x+1}\theta},$

33. $\Delta.\dfrac{2^x}{\sin 3^x\theta} = -2^x\sec 3^x\theta\cot 3^{x+1}\theta,$

34. $\Delta.\dfrac{1}{2^x\sin 3^{-x}\theta} = \dfrac{1}{2^{x+1}}.\cot 3^{-x}\theta\sec 3^{-x-1}\theta,$

35. $\Delta.\dfrac{1}{3^x\sin 3^{-x}\theta} = -\dfrac{4}{3^{x+1}}.\dfrac{\sin^2 3^{-x-1}\theta}{\sin 3^{-x}\theta},$

36. $\Delta.\dfrac{1}{\sin 5^{-x}\theta} = \dfrac{4\cos 5^{-x-1}\theta\cos 3.\,5^{-x-1}\theta}{\sin 5^{-x}\theta}.$

(IV.) $\Delta.\dfrac{1}{h^x\cos k^x\theta} = \dfrac{-\Delta.\,h^x\cos k^{x+1}\theta}{h^{2x+1}\cos k^x\theta\cos k^{x+1}\theta}.$

Thus, 37. $\Delta.\dfrac{1}{\cos 3^{-x}\theta} = -\dfrac{4\sin^2 3^{-x-1}\theta}{\cos 3^{-x}\theta},$

38. $\Delta.\dfrac{1}{(-3)^x\cos 3^{-x}\theta} = \dfrac{4\cos^2 3^{-x-1}\theta}{(-3)^{x+1}\cos 3^{-x}\theta}.$

(V.) $\Delta.\dfrac{1}{h^x\sin^2 k^x\theta} = \dfrac{\Delta.\,h^x\cos k^x.\,2\theta - \Delta\,h^x}{2h^{2x+1}\sin^2 k^x\theta\sin^2 k^{x+1}\theta}.$

Thus, 39. $\Delta.\dfrac{1}{\sin^2 3^{-x}\theta} = \dfrac{4\cos 3^{-x-1}.\,2\theta\sin 3^{-x-1}.\,2\theta}{2\sin^2 3^{-x}\theta\sin^2 3^{-x-1}\theta}$

$= \dfrac{8\cos 3^{-x-1}.\,2\theta\cos^2 3^{-x-1}\theta}{\sin^2 3^{-x}\theta},$

40. $\Delta.\dfrac{1}{2^x\sin^2 2^{-x}\theta} = \dfrac{\cos 2^{-x}\theta}{2^x\sin^2 2^{-x}\theta}.$

(VI.) $\Delta.\dfrac{1}{h^x\cos^2 k^x\theta} = \dfrac{-\Delta.\,h^x\cos k^x.\,2\theta - \Delta\,h^x}{2h^{2x+1}\cos^2 k^x\theta\cos^2 k^{x+1}\theta}.$

Thus, 41. $\Delta.\dfrac{1}{\cos^2 2^{-x}\theta} = \dfrac{-\sin 2^{-x-1}\theta\sin 2^{-x-1}.\,3\theta}{\cos^2 2^{-x}\theta\cos^2 2^{-x-1}.\,\theta}$

$= \dfrac{-16\sin^3 2^{-x-1}\theta\sin 2^{-x-1}.\,3\theta}{\sin^2 2^{-x+1}\theta}$

$$42.\ \Delta.\frac{1}{\cos^2 4^{-x}\theta} = \frac{-\sin 4^{-x-1}.3\theta \sin 4^{-x-1}.5\theta}{\cos 4^{-x}\theta \cos 4^{-x-1}\theta}$$

(VII.) $$\Delta.\ h^x \cot k^x\theta = h^x.\frac{(h-1)\sin k^x(k+1)\theta-(h+1)\sin k^x(k-1)\theta}{2\sin k^x\theta \sin k^{x+1}\theta},$$

Thus, 43. $$\Delta \cot k^x\theta = -\frac{\sin k^x(k-1)\theta}{\sin k^x\theta \sin k^{x+1}\theta},$$

$$44.\ \Delta.3^x \cot 2^x\theta = 3^x\frac{\cos 2^x.3\theta}{\cos 2^x\theta \sin 2^{x+1}\theta},$$

$$45.\quad \Delta \cot 3^{-x}\theta = \frac{\sin 3^{-x-1}.2\theta}{\sin 3^{-x}\theta \sin 3^{-x-1}\theta},$$

$$= \frac{2\cos 3^{-x-1}\theta}{\sin 3^{-x}\theta},$$

$$46.\ \Delta.\ 3^x \cot 3^{-x}\theta = 3^x\frac{2\sin 3^{-x-1}.4\theta + 4\sin 3^{-x-1}.2\theta}{2\sin 3^{-x}\theta \sin 3^{-x-1}\theta}$$

$$= 3^x.\frac{2\sin 3^{-x-1}.2\theta\,(1+\cos 3^{-x-1}.2\theta)}{\sin 3^{-x}\theta \sin 3^{-x-1}\theta}$$

$$= 3^x.\frac{8\cos^3 3^{-x-1}\theta}{\sin 3^{-x}\theta},$$

$$47.\ \Delta.\ 3^{-x}\cot 3^{-x}\theta = \frac{4}{3^{x+1}}.\quad \frac{\sin 3^{-x-1}.2\theta \sin 3^{-x-1}\theta}{\sin 3^{-x}\theta}.$$

(VIII.) $$\Delta.\ h^x \tan k^x\theta = h^x.\frac{(h-1)\sin k^x(k+1)\theta+(h+1)\sin k^x(k-1)\theta}{2\cos k^x\theta \cos k^{x+1}\theta}$$

$$= -k^{2x+1}\tan k^x\theta \tan k^{x+1}\theta\ \Delta.\ h^{-x}\cot k^x\theta.$$

Examples of Trigonometrical Series.

If u_x represent the x^{th} term of a series, s_x the sum of x terms, s the sum of an infinite number of terms, Σ the symbol of finite integration, and c a constant quantity,

$$s_x = u_1 + u_2 + u_3 + \ldots + u_x,$$

$$s_{x+1} = u_1 + u_2 + u_3 + \ldots + u_x + u_{x+1};$$

$$s_{x+1} - s_x = \Delta\ s_x = u_{x+1},$$

and $s_x = c + \Sigma\ u_{x+1}$.

Example 1. Let $s_x = \sin\theta \sin^2\frac{1}{2}\theta + 2\sin\frac{1}{2}\theta \sin^2\frac{1}{4}\theta + \ldots$

$$+ 2^{x-1}\sin\frac{\theta}{2^{x-1}}\sin^2\frac{\theta}{2^x}.$$

$$u_{x+1} = 2^x \sin 2^{-x}\theta \sin^2 2^{-x-1}\theta,$$

and by equation (4)

$$s_x = c + \Sigma.2^x \sin 2^{-x}\sin^2 2^{-x-1}\theta$$

$$= c + 2^{x-2}\sin 2^{-x+1}\theta,$$

$$s_0 = 0 = c + \tfrac{1}{4}\sin 2\theta,$$

$$s_x = 2^{x-2}\sin 2^{-x+1}\theta - \tfrac{1}{4}\sin 2\theta.$$

When $x = \infty$, the angle $2^{-x+1}\theta$ becomes indefinitely small, and its sine becomes equal to the arc, so that, then,

$$2^{x-2}\sin 2^{-x+1}\theta = 2^{x-2}\times 2^{-x+1}\theta = \tfrac{1}{2}\theta,$$

$$s = \tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta.$$

Example 2. Let $s_x = \tan\theta\cos 3\theta + \frac{1}{2}\tan 3\theta\cos 9\theta + \dots + 2^{-x+1}\tan 3^{x-1}\theta\cos 3^x\theta$.

$$u_{x+1} = 2^{-x}\tan 3^x\theta\cos 3^{x+1}\theta$$

and (eq. 6), $$s_x = c + \Sigma\, 2^{-x}\tan 3^x\theta\cos 3^{x+1}\theta$$
$$= c + 2^{-x+1}\sin 3^x\theta$$
$$s_0 = 0 = c + 2\sin\theta$$
$$s_x = 2(2^{-x}\sin 3^x\theta - \sin\theta).$$

Example 3. Let $s_x = \sin\theta\cos^2\theta - \sin 3\theta\cos^2 3\theta + \dots \dots (-1)^{x-1}\sin 3^{x-1}\theta\cos^2 3^{x-1}\theta$.

$$u_{x+1} = (-1)^x\sin 3^x\theta\cos^2 3^x\theta,$$

and (eq. 7), $$s_x = c - \tfrac{1}{4}(-1)^x\sin 3^x\theta;$$
$$s_0 = 0 = c - \tfrac{1}{4}\sin\theta,$$
$$s_x = \tfrac{1}{4}\sin\theta - \tfrac{1}{4}(-1)^x\sin 3^x\theta.$$

Example 4. Let $s_x = \sin\frac{1}{3}\theta\cos\frac{1}{2}\theta + \sin\frac{1}{15}\theta\cos\frac{1}{10}\theta + \dots + \sin 5^{-x+1}.\frac{1}{3}\theta\cos 5^{-x+1}.\frac{1}{2}\theta$

$$u_{x+1} = \sin 5^{-x}.\tfrac{1}{3}\theta\cos 5^{-x}.\tfrac{1}{2}\theta;$$

or, if $\theta = \frac{6}{5}\theta'$, $u_{x+1} = \sin 5^{-x-1}.2\theta'\cos 5^{-x-1}.2\theta'$;

and (eq. 12) $$s_x = c - \tfrac{1}{2}\sin 5^{-x}.\theta' = c - \tfrac{1}{2}\sin 5^{-x+1}.\tfrac{1}{6}\theta,$$
$$s_0 = 0 = c - \tfrac{1}{2}\sin\tfrac{5}{6}\theta,$$
$$s_x = \tfrac{1}{2}\sin\tfrac{5}{6}\theta - \tfrac{1}{2}\sin 5^{-x+1}.\tfrac{1}{6}\theta;$$
and $s = \frac{1}{2}\sin\frac{5}{6}\theta$.

Example 5. Let $s_x = \cos\theta\sin^2\frac{1}{2}\theta + 2\cos\frac{1}{2}\theta\sin^2\frac{1}{4}\theta + \dots\dots, + 2^{x-1}\cos\dfrac{\theta}{2^{x-1}}\sin^2\dfrac{\theta}{2^x}$

$$u_{x+1} = 2^x\cos 2^{-x}\theta\sin^2 2^{-x-1}\theta$$
$$= \tfrac{1}{2}\Delta.\,2^{x-1}\cos 2^{-x+1}\theta - \Delta 2^{x-2}, \qquad \text{(eq. 17)}$$
$$s_x = c + 2^{x-2}\cos 2^{-x+1}\theta - 2^{x-2}$$
$$= c - 2^{x-1}\sin^2 2^{-x}\theta;$$
$$s_0 = c - \tfrac{1}{2}\sin^2\theta = 0,$$
$$s_x = \tfrac{1}{2}(\sin^2\theta - 2^x\sin^2 2^{-x}\theta).$$
$$s = \tfrac{1}{2}\sin^2\theta.$$

Example 6. Let $s_x = \cot\theta\sec\frac{1}{3}\theta + \frac{1}{2}\cot\frac{1}{3}\theta\sec\frac{1}{9}\theta + \dots\dots + \dfrac{1}{2^{x-1}}\cot\dfrac{\theta}{3^{x-1}}\sec\dfrac{\theta}{3^x}$.

$$u_{x+1} = 2^{-x}\cot 3^{-x}\theta\sec 3^{-x-1}\theta,$$

and (eq. 34) $$s_x = c + 2^{-x+1}\operatorname{cosec} 3^{-x}\theta;$$
$$s_0 = c + 2\operatorname{cosec}\theta = 0,$$
$$s_x = 2(2^{-x}\operatorname{cosec} 3^{-x}\theta - \operatorname{cosec}\theta).$$

Example 7. Let $s_x = \dfrac{\sin^2\frac{1}{3}\theta}{\sin\theta} + \dfrac{1}{3}.\dfrac{\sin^2\frac{1}{9}\theta}{\sin\frac{1}{3}\theta} + \dots\dots + \dfrac{1}{3^{x-1}}.\dfrac{\sin^2 3^{-x}\theta}{\sin 3^{-x+1}\theta}$.

$$u_{x+1} = \frac{1}{3^x}.\frac{\sin^2 3^{-x-1}\theta}{\sin 3^{-x}\theta},$$

and (eq. 35). $$s_x = c - \frac{1}{4.3^{x-1}\sin 3^{-x}\theta};$$
$$s_0 = c - \frac{3}{4\sin\theta} = 0,$$
$$s_x = \frac{3}{4}\left(\frac{1}{\sin\theta} - \frac{1}{3^x\sin 3^{-x}\theta}\right).$$

$$s = \tfrac{3}{4}\left(\frac{1}{\sin\theta} - \frac{1}{\theta}\right).$$

Ex. 8. Let $\quad s_x = \dfrac{\cos\frac{1}{5}\theta \cos\frac{3}{5}\theta}{\sin\theta} + \dfrac{\cos\frac{1}{25}\theta \cos\frac{3}{25}\theta}{\sin\frac{1}{5}\theta} + \ldots$

$$+ \frac{\cos 5^{-x}\theta \cos 3.\,5^{-x}\theta}{\sin 5^{-x+1}\theta}.$$

$$u_{x+1} = \frac{\cos 5^{-x-1}\theta \cos 3.\,5^{-x-1}\theta}{\sin 5^{-x}\theta};$$

and (eq. 36), $\quad s_x = c + \dfrac{1}{4\sin 5^{-x}\theta}$

$$= \tfrac{1}{4}\left(\frac{1}{\sin 5^{-x}\theta} - \frac{1}{\sin\theta}\right).$$

Ex. 9. Let $\quad s_x = \dfrac{\cos^2\frac{1}{3}\theta}{\cos\theta} - \frac{1}{3}.\dfrac{\cos^2\frac{1}{9}\theta}{\cos\frac{1}{3}\theta} + \ldots\ldots + (-\tfrac{1}{3})^{x-1}.\dfrac{\cos^2 3^{-x}\theta}{\cos 3^{-x+1}\theta}.$

$$u_{x+1} = \frac{1}{(-3)^x}.\frac{\cos^2 3^{-x-1}\theta}{\cos 3^{-x}\theta},$$

and (eq. 38), $\quad s_x = c + \dfrac{1}{4.\,(-3)^{x-1}\cos 3^{-x}\theta}$

$$= \tfrac{3}{4}\left(\frac{1}{\cos\theta} - \frac{1}{(-3)^x \cos 3^{-x}\theta}\right). \qquad s = \tfrac{3}{4}\sec\theta$$

Ex. 10. Let $\quad s_x = \dfrac{\sin^2\frac{1}{3}\theta}{\cos\theta} + \dfrac{\sin^2\frac{1}{9}\theta}{\cos\frac{1}{3}\theta} + \ldots + \dfrac{\sin^2 3^{-x}\theta}{\cos 3^{-x+1}\theta}.$

$$u_{x+1} = \frac{\sin^2 3^{-x-1}\theta}{\cos 3^{-x}\theta},$$

and (eq. 37.) $\quad s_x = c - \dfrac{1}{4\cos 3^{-x}\theta}$

$$= \tfrac{1}{4}\left(\frac{1}{\cos\theta} - \frac{1}{\cos 3^{-x}\theta}\right). \qquad s = \tfrac{1}{4}(\sec\theta - 1).$$

Ex. 11. Let $\quad s_x = \dfrac{\cos^2\frac{1}{3}\theta \cos\frac{2}{3}\theta}{\sin^2\theta} + \dfrac{\cos^2\frac{1}{9}\theta \cos\frac{2}{9}\theta}{\sin^2\frac{1}{3}\theta} + \ldots\ldots$

$$+ \frac{\cos^2 3^{-x}\theta \cos 3^{-x}.\,2\theta}{\sin^2 3^{-x+1}\theta},$$

$$u_{x+1} = \frac{\cos^2 3^{-x-1}\theta \cos 3^{-x-1}.\,2\theta}{\sin^2 3^{-x}\theta},$$

and (eq. 39) $\quad s_x = c + \frac{1}{8}\operatorname{cosec}^2 3^{-x}\theta$

$$= \tfrac{1}{8}(\operatorname{cosec}^2 3^{-x}\theta - \operatorname{cosec}^2\theta).$$

Ex. 12. Let $\quad s_x = \dfrac{\cos\theta}{\sin^2\theta} + \frac{1}{2}.\dfrac{\cos\frac{1}{2}\theta}{\sin^2\frac{1}{2}\theta} + \ldots\ldots + \dfrac{1}{2^{x-1}}.\dfrac{\cos 2^{-x+1}\theta}{\sin^2 2^{-x+1}\theta}.$

$$u_{x+1} = \frac{\cos 2^{-x}\theta}{2^x \sin^2 2^{-x}\theta},$$

and (eq. 40) $\quad s_x = c + \dfrac{1}{2^x \sin^2 2^{-x}\theta}$

$$= \frac{1}{2^x \sin^2 2^{-x}\theta} - \frac{1}{\sin^2\theta}.$$

Ex. 13. Let
$$s_x = \frac{\sin^3 \frac{1}{2}\theta \sin \frac{3}{2}\theta}{\sin^4 \frac{1}{2}\theta} + \frac{\sin^3 \frac{1}{4}\theta \sin \frac{3}{4}\theta}{\sin^2 2\theta} + \ldots\ldots + \frac{\sin^3 2^{-x}\theta \sin 2^{-x}.3\theta}{\sin^2 2^{-x+2}\theta}$$

$$u_{x+1} = \frac{\sin^3 2^{-x-1}\theta \sin 2^{-x-1}.3\theta}{\sin^2 2^{-x+1}\theta},$$

and (eq. 41)
$$s_x = c - \tfrac{1}{16}\sec^2 2^{-x}\theta$$
$$= \tfrac{1}{16}(\sec^2\theta - \sec^2 2^{-x}\theta) \qquad s = \tfrac{1}{16}\tan^2\theta$$

Ex. 14. Let
$$s_x = \frac{\sin \frac{3}{4}\theta \sin \frac{5}{4}\theta}{\cos^2\theta \cos^2 \frac{1}{4}\theta} + \frac{\sin \frac{3}{16}\theta \sin \frac{5}{16}\theta}{\cos^2 \frac{1}{4}\theta \cos^2 \frac{1}{16}\theta} + \ldots\ldots + \frac{\sin 4^{-x}.3\theta \sin 4^{-x}.5\theta}{\cos^2 4^{-x+1}\theta \cos^2 4^{-x}\theta}.$$

$$u_{x+1} = \frac{\sin 4^{-x-1}.3\theta \sin 4^{-x-1}.5\theta}{\cos^2 4^{-x}\theta \cos^2 4^{-x-1}\theta},$$

and (eq. 42)
$$s_x = c - \sec^2 4^{-x}\theta$$
$$= \sec^2\theta - \sec^2 4^{-x}\theta. \qquad s = \tan^2\theta$$

Ex. 15. Let
$$s_x = \frac{\cos \frac{1}{3}\theta}{\sin\theta} + \frac{\cos \frac{1}{9}\theta}{\sin \frac{1}{3}\theta} + \ldots\ldots + \frac{\cos 3^{-x}\theta}{\sin 3^{-x+1}\theta}.$$

$$u_{x+1} = \frac{\cos 3^{-x-1}\theta}{\sin 3^{-x}\theta},$$

and (eq. 45)
$$s_x = c + \tfrac{1}{2}\cot 3^{-x}\theta$$
$$= \tfrac{1}{2}(\cot 3^{-x}\theta - \cot\theta).$$

Ex. 16. Let
$$s_x = \frac{\cos^3 \frac{1}{3}\theta}{\sin\theta} + 3.\frac{\cos^3 \frac{1}{9}\theta}{\sin \frac{1}{3}\theta} + \ldots\ldots + 3^{x-1}.\frac{\cos^3 3^{-x}\theta}{\sin 3^{-x+1}\theta}.$$

$$u_{x+1} = 3^x.\frac{\cos^3 3^{-x-1}\theta}{\sin 3^{-x}\theta},$$

and (eq. 46)
$$s_x = c + \tfrac{1}{8}.3^x \cot 3^{-x}\theta$$
$$= \tfrac{1}{8}(3^x \cot 3^{-x}\theta - \cot\theta).$$

Ex. 17. Let
$$s_x = \frac{\sin \frac{2}{3}\theta \sin \frac{1}{3}\theta}{\sin\theta} + \tfrac{1}{3}.\frac{\sin \frac{2}{9}\theta \sin \frac{1}{9}\theta}{\sin \frac{1}{3}\theta} + \ldots\ldots \frac{1}{3^{x-1}}.\frac{\sin 3^{-x}.2\theta \sin 3^{-x}\theta}{\sin 3^{-x+1}}.$$

$$u_{x+1} = \frac{1}{3^x}\,\frac{\sin 3^{-x-1}.2\theta \sin 3^{-x-1}\theta}{\sin 3^{-x}\theta},$$

and (eq. 47)
$$s_x = c + \tfrac{3}{4}.3^{-x}\cot 3^{-x}\theta$$
$$= \tfrac{3}{4}(3^{-x}\cot 3^{-x}\theta - \cot\theta);$$
$$s = \tfrac{3}{4}\left(\frac{1}{\theta} - \cot\theta\right).$$

Corollary. The series in *Ex.* 15 is evidently one fourth of the sum of the two series

$$\cot\theta + 3\cot \tfrac{1}{3}\theta + 9\cot \tfrac{1}{9}\theta + \ldots\ldots 3^{x-1}.\cot 3^{-x+1}\theta,$$

and $\dfrac{3\cos \frac{1}{3}\theta}{\sin\theta} + \dfrac{9\cos \frac{1}{9}\theta}{\sin \frac{1}{3}\theta} + \ldots\ldots\ldots 3^x.\dfrac{\cos 3^{-x}\theta}{\sin 3^{-x+1}\theta};$

and that in *Ex.* 16 is half the difference of the two

$$\frac{\cos\frac{1}{3}\theta}{\sin\theta}+\tfrac{1}{3}.\frac{\cos\frac{1}{9}\theta}{\sin\frac{1}{3}\theta}+\tfrac{1}{9}.\frac{\cos\frac{1}{27}\theta}{\sin\frac{1}{9}\theta}+\ldots\ldots\frac{1}{3^{x-1}}.\frac{\cos 3^{-x}\theta}{\sin 3^{-x+1}\theta},$$

and $\quad \cot\theta+\frac{1}{3}.\cot\frac{1}{3}\theta+\frac{1}{9}\cot\frac{1}{9}\theta+\ldots\, 3^{-x+1}.\cot 3^{-x+1}\theta.$

Examples of the products of Trigonometrical factors.

If
$$P_x = u_1 u_2 u_3 \ldots\ldots u_x,$$
$$P_{x+1} = u_1 u_2 u_3 \ldots\ldots u_{x+1};$$
$$\frac{P_{x+1}}{P_x} = u_{x+1},$$
$$\frac{P_{x+1}}{P_x} - 1 = \frac{P_{x+1} - P_x}{P_x} = \frac{\Delta P_x}{P_x} = u_{x+1} - 1.$$

Hence, if we have, $\varphi(x)$ being any function of x,

$$u_{x+1} - 1 = \frac{\Delta.\varphi(x)}{\varphi(x)},$$

we shall have
$$P_x = c\,\varphi(x);$$
$$P_1 = u_1 = c\,\varphi(1).$$

But
$$u_{x+1} = \frac{\Delta.\varphi(x)}{\varphi(x)} + 1$$
$$= \frac{\varphi(x+1)}{\varphi(x)},$$
$$u_1 = \frac{\varphi(1)}{\varphi(0)};$$

therefore
$$1 = c\,\varphi(0),$$
$$P_x = \frac{\varphi(x)}{\varphi(0)}.$$

Example 1. Let
$$u_{x+1} - 1 = \cos 2^x\theta - 1 = \frac{\Delta.\,2^{-x}\sin 2^x\theta}{2^{-x}\sin 2^x\theta},\ \text{(eq. 2)}$$

$$u_{x+1} = \cos 2^x\theta,$$
$$u_x = \cos 2^{x-1}\theta;$$
$$\varphi(x) = 2^{-x}\sin 2^x\theta,$$
$$\varphi(0) = \sin\theta;$$

$$P_x = \cos\theta\cos 2\theta\cos 4\theta\ldots\ldots\cos 2^{x-1}\theta = \frac{\varphi(x)}{\varphi(0)} = \frac{1}{2^x}.\frac{\sin 2^x\theta}{\sin\theta}.$$

Corollary. Write $2^{-x+1}\theta$ for θ, and this becomes

$$P_x = \cos\theta\cos\tfrac{1}{2}\theta\cos\tfrac{1}{4}\theta\ldots\ldots\cos 2^{-x+1}\theta = \frac{1}{2^x}.\frac{\sin 2\theta}{\sin 2^{-x+1}\theta},$$

and $\quad P_\infty = \cos\theta\cos\frac{1}{2}\theta\cos\frac{1}{4}\theta\ldots\ldots\ldots = \dfrac{\sin 2\theta}{2\theta},$

which might also be deduced from equation 4.

Example 2. Let $\quad u_{x+1} - 1 = 2\cos 3^x\theta = \dfrac{\Delta\sin 3^x.\frac{1}{2}\theta}{\sin 3^x.\frac{1}{2}\theta}$, (eq. 5)

$$u_x = 1 + 2\cos 3^{x-1}\theta$$
$$\varphi(x) = \sin 3^x.\tfrac{1}{2}\theta, \qquad \varphi(0) = \sin\tfrac{1}{2}\theta$$

$$P_x = (1+2\cos\theta)(1+2\cos 3\theta)(1+2\cos 9\theta)\ldots\ldots(1+2\cos 3^{x-1}\theta) = \frac{\sin 3^x.\frac{1}{2}\theta}{\sin\frac{1}{2}\theta},$$

which is question (92) of the Miscellany.

Corollary. Write $3^{-x+1}\theta$ for θ, and this becomes

$$P_x = (1+2\cos\theta)(1+2\cos\tfrac{1}{3}\theta)(1+2\cos\tfrac{1}{9}\theta)\ldots\ldots(1+2\cos 3^{-x+1}\theta) = \frac{\sin\frac{3}{2}\theta}{\sin 3^{-x+1}.\frac{1}{2}\theta}$$

Example 3. Let $u_{x+1}-1=-2\cos 3^x\theta = \frac{\Delta(-1)^x\cos 3^x.\frac{1}{2}\theta}{(-1)^x\cos 3^x.\frac{1}{2}\theta}$ (eq. 19)

$$u_x = 1-2\cos 3^{x-1}\theta,$$

$$\varphi(x) = (-1)^x\cos 3^x.\tfrac{1}{2}\theta, \qquad \varphi(0) = \cos\tfrac{1}{2}\theta;$$

$$P_x = (1-2\cos\theta)(1-2\cos 3\theta)\ldots\ldots(1-2\cos 3^{x-1}\theta) = (-1)^x.\frac{\cos 3^x.\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}.$$

Corollary. Write $3^{-x+1}\theta$ for θ, and this becomes

$$P_x = (1-2\cos\theta)(1-2\cos\tfrac{1}{3}\theta)\ldots\ldots\left(1-2\cos\frac{\theta}{3^{x-1}}\right) = (-1)^x.\frac{\cos\frac{3}{2}\theta}{\cos 3^{-x+1}.\frac{1}{2}\theta},$$

Example 4. Let $u_{x+1}-1 = \frac{2\cos 3^{-x-1}\theta}{\cos 3^{-x}\theta} = \frac{\Delta.\cot 3^{-x}\theta}{\cot 3^{-x}\theta}$, (eq. 45),

$$u_x = 1+2.\frac{\cos 3^{-x}\theta}{\cos 3^{-x+1}\theta},$$

$$\varphi(x) = \cot 3^{-x}\theta, \qquad \varphi(0) = \cot\theta$$

$$P_x = \left(1+2\frac{\cos\frac{1}{3}\theta}{\cos\theta}\right)\left(1+2\frac{\cos\frac{1}{9}\theta}{\cos\frac{1}{3}\theta}\right)\ldots\ldots\left(1+2\frac{\cos 3^{-x}\theta}{\cos 3^{-x+1}\theta}\right) = \frac{\cot 3^{-x}\theta}{\cot\theta}.$$

Corollary. Write $3^x\theta$ for θ, and this becomes

$$P_x = \left(1+2\frac{\cos\theta}{\cos 3\theta}\right)\left(1+2\frac{\cos 3\theta}{\cos 9\theta}\right)\ldots\ldots\left(1+2\frac{\cos 3^{x-1}\theta}{\cos 3^x\theta}\right) = \frac{\cot\theta}{\cot 3^x\theta}$$

Example 5. Let $u_{x+1}-1 = \frac{\cos 2^x.3\theta}{2\cos^3 2^x\theta} = \frac{\Delta.3^x\cot 2^x\theta}{3^x\cot 2^x\theta}$, (eq. 44);

$$u_x = 1+\tfrac{1}{2}.\frac{\cos 2^{x-1}.3\theta}{\cos^3 2^{x-1}\theta}$$

$$P_x = \left(1+\tfrac{1}{2}\frac{\cos 3\theta}{\cos^3\theta}\right)\left(1+\tfrac{1}{2}\frac{\cos 6\theta}{\cos^3 2\theta}\right)\ldots\ldots\left(1+\tfrac{1}{2}\frac{\cos 2^{x-1}3\theta}{\cos^3 2^{x-1}\theta}\right) = 3^x.\frac{\cot 2^x\theta}{\cot\theta}$$

Corollary. Write $2^{-x+1}\theta$ for θ, and multiply by 2^x,

$$P_x = \left(2+\frac{\cos 3\theta}{\cos^3\theta}\right)\left(2+\frac{\cos\frac{3}{2}\theta}{\cos^3\frac{1}{2}\theta}\right)\ldots\ldots\left(2+\frac{\cos 2^{-x+1}.3\theta}{\cos^3 2^{-x+1}\theta}\right) = \frac{6^x\cot 2\theta}{\cot 2^{-x+1}\theta}.$$

ARTICLE VII.

DIOPHANTINE SPECULATIONS.

By Wm. Lenhart, Esq. York, Pa.

NUMBER THREE.

In the Mathematical Diary, Vol. II, page 185, it is said, that, in No. 12, for June 1830, of the Annals of Mathematics pure and applied by M. Gergonne, M. Pagliani resolves this problem: "To find 1000 consecutive numbers of the natural series, such that the sum of their cubes shall be itself a cube," and also that M. Pagliani thinks that no other solution can be deduced, except by a very complicated analysis. We have not had the pleasure of seeing the number alluded to, nor, indeed, any of the numbers of M. Gergonne's celebrated work, and are therefore not aware of the method M. P. has pursued to obtain his numbers. As we deem the Problem, when generalized, very beautiful, we propose laying it before the readers of the Miscellany, accompanied by an ample solution, entirely free from a complicated analysis; and for which we claim their indulgence.

Problem. To find m numbers in arithmetical progression such that the sum of their cubes shall be itself a cube; and give examples when the numbers are consecutive numbers of the natural series.

SOLUTION.

Case I. When m is of the form $2n$, or an even number.

Let us assume the equation $x^3+y^3=(x+y)\,\{x(x-y)+y^2\}$. (*a*)

and suppose $\begin{cases} x=s+1,\ s+2,\ s+3,\ \&c \\ y=s\quad s-1,\ s-2,\ \&c. \end{cases}$ respectively.

Then will $x+y=2s+1$, and $x-y=1, 3, 5, 7,$ &c.; and by substituting the different values of x and y in equation (*a*) we shall find

$$(s+1)^3+(s)^3=(2s+1)\,.\,(s^2+s+1) \quad . \quad . \quad . \quad (1),$$
$$(s+2)^3+(s-1)^3=(2s+1)\,.\,(s^2+s+7) \quad . \quad . \quad (2),$$
$$(s+3)^3+(s-2)^3=(2s+1)\,.\,(s^2+s+19) \quad . \quad . \quad (3),$$
$$\&c. \qquad\qquad \&c.$$

Now the sum of . . . . . (1). (2) or of 4 cubes is $(2s+1)\,.\,(2s^2+2s+8)$,

of . . . . (1). (2). (3) " 6 " $(2s+1)\,.\,(3s^2+3s+27)$,

of . . (1).(2).(3) to (n)" $2n$ " $(2s+1)\,.\,(ns^2+ns+n^3)$;

Or making the multiplication

$$2ns^3+3ns^2+(2n^3+n)\,s+n^3 \quad . \quad . \quad . \quad . \quad (b)$$

which is a general expression for the sum of an even number of cubes, and which, by the problem, is to be made a cube. Put $n=4n'^3$, and divide by $(2n')^3$ then

$$s^3+\frac{3s^2}{2}+\left(\frac{32n'^6+1}{2}\right)s+8n'^6=\text{cube}=(s+2n'^2)^3:$$

This reduced will give

$$s=\frac{32n'^6-24n'^4+1}{12n'^2-3}=\frac{8n'^4-4n'^2-1}{3}.$$

Now, in order that the above series of values of x and y may be consecutive numbers of the natural series, it is evident that s must be a whole number greater than unity, which, as may be easily proved, will always be the case when n' is prime to or not divisible by 3.*

Let us suppose $n'=2$, then $n=32$, $m=64$, $s=37$ and $s+1=38$; and as 37 and 38 are the means of the 64 numbers we shall of course have 6 and 69 for the extremes; that is, 6 will be the first, and 69 the last of 64 consecutive numbers of the natural series such that the sum of their cubes shall be itself a cube: and its root $2n'(s+2n'^2)=180$.

Suppose $n'=5$, then $n=500$, $m=1000$ and $s=1633$, $s+1=1634$. Consequently 1134 will be the first, and 2133 the last of 1000 consecutive numbers of the natural series having the same properties: and $2n'(s+2n'\ )=16830$ will be the root of the cube to which the sum of their cubes is equal. These numbers are the same as those named in the Diary, as having been found by M. Pagliani.

But let us return to the original formula

$$2ns^3+3ns^2+(2n^3+n)\,s+n^3,$$

and make a cube of it by assuming the root $n+\left(\dfrac{2n^2+1}{3n}\right)s$.

From this assumption by reduction, we find

$$s=\frac{9n^2(-4n^4+5n^2-1)}{8n^6-42n^4+6n^2+1}=\frac{9n^2(n+1).(n-1)}{2n^2(5-n^2)+1}$$

in which n may be taken for any number $>$ than unity. If $n=2$, then $m=4$, $s=12$, $s+1=13$, and the four roots will be the consecutive numbers 11, 12, 13 and 14. If $n=3$, then $m=6$; $s=-\frac{648}{71}$, $s+1=-\frac{577}{71}$ and thence the progression 435, 506, 577, 648, 719, 790, having a common difference 71, and the sum of their cubes $(1155)^3$.

Case II. When m is of the form $2n+1$, or an odd number.

By adding $(s-n)^3$ to formula (b) we shall find

$$(2n+1)\,s^3+(2n+1).(n+1)\,ns \quad . \quad . \quad . \quad (c),$$

which is evidently a general expression for the sum of an odd number of cubes. It may be simplified by putting m, which, in this case, is an odd number in the place of $2n+1$. Formula (c) then becomes

$$ms^3+\left(\frac{m(m^2-1)}{4}\right)s \quad . \quad . \quad . \quad . \quad (d),$$

which by the problem, is to be made a cube. It is a cube when $s=\frac{1}{2}$; put therefore $s=\frac{1}{2}+t$, and (d) becomes

$$\frac{m^3}{8}+\left(\frac{m^3+2m}{4}\right)t+\frac{3m}{2}t^2+mt^3 \quad . \quad . \quad (f),$$

or, putting $m=m'^2$, and dividing by m'^3

$$\frac{m'^6}{8}+\left(\frac{m'^4+2}{4}\right)t+\frac{3t^2}{2}+t^3=\text{cube}=\left(\frac{m'^2}{2}+t\right)^3.$$

---

* If, however, we take $n'=1$, we shall have $n=4$, $m=8$, $s=1$, $s+1=2$, and the eight roots will be 2, 1, 3, 0, 4, -1, 5 and -2; but four of these cancel each other, and one of them is 0, there will therefore only remain the consecutive roots 3, 4 and 5. And thus we curiously obtain the well known equation $(3)^3+(4)^3+(5)^3=(6)^3$.

This being reduced will furnish

$$t = \frac{m'^6 - 3m'^4 + 2}{6(m'^2 - 1)} = \frac{m'^4 - 2m'^2 - 2}{6} \text{ and therefore}$$

$$s = \tfrac{1}{2} + t = \frac{(m'^2 - 1)^2}{6},$$

which will evidently always be a whole number when m' is an odd number $>$ than unity, and prime to, or not divisible by 3.* Assume $m' = 5$ then will $m = 125$, $s = 96$, $s + 1 = 97$, $n = 62$, $s - n = 34$, $s + n = 158$; and therefore 34 will be the first, and 158 the last, of 125 consecutive numbers of the natural series such that the sum of their cubes shall be itself a cube; and $m'\left(\frac{m'^2}{2} + t\right) = 540$ will be its root. If we take $m' = 9$ we shall find 2108 to be the first, and 4292 the last of 729 numbers in arithmetical progression, having 3 for a common difference, and the sum of their cubes $(9960)^3$.

Again: equate (f) to $\left(\frac{m}{2} + \left(\frac{m^2+2}{3m}\right) t\right)^3$, and we shall find

$s = \frac{4(m^2-1)^2}{9(2m^2+1) - (m^2-1)^2}$, in which any odd number $>$ than unity may be taken for m, and thence integers obtained to answer.

There is another, and, perhaps, a preferable method of making formula (d) a cube. It is briefly this: put $ms^3 + \left(\frac{m(m^2-1)}{4}\right)s = p^3 m^3 s^3$, then, by dividing by ms, and reducing, &c., we shall find

$$4s^2 = \frac{m^2-1}{p^3 m^2 - 1} = \square \quad . \quad . \quad . \quad (g).$$

Now (g) is plainly a square when $p = 1$; put then $p = 1 + r$, and from the reciprocal of (g) we shall have

$$1 + \frac{3m^2}{m^2-1} r + \frac{3m^2}{m^2-1} r^2 + \frac{m^2}{m^2-1} r^3 = \square = \left(1 + \frac{3m^2}{2(m^2-1)} r\right)^2.$$

$$r = \frac{3(4 - m^2)}{4(m^2-1)}; \text{ then } p = 1 + r = \frac{m^2+8}{4(m^2-1)} \text{ and consequently}$$

$$s = \frac{4(m^2-1)^2}{9(2m^2+1) - (m^2-1)^2}, \text{ the same as before.}$$

Equation (g) is also a square when $m = 3$ and $p = \frac{1}{2}$. Then $s = 4$, and we obtain, as before, the three consecutive numbers 3, 4 and 5.

Problem. Find three square numbers such that the difference of every two of them may be squares.

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* We shall take this occasion to note that Barlow in his miscellaneous propositions, page 258 of his Theory of Numbers, has the following proposition, namely: "If m be a prime number greater than 3, then will $m^2 - 1$ be divisible by 24." Now this is certainly true; but it is also true if m be any odd number not divisible by 3, which, if thus enunciated, would make the proposition more general.

Solution. If we suppose the roots of the three required squares to be represented by $\frac{x^2+y^2}{x^2-y^2}$, $\frac{v^2+w^2}{v^2-w^2}$ and unity; or their reciprocals, unity, $\frac{v^2-w^2}{v^2+w^2}$ and $\frac{x^2-y^2}{x^2+y^2}$. Or $\frac{v^2+w^2}{2vw}$, $\frac{x^2+y^2}{2xy}$ and unity; or their reciprocals, unity, $\frac{2xy}{x^2+y^2}$ and $\frac{2vw}{v^2+w^2}$; then will the following conditions be answered, namely:

$$\left(\frac{x^2+y^2}{x^2-y^2}\right)^2-1=\left(\frac{2xy}{x^2-y^2}\right)^2;\ \left(\frac{v^2+w^2}{v^2-w^2}\right)^2-1=\left(\frac{2vw}{v^2-w^2}\right)^2;$$

$$1-\left(\frac{v^2-w^2}{v^2+w^2}\right)^2=\left(\frac{2vw}{v^2+w^2}\right)^2;\ 1-\left(\frac{x^2-y^2}{x^2+y^2}\right)^2=\left(\frac{2xy}{x^2+y^2}\right)^2;$$

$$\left(\frac{v^2+w^2}{2vw}\right)^2-1=\left(\frac{v^2-w^2}{2vw}\right)^2;\ \left(\frac{x^2+y^2}{2xy}\right)^2-1=\left(\frac{x^2-y^2}{2xy}\right)^2;$$

$$1-\left(\frac{2xy}{x^2+y^2}\right)^2=\left(\frac{x^2-y^2}{x^2+y^2}\right)^2 \text{ and } 1-\left(\frac{2vw}{v^2+w^2}\right)^2=\left(\frac{v^2-w^2}{v^2+w^2}\right)^2:$$

and it will therefore only remain to make squares of the formulas following, viz.:

$$\text{I.}\qquad \left(\frac{x^2+y^2}{x^2-y^2}\right)^2-\left(\frac{v^2+w^2}{v^2-w^2}\right)^2$$

$$\text{II.}\qquad \left(\frac{v^2-w^2}{v^2+w^2}\right)^2-\left(\frac{x^2-y^2}{x^2+y^2}\right)^2$$

$$\text{III.}\qquad \left(\frac{v^2+w^2}{2vw}\right)^2-\left(\frac{x^2+y^2}{2xy}\right)^2$$

$$\text{IV.}\qquad \left(\frac{2xy}{x^2+y^2}\right)^2-\left(\frac{2vw}{v^2+w^2}\right)^2$$

Now, if I, II, III and IV, be properly reduced, and the square factors and denominators be rejected, it will be found that each will require the same formula to be made a square, namely:

$$(vx+wy).(vx-wy).(vy+wx).(vy-wx) \quad . \quad . \quad . \quad (5)$$

and therefore the same values of v, w, x and y, that shall render (5) a square, will also enable us at once to obtain four sets of integers to answer, which is exceedingly curious; and, as we presume, something entirely new on this old and often discussed subject.

To make (5) a square, we shall proceed thus: put

$$vy+wx=t\,(vx-wy) \quad . \quad . \quad . \quad . \quad . \quad (6)$$

then substituting in (5) and dividing by $(vx-wy)^2$ we shall have to make

$$t\,(vx+wy).(vy-wx)=\square \quad . \quad . \quad . \quad . \quad (7).$$

From (6) we obtain $\frac{v}{w}=\frac{ty+x}{tx-y}$, therefore $v=ty+x$ and $w=tx-y$

$$t\,(x^2-y^2+2txy).(ty^2-tx^2+2xy)=\square,\ \text{Consequently}$$

$$x^2-y^2+2txy=\square \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

$$y^2-x^2+\frac{2xy}{t}=\square \quad . \quad . \quad . \quad . \quad . \quad (9).$$

These formulas are identical with those in the Ladies' Diary, Vol. IV, page 346, and in the American edition of Young's Algebra, page 350, to which we might therefore refer, but as they are there rather laboriously and not neatly reduced, we have concluded to resolve them here at length.

In the first place, then (9) is a square, when $x=\frac{2y}{t}$; consequently (8) becomes $4+3t^2=\square=(2-pt)^2$; from which $t=\frac{4p}{p^2-3}$, therefore $x=p^2-3$ and $y=2p$. Then $v=(p^2+1)^2+8$ and $w=2p(p^2-3)$. If $p=2$, then $x=1$, $y=4$, $v=33$, $w=4$ and the required roots will be

$\frac{17}{15}$, $\frac{1105}{1073}$ and 1; $\frac{15}{17}$, $\frac{1073}{1105}$ and 1; $\frac{1105}{8\times 33}$, $\frac{17}{8}$ and 1; $\frac{8\times 33}{1105}$, $\frac{8}{17}$ and 1; Or, reduced to integers

16095, 16575 and 18241; 264, 561 and 1105;
975, 1073 " 1105; 264, 520 " 1105.

In the second place, put formula (8) or

$$x^2-y^2+2txy=\square=(x-py)^2;$$

then $x=p^2+1$, aud $y=2(p+t)$: assume also

$$t^2y^2-t^2x^2+2t\,xy=\square=(ty-r)^2$$

then $-t^2x^2+2\,txy=-2rty+r^2$; or substituting for y its value $2(p+t)$; $-t^2x^2+4ptx+4t^2x=-4pxt-4rt^2+r^2$.

Now, put $-t^2x^2+4t^2x=-4rt^2$ and $4ptx=-4prt+r^2$, then

$$\text{will } r=\frac{x^2-4x}{4} \text{ and } t=\frac{r^2}{4p(r+x)}=\frac{r^2}{px^2}=\frac{(p^2-3)^2}{16p}.$$

$$\text{Consequently } x=p^2+1, y=2(p+t)=\frac{(p^2-3)^2+16p^2}{8p}=\frac{(p^2+1).(p^2+9)}{8p}$$

or dividing by $\frac{p^2+1}{8p}$, $x=8p$ and $y=p^2+9$.

$$\text{Again: } v=ty+x=\frac{(p^2-3)^2.(p^2+9)+128p^2}{16p}=\frac{(p^2+1).(p^4+2p^2+81)}{16p}$$

$$\text{and } w=tx-y=\frac{(p^2-3)^2}{2}-p^2-9=\frac{p^4-8p^2-9}{2}=\frac{(p^2+1).(p^2-9)}{2}$$

or, dividing by $\frac{p^2+1}{16p}$ we shall get $v=p^4+2p^2+81=(p^2+4p+9).(p^2-4p+9)$, and $w=8p(p^2-9)$.

Now, if $p=1$, then $x=4$, $y=5$, $v=21$, $w=16$ and thence the roots,

$\frac{41}{9}$, $\frac{697}{185}$ and 1; $\frac{9}{41}$, $\frac{185}{697}$ and 1; $\frac{697}{672}$, $\frac{41}{40}$ and 1; $\frac{672}{697}$, $\frac{40}{41}$ and 1:

Or, in integers

1665, 6273 and 7585; 3360, 3444 and 3485;
672, 680 and 697; 153, 185 and 697.

We may also make (7) a square in the following manner:

From (6) we get $\frac{x}{y}=\frac{tw+v}{tv-w}$, or $x=tw+v$ and $y=tv-w$.

Hence (7) becomes $t\,(v^2-w^2+2tw).\,(tv^2-tw^2-2vw)=\square$, and therefore $v^2-w^2+2tvw=\square=\text{A}^2$ . . . . (10)

$$v^2-w^2-\frac{2vw}{t}=\square=\text{B}^2 \quad . \quad . \quad . \quad . \qquad (11)$$

Now $(10)-(11)=(\text{A}+\text{B}).\,(\text{A}-\text{B})=2v\times\frac{w(t^2+1)}{t}$, and putting $\text{A}+\text{B}=2v$ and $\text{A}-\text{B}=\frac{w(t^2+1)}{t}$ we shall find $\text{A}=v+\frac{w\,(t^2+1)}{2t}$ and thence from (10), $v=(t^2+1)^2+4t^2$ & $w=4t(t^2-1)$; also $x=tw+v=(t^2+1)^2+4t^4$ and $y=tv-w=t\,(\,(t^2+1)^2+4)$, in which $t>1$. If $t=3$, then $v=17$, $w=12$, $x=53$, $y=39$, and the two least sets of roots will be found to be

644, 725 and 2165; and 2040, 2067 and 2165.

Problem Find three squares such that the sum of every two shall make a square.

Solution. Suppose $\left(\frac{p^2-1}{2p}\right)^2$, $\left(\frac{2q}{q^2-1}\right)^2$ and unity, to be the three required squares, then will

$$\left(\frac{p^2-1}{2p}\right)^2+1=\left(\frac{p^2+1}{2p}\right)^2=\square,\text{ and }\left(\frac{2q}{q^2-1}\right)^2+1=\left(\frac{q^2+1}{q^2-1}\right)^2=\square;$$

and therefore it only remains to make

$$\left(\frac{p^2-1}{2p}\right)^2+\left(\frac{2q}{q^2-1}\right)^2=\square;$$

Or, actually performing the addition, and rejecting the square denominator, to make

$(p^2-1)^2.\,(q^2-1)^2+16\,p^2q^2=\square=(\,(p^2-1).\,(q^2-1)+8)^2$ by assumption.

This, by a proper reduction, furnishes the extraordinary equation

$$p^2+q^2=5=(1)^2+(2)^2,$$

which is resolved in all the elementary books, and furnishes

$$p=\frac{s^2+4s-1}{s^2+1}\text{ and }q=\frac{2(s^2-s-1)}{s^2+1}$$

in which s may be any number other than unity or three. If $s=2$, then

$$p=\frac{11}{5},\ q=\frac{2}{5}\text{ and the required squares will be}$$

$\left(\frac{48}{55}\right)^2$, $\left(\frac{20}{21}\right)^2$ and unity; or their reciprocals $\left(\frac{55}{48}\right)^2$, $\left(\frac{21}{20}\right)^2$ and 1.

Or integers

$(1008)^2$, $(1100)^2$ and $(1155)^2$; $(240)^2$, $(252)^2$ and $(275)^2$.

WM. LENHART.

YORK, Penn., *January*, 1839.

ARTICLE VIII.

ON THE APPLICATION OF STURM'S THEOREM.

I. *To Equations of the Fourth Degree.*

Any equation of the fourth degree may be put into the form

$$\mathrm{X} = x^4 + ax^2 + bx + c = 0;$$

and applying the process of Sturm, we find

$$\begin{aligned}
\mathrm{X}_1 &= 4x^3 + 2ax + b,\\
\mathrm{X}_2 &= -\,2ax^2 - 3bx - 4c,\\
\mathrm{X}_3 &= (8ac - 2a^3 - 9b^2)\,x - b\,(a^2 + 12c)\\
&= \mathrm{A}x + \mathrm{B},\\
\mathrm{X}_4 &= 4c\mathrm{A}^2 - 3b\mathrm{AB} + 2a\mathrm{B}^2\\
&= 4\,(a^2 + 12c)^3 - (27b^2 + 2a^3 - 72ac)^2.
\end{aligned}$$

Making, for the co-efficients of the polynomial X_3,

$$\begin{aligned}
\mathrm{A} &= 8ac - 2a^3 - 9b^2,\\
\mathrm{B} &= -\,b\,(a^2 + 12c).
\end{aligned}$$

In calculating these co-efficients, if those of any one polynomial, as, for instance, A and B, have a common divisor, that divisor may be neglected, if it is a numerical quantity, or any quantity essentially positive. In cases also where the ordinary methods of approximation might be advantageously employed, that is, when the roots are widely separated, approximate values of these co-efficients may be used which are nearly proportional to them. If any of the leading or terminating coefficients of the succeeding polynomials are found to approximate to zero, the true values must be recurred to; for under such circumstances, there may be roots either equal, or having a very small difference.

Now for $x = \infty$, these polynomials will become

$$\mathrm{X} = +\infty,\quad \mathrm{X}_1 = +\infty,\quad \mathrm{X}_2 = -a.\infty,\quad \mathrm{X}_3 = \mathrm{A}.\infty,\quad \mathrm{X}_4 = \text{const.}$$

Let i be the number of changes of sign in these quantities, and represent by m the number of real roots of the equation $\mathrm{X} = 0$; then, as a necessary consequence of Sturm's Theorem, when the number of polynomials is complete,

$$m = \pm\,(4 - 2\,i) = \pm\,2\,(2 - i).$$

Thus we may form the following Table of conditions: when

		$\mathrm{X}_4 > 0$		$\mathrm{X}_4 < 0$	
$a > 0$	$\mathrm{A} > 0$	$i = 2$	$m = 0$	$i = 3$	$m = 2,$
"	$\mathrm{A} < 0$	2	0	1	2,
$a < 0$	$\mathrm{A} > 0$	0	4	1	2,
"	$\mathrm{A} < 0$	2	0	1	2,

So that, if $\mathrm{X}_4 < 0$, or if

$$(27b^2 + 2a^3 - 7ac)^2 > 4\,(a^2 + 12c)^3,$$

the equation has two real roots and no more; while if $\mathrm{X}_4 > 0$, or

$$(27b^2 + 2a^3 - 7ac)^2 < 4\,(a^2 + 12c)^3,$$

the equation has either four real roots, or none at all, and it has four if, at the same time,

$$a < 0 \text{ and } \mathrm{A} = 8\,ac - 2\,a^3 - 9b^2 > 0,$$

It may be remarked that we cannot have $i = 3$; for X_4 may be put into the form

$$\text{X}_4 = \tfrac{1}{a}\{64\, a^6 b^2 + 2a^3\,(216\, b^4 + 36 b^2\, \text{A} + \text{A}^2) + (9b^2 + \text{A})^3\};$$

so that if $$a > 0 \text{ and } \text{A} > 0,$$
we necessarily have $$\text{X}_4 > 0.$$

To find the nature of these roots, let $x = 0$, and the polynomials are

$$\text{X} = c, \quad \text{X}_1 = b, \quad \text{X}_2 = -c, \quad \text{X}_3 = \text{B}, \quad \text{X}_4 = \text{const}.$$

Let k be the number of changes of sign in these quantities, which is obviously not affected by any change in the value of b, otherwise than as this change may affect the sign of B, and m' the number of *positive* roots in the equation $\text{X} = 0$. Then will

$$m' = \pm\,(i - k);$$

and the only cases we have to examine are, 1°. when there are four real roots, or $\text{X}_4 > 0$ and $i = 0$, and, 2°. when there are two real roots or $\text{X}_4 < 0$ and $i = 1$. Then when

		$\text{X}_4 > 0,\ i = 0$		$\text{X}_4 < 0$ and $i = 1$,	
$c > 0$	$\text{B} > 0$	$k = 2$	$m' = 2$	$k = 3$	$m' = 2$,
"	$\text{B} < 0$	$= 2$	2	1	0,
$c < 0$	$\text{B} > 0$	$= 1$	1	2	1,
"	$\text{B} < 0$	$= 3$	3	2	1,

so that, if all the four roots are real,

2 are positive and 2 negative, if $c > 0$;
1 is positive and 3 negative, if $c < 0$ and $\text{B} > 0$ or $b\,(a^2 + 12c) < 0$;
3 are positive and one negative, if $c < 0$ and $\text{B} < 0$ or $b\,(a^2 + 12c) > 0$.

If only two of the roots are real,

1 of them is positive and 1 negative, if $c < 0$;
both are positive, if $c > 0$ and $\text{B} > 0$ or $b\,(a^2 + 12c) < 0$;
both are negative, if $c > 0$ and $\text{B} < 0$ or $b\,(a^2 + 12c) > 0$.

It is evident that if $\text{X}_4 = 0$, or

$$(27b^2 + 2a^3 - 7ac)^2 = 4\,(a^2 + 12c)^3,$$

two of the roots are equal, and these are each

$$= -\frac{\text{B}}{\text{A}} = \frac{b\,(a^2 + 12c)}{8\,ac - 2a^3 - 9b^2},$$

and the other roots are those of the equation

$$(\text{A}x - \text{B})^2 + \text{A}^2 a + 2\text{B}^2 = 0.$$

If $\text{A} = 0$ and $\text{B} = 0$, there are two pairs of equal roots, which are equal to the roots of the equation

$$2ax^2 + 3bx + 4c = 0;$$

except when the roots of this last equation are equal, that is when

$$9b^2 - 32\, ac = 0,$$

and then there are three roots each $= -\dfrac{3b}{4a}$, the fourth one being $\dfrac{9b}{4a}$.

If, $\text{A} = 0$, while B is finite, $\text{X}_3 = \text{B}$, and the Theorem gives in this case two real roots when a and B have contrary signs, and no real roots otherwise; these two roots are

both positive if B and c are both positive;
both negative if B and c are both negative;
one positive and one negative if B and c have different signs.

If $a=0$, or the equation is

$$X = x^4 + bx + c = 0,$$

the previous conclusions are no longer applicable; for then

$$X_1 = 4x^3 + b,$$
$$X_2 = -3bx - 4c,$$
$$X_3 = b\,(256c^3 - 27b^4);$$

and it is easily seen that the equation can never have more than two real roots, and two only when b and X_3 have different signs, that is when

$$256c^3 - 27.\,b^4 < 0;$$

also, these two roots have different signs if $c<0$;

they are both positive if $c>0$, $b<0$;

they are both negative if $c>0$, $b>0$;

and they are each $= -\frac{4c}{3b}$, if $256c^3 - 27\,b^4 = 0$.

EXAMPLE.

Take the equation (8) in the solution of question (107),

$$2t^4 - kt^3 + 3t^2 - 2kt + 1 = 0,$$

and put $k = 2h$, $t = \frac{1}{x+h}$, then

$$X = x^4 + 3\,(1-2h^2)\,x^2 + 4\,(1-2h^2)\,hx + (1-h^2)\,(2+3h^2) = 0;$$
$$a = 3\,(1-2h^2), \quad b = 4h\,(1-2h^2), \quad c = (1-h\ )\,(2+3h^2);$$
$$A = (1-2h^2)(16h^2-1), \quad B = 2h\,(1-2h^2)(8h^2-11);$$
$$X_4 = -\ 128\,h^6 - 795\,h^4 + 597\,h^2 + 2.$$

The equation $X_4 = 0$, will be found, by the Theorem, to have three real values of h^2, two negative and one positive; the latter one, which is the only one that is here admissible, is

$$h^2 = ,680158;$$

so that, if $h^2 <,680158$, or $k^2 < 2,720632$; X_4 is >0.

But then, if $h^2 > \frac{1}{2}$; $a<0$, $A<0$;

if $h^2 < \frac{1}{2} > \frac{1}{16}$; $a>0$, $A<0$;

if $h^2 < \frac{1}{16}$; $a>0$, $A>0$,

so that the equation has no real roots.

If $h^2 = ,680158$ or $k^2 = 2,720632$; $X_4 = 0$

and the equation has a pair of real roots, each $= ,463888$.

If $h^2 >,680158$ or $k^2 > 2,720632$; $X_4 < 0$,

and the equation has two real roots; also

if $h<1$, or $k<2$; $c>0$, $B>0$,

and both roots are positive;

if $h=1$, or $k=2$; $c=0$,

and one root is zero, then $t = \frac{1}{h} = 1$;

if $h>1$, or $k>2$; $c<0$.

and the roots have different signs.

The negative value of x is always $> -h$; for since $c<0$, $X_4 < 0$, the polynomials have two changes of sign by the Table, when $x=0$; and when $x = -h$, h varying from 1 to ∞, the polynomials are

$X=2>0$, $X_1=-h<0$, $X_2=h^2-2$, $X_3=2h^3-h>0$, $X_4<0$; so that there are three changes of sign, and consequently one real root between 0 and $-h$ while h is >1; it follows that both the resulting values of t are positive.

---

2. *To equations of the fifth Degree.*

Every equation of the fifth degree may be put into the form

$$X = x^5+ax^3+bx^2+cx+d=0,$$

and for this equation, we find

$$\begin{aligned}
X_1 &= 5x^4+3ax^2+2bx+c,\\
X_2 &= -2ax^3-3bx^2-4cx-5d.\\
X_3 &= Ax^2+Bx+C,\\
X_4 &= Dx+E,\\
X_5 &= BDE-CD^2-AE^2\,;
\end{aligned}$$

making thus, for the co-efficients of the polynomial X_3,

$$\begin{aligned}
A &= 40ac-12a^3-45b^2,\\
B &= 50ad-8a^2b-60bc,\\
C &= -4a^2c-75bd\,;
\end{aligned}$$

and, for those of the polynomial X_4,

$$\begin{aligned}
D &= 4cA^2+2aB^2-2aAC-3bAB,\\
E &= 5dA^2+2aBC-3bAC\,;
\end{aligned}$$

the same remarks apply to the calculation of these co-efficients, as to those of the fourth degree. If $x=\infty$,

$X=+\infty$, $X_1=+\infty$, $X_2=-a.\infty$, $X_3=A.\infty$, $X_4=D.\infty$, $X_5=$const.; and using i and m as before, having in this case

$$m=\pm(5-2i),$$

the table of conditions is

			$X_5>0$		$X_5<0$	
			$i=$	$m=$	$i=$	$m=$
$a>0$	$A>0$	$D>0$	2	1	3	1,
"	"	$D<0$	4	3	3	1,
"	$A<0$	$D>0$	2	1	3	1,
"	"	$D<0$	2	1	1	3,
$a<0$	$A>0$	$D>0$	0	5	1	3,
"	"	$D<0$	2	1	1	3,
"	$A<0$	$D>0$	2	1	3	1,
"	"	$D<0$	2	1	1	3.

The second of this series of conditions is impossible; since,

$$\text{if } a>0, \qquad A>0\,;$$

$$40\,ac=A+12a^3+45b^2>0, \text{ and } c>0\,;$$

$$\begin{aligned}
\text{then } D &= 4cA^2+2aB^2-A\,(2\,aC+3bB)\\
&= 4cA^2+2aB^2+4cA\,(2a^3+45b^2)+24a^2b^2A\\
&> 0.
\end{aligned}$$

It follows that, if $X_5>0$, there must be either five real roots, or only one; and there are five, only when, at the same time,

$$a<0, \qquad A>0, \qquad D>0.$$

If $X_5 < 0$, there may be one or three real roots, but no more; and there will be three when, at the same time, either

$$1^\circ, \quad A < 0, \quad D < 0,$$
$$\text{or } 2^\circ, \quad a < 0, \quad A > 0.$$

To find the nature of these roots: if $x=0$, the polynomials become

$$X = d, \quad X_1 = c, \quad X_2 = -d, \quad X_3 = C, \quad X_4 = E, \quad X_5 = \text{const.}$$

Now, when there is only one real root, its sign, from simple considerations, is opposite to that of d; and therefore we need only examine the two cases of five real roots, indicated by $X_5 > 0$ and $i = 0$, and three real roots, indicated by $X_5 < 0$ and $i = 1$. Then using k and m' as before, so that $m' = \pm (i - k)$, the Table of conditions is

			$X_5 > 0$ and $i = 0$		$X_5 < 0$ and $i = 1$	
$d > 0$	$C > 0$	$E > 0$	$k = 2$	$m' = 2$	$k = 3$	$m' = 2,$
"	"	$E < 0$	4	4	3	2,
"	$C < 0$	$E > 0$	2	2	3	2,
"	"	$E < 0$	2	2	1	0,
$d < 0$	$C > 0$	$E > 0$	1	1	2	1,
"	"	$E < 0$	3	3	2	1,
"	$C < 0$	$E > 0$	3	3	4	3,
"	"	$E < 0$	3	3	2	1,

Then when there are five real roots,
if $d > 0$, 4 are positive and 1 negative, when $C > 0$, $E < 0$;
2 are positive and 3 negative, in all other cases:
if $d < 0$, 1 is positive and 4 negative, when $C > 0$, $E > 0$;
3 are positive and 2 negative, in all other cases.
Also, when there are three real roots,
if $d > 0$, all of them are negative, when $C < 0$, $E < 0$;
2 are positive and 1 negative, in all other cases.
if $d < 0$, all of them are positive, when $C < 0$, $E > 0$;
1 is positive and 2 negative, in all other cases.

If $X_5 = 0$, there are a pair of equal roots, each $= -\frac{E}{D}$, and the other three are the roots of the equation

$$D^2x^3 - 2D^2Ex^2 + (aD^2 + 3E^2)\,Dx + BD^2 - 2aD^2E - 4E^3 = 0.$$

If $D = 0$ and $E = 0$, there are two pairs of equal roots, which are the roots of the equation

$$Ax^2 + Bx + C = 0,$$

the fifth root being $= -\frac{2B}{A}$; except when the roots of this equation are equal, or when

$$B^2 - 4AC = 0,$$

and then there are three roots, each $= -\frac{B}{2A}$, and the other two are those of the equation

$$2\,AB^2x^2 - 3B^4x + 16A^4d = 0$$

If $A=0$, $B=0$ and $C=0$, which could only be when, either

$$a=10a'^2,\quad b=\pm 20a'^3,\quad c=-15a'^4,\quad d=\pm 4a'^5,$$

and then 4 roots are each $=\pm a'$ and the fifth $=\mp 4a'$; or

$$a=-15a'^2,\quad b=\pm 10a'^3,\quad c=60a'^4,\quad d=\mp 72a'^5,$$

and then 3 roots are each $=\pm 2a'$ and 2 are each $=\mp 3a'$.

If $D=0$, $X_4=E$, and the general results do not apply; it will be found that the equation has then three real roots,

$$\text{if } a<0,\quad A>0;$$

and that, of these roots,

if $d>0$, all of them are negative when $C<0$, $E<0$;
two positive and one negative in other cases:
if $d<0$, all of them are positive when $C<0$, $E>0$;
one positive and two negative in other cases;

otherwise, the equation has but one real root which has the opposite sign to d.

If $A=0$, then $X_3=Bx+C$,
$X_4=B\,(3bBC^2+5dB^3-2aC^3-4cB^2C)$;

and the equation has three real roots

$$\text{if } a<0,\quad B>0;$$

these roots being,

if $d>0$, all negative when $C<0$, $X_4<0$,
2 positive and 1 negative in other cases;
if $d<0$, all positive when $C<0$, $X_4>0$,
1 positive and two negative in other cases:

otherwise, the equation has but one real root which has the opposite sign to d.

If $A=0$ and $B=0$; then $X_3=C$, and the equation has three real roots

$$\text{if } a<0,\quad C>0;$$

of which, if $d>0$, two are positive,
if $d<0$, one is positive,

otherwise it has only one real root.

If $a=0$, the equation and polynomials are

$$\begin{aligned}
X &= x^5+bx^2+cx+d=0,\\
X_1 &= 5x^4+2bx+c,\\
X_2 &= -3bx^2-4cx-5d,\\
X_3 &= Ax+B,\\
X_4 &= 5dA^2-4cAB+3bB^2;
\end{aligned}$$

where $A=b\,(320c^3-600bcd-54b^4)$,
$B=b\,(400c\,d-375bd^2-27b^3c)$.

It is found that this equation has three real roots, if A and X_4 have the same sign, provided b have a contrary sign; and that of these roots,

if $d>0$, all are negative when $B<0$, $X_4<0$,
two positive in other cases;
if $d<0$, all are positive if $B<0$. $X^4>0$,
1 is positive and 2 negative in other cases;

otherwise there is but one real root, which has the contrary sign to d.

If $a=0$ and $b=0$, the equation and the polynomials are

$$\begin{aligned} X &= x^5+cx+d=0,\\ X_1 &= 5x^4+c,\\ X_2 &= -4cx-5d,\\ X_3 &= -(\tfrac{1}{5}c)^5-(\tfrac{1}{4}d)^4. \end{aligned}$$

and this equation has three real roots when

$$X_3>0, \text{ or } -(\tfrac{1}{5}c)^5>(\tfrac{1}{4}d)^4,$$

of which only one is positive when $d<0$, and two are positive when $d>0$; otherwise it has only one real root.

As it may often spare the labor of a transformation, we shall also give, while on the subject, the application of the Theorem

3. *To equations of the Third Degree,*

in their most general form, that is when

$$\begin{aligned} X &= x^3+ax^2+bx+c=0,\\ X_1 &= 3x^2+2ax+b,\\ X_2 &= 2\,(a^2-3b)x+ab-9c\\ &= Ax+B,\\ X_3 &= -\,bA^2+2aAB-3B^2;\\ &= 9\,(a^2b^2+18abc-4a^3c-27c^2-4b3)\\ &= 3\,\{4\,(a^2-3b)(b^2-3ac)-(ab-9c)^2\}\\ &= \tfrac{1}{4}\,\{4\,(a^2-3b)^3-(2a^3-9ab+27c)^2\}. \end{aligned}$$

If $x=\infty$, we have

$$X=+\infty,\quad X_1=+\infty,\quad X_2=A.\infty,\quad X_3=\text{const.},$$

And since, in this case, $m=\pm\,(3-2i)$, when

$A>0$	$X_3>0$	$i=0$	$m=3,$
>0	<0	1	$1,$
<0	<0	1	$1.$

These are all the conditions that can obtain, since if $A<0$, we necessarily have $X_3<0$, from its last form; it follows also that if $X_3>0$, A will also be >0; hence the equation has three real roots when

$$X_3>0,\qquad \text{or}\quad a^2b^2+18abc-4a^3c-27c^2-4b^3>0,$$

and but one when

$$X_3<0,\qquad \text{or}\quad a^2b^2+18abc-4a^3c-27c^2-4b^3<0.$$

If $x=0$, the polynomials become

$$X=c,\qquad X_1=b,\qquad X_2=B,\qquad X_3=\text{const.}$$

When there is only one real root, its sign is contrary to that of c, but when there are three real roots, or $X_3>0$ and $i=0$, the number of positive roots is $=m'=k$, the number of changes of signs in these last quantities, and by proceeding as before we find that

if $c>0$, the three roots are all negative, when $b>0$, and $ab>9c$,
2 are positive and 1 negative, in other cases;

if $c<0$, the three roots are all positive, when $b>0$, and $ab<9c$,
1 is positive and 2 are negative, in other cases.

If $a=0$, the properties are reduced to the known ones, that the equation has three real roots when

$$-4b^3-27c^2>0;$$

two of which are positive when $c > 0$,
and only one is positive when $c < 0$;
otherwise, it has only one real root, which has a different sign to that of c.

If $X_3 = 0$, or $a^2b^2+18abc-4a^3c-27c^2-4b^3=0$,
the equation has a pair of roots $= \frac{9c-ab}{2a^2-6b}$, and the third is $\frac{4ab-a^3-9c}{a^2-3b}$.

If B $= 0$, or $ab = 9c$, then $X_3 = -b$, and the equation will have three real roots, when $b < 0$, or when a and c have contrary signs; two of which are positive, when $a < 0$, and one positive when $a > 0$.

If A $= 0$, or $a^2 = 3b$, then $X_2 =$ B $= ab - 9c = \frac{1}{3}(a^3-27c)$, and the equation has only one real root.

If A $= 0$ and B $= 0$, which can only be when $b = \frac{1}{3}a^2$, and $c = \frac{1}{27}a^3$, the three roots of the equation are each $= -\frac{1}{3}a$.

Δ.

ARTICLE IX.

NOTE, ON A CONTINUED PRODUCT.

BY THE EDITOR.

In equation (5), page 381, Vol. II, if we put

$$k' = -k = \frac{h^2+1}{h},$$

write for r_n its value, and divide the equation by $-P$, we get

$$\Sigma.\frac{1}{k'-2\cos 2(x\beta+\theta)} = \frac{nh}{1-h^2}.\quad\frac{1-h^{2n}}{1-2h^n\cos 2n\theta+h^{2n}}.$$

Multiply by $\qquad dk' = -\frac{1-h^2}{h^2}.\,dh,\qquad$ and integrate,

$$\Sigma.\log\{k'-2\cos 2(x\beta+\theta)\} = \int -\frac{ndh}{h}.\,\frac{1-h^{2n}}{1-2h^n\cos 2n\theta+h^{2n}}$$

$$= \int -\frac{d.h^n}{h^n}\left\{1-\frac{2h^{2n}-2h^n\cos 2n\theta}{h^{2n}-2h^n\cos 2n\theta+1}\right\}$$

$$= \int -\frac{d.h^n}{h^n}+\int\frac{d.h^n.(2h^n-2\cos 2n\theta)}{h^{2n}-2h^n\cos 2n\theta+1}$$

$$= -\log h^n + \log(h^{2n}-2h^n\cos 2n\theta+1),$$

and, by restoring the value of k',

$$\Sigma.\log\{h^2-2h\cos 2(x\beta+\theta)+1\} = \log(h^{2n}-2h^n\cos 2n\theta+1);$$

or eliminating the logarithms,

$$(h^2-2h\cos 2.\overline{\beta+\theta}+1)(h^2-2h\cos 2.\overline{2\beta+\theta}+1)\ldots\ldots(h^2-2h\cos 2.\overline{n\beta+\theta}+1)$$
$$= h^{2n}-2h^n\cos 2n\theta+1.$$

Here $n\beta = i\pi$; and if we take $i = 1$, and put $2n\,\theta = \theta'$, this equation becomes, after transposing the last factor,

$$\left(h^2 - 2h\cos\frac{\theta'}{n} + 1\right)\left(h^2 - 2h\cos\frac{2\pi+\theta'}{n} + 1\right)\ldots\ldots$$

$$\left(h^2 - 2h\cos\frac{2(n-1)\pi+\theta'}{n} + 1\right)$$

$$= h^{2n} - 2h^n\cos\theta' + 1.$$

This is the well known Theorem of Moivre, from which, had it occurred to me at the proper time, the sum in question (79) might have been deduced. The form in which I have deduced the Theorem, shows that any multiple of π, prime to n, might be used in it, instead of π, as is otherwise obvious.

A form of the theorem of Cotes, immediately deducible from this, is

$$\left(h^2 - 2h\cos\frac{\pi}{2n} + 1\right)\left(h^2 - 2h\cos\frac{3\pi}{2n} + 1\right)\ldots\ldots$$

$$\left(h^2 - 2h\cos\frac{2n-1}{2n}\pi + 1\right)$$

$$= h^{2n} + 1;$$

and from this M. Delaunay deduces a definite integral by a very ingenious process, which may be useful in other cases. It is this:

Each factor in the first member is comprised in the general formula

$$h^2 - 2h\cos\frac{2i+1}{2n}\pi + 1,$$

in which there must be given to i the n successive values

$$0,\ 1,\ 2,\ 3,\ \ldots\ldots\ \overline{n-1};$$

and in order to pass from one factor to the following one, the arc comprised under the sign *cos.* must be increased by the constant quantity $\frac{\pi}{n} = \omega$.

By taking the logarithms, it becomes

$$\Sigma.\log\left(h^2 - 2h\cos\frac{2i+1}{2n}\pi + 1\right) = \log(h^{2n}+1).$$

Multiply each member by ω, and put $(2i+1)\pi = 2nx$: then

$$\Sigma.\,\omega\log(h^2 - 2h\cos x + 1) = \omega\log(h^{2n}+1)$$

$$= \log(h^{2n}+1)^{\frac{\pi}{n}};$$

the sign Σ indicating a sum taken relatively to the variable x, which increases from $x = \frac{\pi}{2n}$, to $x = \frac{2n-1}{2n}\pi$, by constant differences, equal to ω.

If, now, we suppose that n becomes infinite, ω will become dx, the sum Σ will be changed into a definite integral taken between the limits $x=0$ and $x = \pi$, and the first member of the equation becomes

$$\int_0^{\pi}\log(h^2 - 2h\cos x + 1)\,dx.$$

To determine the second member, two cases must be distinguished:

1°. If $h<1$, $(h^{2n}+1)^{\frac{\pi}{n}}$ is reduced to 1, and we have

$$\int_0^\pi \log(h^2-2h\cos x+1)\,dx=0;$$

2°. If $h>1$, $(h^{2n}+1)^{\frac{\pi}{n}}$ becomes $h^{2\pi}$, and we have

$$\int_0^\pi \log(h^2-2h\cos x+1)\,dx=2\pi\log h.$$

M. Poisson finds the same integrals by a very different process in the *Journal de l'Ecole Polytechnique*, Cahier XVII.; and deduces the following, among many others, from them:

Integration by parts gives, in general,

$$\int \log(h^2-2h\cos x+1)\,dx=x\log(h^2-2h\cos x+1)$$

$$-2h\int\frac{x\sin x.dx}{h^2-2h\cos x+1};$$

hence, whence $h<1$,

$$\int_0^\pi \frac{x\sin x.\,dx}{h^2-2h\cos x+1}=\frac{\pi}{h}.\ \log(1+h),$$

and, when $h>1$,

$$\int_0^\pi \frac{x\sin x.\,dx}{h^2-2h\cos x+1}=\frac{\pi}{h}.\ \log\left(1+\frac{1}{h}\right).$$

If $h=1$, these two formulas will be coincident, and give a known result, namely,

$$\int_0^\pi \frac{x\sin x.\,dx}{2(1-\cos x)}=\pi.\ \log 2;$$

or, by making $x=2z$, and reducing,

$$\int_0^{\frac{1}{2}\pi} z\cot z.\,dz=\tfrac{1}{2}\pi.\ \log 2.$$

In these integrals also, h may be imaginary, and if we make

$$h=r(\cos\alpha+\sin\alpha\sqrt{-1})$$

the first or second, of the several pairs of equations, must be used, accordingly as we have $r<1$, or $r>1$.